

CHAPTER TWO
UNIVERSAL EXTENSIONS AND CRYSTALS

In this chapter we describe the crystalline nature of the universal extension. More precisely we associate with an abelian scheme (resp. Barsotti-Tate group) G/S a crystal, $\mathbb{E}^*(G)$, on S whose value of S , $\mathbb{E}^*(G)_S$, is the universal extension $E(G^*)$ of G^* by a vector group. By applying the functor Lie we then obtain a crystal in locally-free modules, $\mathbb{D}^*(G)$. If $f: G \rightarrow S$ is an abelian scheme then $\mathbb{D}^*(G)$ is nothing but the usual crystalline cohomology, $R^1 f_{\text{crys}*}(\mathcal{O}_{(G/S)}^{\text{crys}})$. On the other hand when G is a Barsotti-Tate group, $\mathbb{D}^*(G)$ is the generalized Dieudonné module associated to G .

One procedure for constructing crystals from the universal extension was given in [16]. Here we shall use a completely different approach allowing us to construct the crystals intrinsically without making use of liftings. Unfortunately, it seems that in order to verify that our crystals have reasonable properties (and in fact that the sheaves constructed are crystals) we must fall back on liftings.

We shall discuss separately the constructions for abelian schemes and for Barsotti-Tate groups. For abelian schemes the construction is straightforward. The procedure for Barsotti-Tate groups is more technical. The reason for the additional complications is the following: For G an abelian scheme our description of $E(G^*)$ uses exclusively the whole group G , while for G a Barsotti-Tate group we use the individual $G(n)$'s as well. But while G is smooth (resp. formally smooth)

and hence amenable to standard crystalline techniques, the individual $G(n)$'s are not usually smooth. We assume some familiarity with crystalline theory [2,3].

§1. THE CRYSTALLINE NATURE OF THE UNIVERSAL EXTENSION OF AN
ABELIAN SCHEME

Let $S_0 \hookrightarrow S$ be a (locally)-nilpotent immersion defined by an ideal I , endowed with (locally) nilpotent divided powers $(\gamma_n)_{n \geq 0}$. Let A and B be abelian schemes on S and $f_0: A_0 \rightarrow B_0$ a homomorphism between their reductions to S_0 . f_0 induces a map on the dual abelian schemes $f_0^*: B_0^* \rightarrow A_0^*$ and hence a map on the corresponding universal extensions $E(B_0^*) \rightarrow E(A_0^*)$. We've shown in chapter I [2.6.7, 3.2.3] that this is the map

$$(1.1) \quad \underline{\text{Ext}}^{\mathbb{Z}}(B_0, G_m) \rightarrow \underline{\text{Ext}}^{\mathbb{Z}}(A_0, G_m)$$

induced by f_0 .

We shall construct a homomorphism $E(B^*) \rightarrow E(A^*)$ lifting (1.1). Although this morphism depends on the triple (A, B, f_0) we shall denote it by $E_S^*(f_0)$. From the construction it follows that these homomorphisms enjoy the following properties:

(i) transitivity (= functoriality):

$$\text{Given } A, B, C, \quad A_0 \xrightarrow{f_0} B_0 \xrightarrow{g_0} C_0, \\ E_S^*(g_0 \circ f_0) = E_S^*(f_0) \circ E_S^*(g_0)$$

(ii) additivity:

$$\text{Given two homomorphisms } f_0, f_1: A_0 \rightarrow B_0 \\ E_S^*(f_0 + f_1) = E_S^*(f_0) + E_S^*(f_1)$$

(iii) functoriality in S :

Assume given a commutative diagram

$$\begin{array}{ccc} S_0 & \xrightarrow{\hookrightarrow} & S \\ & \searrow & \downarrow \phi \\ & & S' \end{array}$$

where ϕ is a divided power morphism [2,3].

Let A', B' be abelian schemes on S' with $A = \phi^*(A')$, $B = \phi^*(B')$ and let $f_0: A_0 \rightarrow B_0$ be a homomorphism as above. The following diagram commutes:

$$\begin{array}{ccc} E(B^*) & \xrightarrow{E_S^*(f_0)} & E(A^*) \\ \downarrow \wr & & \downarrow \wr \\ \phi^*(E(B'^*)) & \xrightarrow{\phi^*(E_{S'}^*(f_0))} & \phi^*(E(A'^*)) \end{array}$$

(iv) compatibility with liftable maps:

Given a homomorphism $f: A \rightarrow B$ with reduction $f_0: A_0 \rightarrow B_0$,

$$E(f^*) = E_S^*(f_0)$$

(1.2) Remarks

(i) Conditions (i) and (iv) imply $E_S^*(f_0)$ is an isomorphism when f_0 is (take $f = \text{id}_A$ in (iv))

(ii) Note we do not assert and in general it will not be true that $E_S^*(f_0)$ induces a morphism of extensions.

(1.3) The construction of $E_S^*(f_0)$.

We construct for each flat S -scheme, T , a homomorphism $\underline{\text{Ext}}^{\mathbb{Z}}(B_T, G_m) \rightarrow \underline{\text{Ext}}^{\mathbb{Z}}(A_T, G_m)$. It is functorial in T and passing to the associated Zariski sheaves yields a homomorphism between sheaves on the small flat site of S :

$$\underline{\text{Ext}}^{\mathbb{Z}}(B, G_m) \rightarrow \underline{\text{Ext}}^{\mathbb{Z}}(A, G_m).$$

But because $E(B^*)$ is a flat S -scheme, the map "restriction

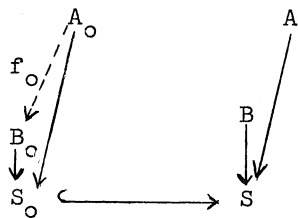
to the small flat site":

$$\text{Hom}(E(B^*), E(A^*)) \rightarrow \text{Hom}_{S_{\text{flat}}}(E(B^*), E(A^*))$$

is bijective. Thus we obtain our homomorphism

$$E_S^*(f_0): E(B^*) \rightarrow E(A^*) .$$

Because the construction of the map $\text{Ext}^{\mathbb{H}}(B_T, G_m) \rightarrow \text{Ext}^{\mathbb{H}}(A_T, G_m)$ is functorial in T , we shall assume that $T = S$. Consider the following diagram



Recall that if X is any smooth S -scheme, the category of line bundles with integrable connection on X is equivalent to the category of invertible modules on the nilpotent crystalline site of X/S . This equivalence is functorial in the smooth S -scheme X .

Also it preserves the algebraic structure inherent in these categories, i.e. it is an equivalence of Picard categories [7]. In particular when we pass to the groups of isomorphism classes of objects, we obtain a canonical isomorphism.

On the other hand since the ideal of the thickening $S_0 \hookrightarrow S$ has nilpotent divided powers, there is, for any stack \mathcal{J} , an equivalence of categories between \mathcal{J} -crystals on $X \times_S S_0/S$ and \mathcal{J} -crystals on X/S . In particular, with $\mathcal{J} =$ invertible modules, we find invertible modules on $(X_0/S)_{\text{crys}} \xrightarrow{\cong}$ invertible modules on $(X/S)_{\text{crys}}$. Once again this equivalence is

functorial in X and preserves the algebraic structure.

Consider the map

$$H^1(B, \mathcal{O}_{B/S_{\text{crys}}}^*) \xrightarrow{\cong} H^1(B_0, \mathcal{O}_{(B_0/S)_{\text{crys}}}^*) \xrightarrow{f_0^*} H^1(A_0, \mathcal{O}_{(A_0/S)_{\text{crys}}}^*) \xrightarrow{\cong} H^1(A, \mathcal{O}_{(A/S)_{\text{crys}}}^*)$$

The fact that f_0 is a group homomorphism, plus the functoriality (indicated above) applied to the "primitivity maps" $s^* - p_1^* - p_2^*$, shows that our composite maps $\text{Ext}^{\mathbb{H}}(B, G_m)$ to $\text{Ext}^{\mathbb{H}}(A, G_m)$. This is the desired homomorphism.

(1.4) Remark: Given $S_0 \hookrightarrow S$ as above and A_0 an abelian scheme on S_0 , we can define for a flat S -scheme T , an abelian group

$$\text{Prim}[H^1(A_{0T_0}, \mathcal{O}_{(A_{0T_0}/T)_{\text{crys}}}^*)] \subseteq H^1(A_{0T_0}, \mathcal{O}_{(A_{0T_0}/T)_{\text{crys}}}^*)$$

to be the kernel of $s^* - p_1^* - p_2^*$.

Passing to the associated sheaf for the Zariski topology we obtain a group which is canonically identified with the universal extension of (the dual of) any lifting of A_0 . This is an example of an "intrinsic" definition of the crystal alluded to above.

(1.5) Now pass to tangent spaces. We've already seen that $\text{Lie}(E(A^*))$ is canonically isomorphic to $H_{\text{DR}}^1(A/S)$. The general crystalline machine, [2] tells us that this module is $H^1(\mathcal{O}_{(A/S)_{\text{crys}}})$. Alternatively, this result can be deduced in the standard way from the fact that the tangent space to G_m is G_a :

Consider the commutative diagram

$$\begin{array}{ccc}
 A & \xleftarrow{\pi} & A[\epsilon] \\
 \downarrow & & \downarrow \\
 S & \xleftarrow{\quad} & S[\epsilon]
 \end{array}$$

defining $A[\epsilon]$. Since A is smooth we can assume that I is the zero ideal. The morphism of topoi $(A[\epsilon]/S[\epsilon])_{\text{crys}} \rightarrow (A/S)_{\text{crys}}$ induced by π is easily understood (because $S[\epsilon] \rightarrow S$ is flat): For F a sheaf on $A[\epsilon]/S[\epsilon]$, $\pi_*(F)(U \hookrightarrow T, J, \psi) = F(U[\epsilon] \hookrightarrow T[\epsilon], \dots)$. Visibly π_* is exact. For any $(U \hookrightarrow T, J, \psi)$ in the crystalline site of A/S we have a split exact sequence of ordinary sheaves (on T)

$$0 \rightarrow \mathcal{O}_T \rightarrow \pi_*(\mathcal{O}_{A[\epsilon]}^*)_{U \hookrightarrow T} \rightarrow \mathcal{O}_T^* \rightarrow 0.$$

This tells us we have a split exact sequence

$$0 \rightarrow \mathcal{O}_{A_{\text{crys}}} \rightarrow \pi_*(\mathcal{O}_{A[\epsilon]_{\text{crys}}}^*) \rightarrow \mathcal{O}_{A_{\text{crys}}}^* \rightarrow 0$$

Applying H^1 and using the exactness of π_* to know $H^1(\pi_*(\mathcal{O}_{A[\epsilon]}^*)) = H^1(\mathcal{O}_{A[\epsilon]}^*)$ we conclude.

(1.6) Remark: In particular we see that the map $H_{\text{DR}}(B) \rightarrow H_{\text{DR}}(A)$ furnished by crystalline cohomology is precisely the map obtained from $E_S^*(f_0)$ by applying the functor Lie.

§2. STABILITY OF $(\epsilon_{m,n})$ -HOM⁷

Fix a prime p , In §2-§6 below we shall work with a pair (S, N) where S is a scheme, and N a number such that $p^N \cdot 1_S = 0$.

Let G be a Barsotti-Tate group on S and

$$(\epsilon_{m,n}) \quad 0 \rightarrow G(n) \rightarrow G(m+n) \xrightarrow{p^n} G(m) \rightarrow 0$$

the doubly indexed family of exact sequences.

We have the push out maps

$$\begin{array}{ccccccc}
 (\epsilon_{m,n+1}) & : 0 \rightarrow G(n+1) \rightarrow G(m+n+1) & \xrightarrow{p^{n+1}} & G(m) \rightarrow 0 \\
 (2.0.1) \downarrow & \downarrow p & \downarrow p & \downarrow \parallel \\
 (\epsilon_{m,n}) & : 0 \rightarrow G(n) \rightarrow G(m+n) & \xrightarrow{p^n} & G(m) \rightarrow 0
 \end{array}$$

and the pullback maps

$$\begin{array}{ccccccc}
 (\epsilon_{m+1,n}) & : 0 \rightarrow G(n) \rightarrow G(m+n+1) \rightarrow G(m+1) \rightarrow 0 \\
 (2.0.2) \downarrow & \parallel & \downarrow p & \downarrow p & & & \\
 (\epsilon_{m,n}) & : 0 \rightarrow G(n) \rightarrow G(m+n) \rightarrow G(m) \rightarrow 0
 \end{array}$$

(2.1) Lemma (Stability in the second index):

For $n \geq N$, the maps

$$(i) \quad (\epsilon_{N,n})\text{-Hom}^7(G(n), G_a) \rightarrow (\epsilon_{N,n+1})\text{-Hom}^7(G(n+1), G_a)$$

$$\text{and } (ii) \quad \underline{\text{Hom}}(G(n), G_a) \rightarrow \underline{\text{Hom}}(G(n+1), G_a)$$

are isomorphisms.

Proof: By the five-lemma it suffices to show the maps (ii) are isomorphisms. Consider the sequence $0 \rightarrow G(1) \rightarrow G(n+1) \xrightarrow{p} G(n) \rightarrow 0$. We must show $\underline{\text{Hom}}(G(n+1), G_a) \rightarrow \underline{\text{Hom}}(G(1), G_a)$ is the zero map.

But because $G(n+1) \xrightarrow{p^n} G(1)$ is an epimorphism, it suffices to note that $\text{Hom}(G(n+1), G_a) \xrightarrow{p^n} \text{Hom}(G(n+1), G_a)$ is zero since p^n kills S .

(2.2) Lemma (Stability in the first index)

$$(\epsilon_{m,n})\text{-Hom}^{\mathcal{L}}(G(n), G_a) \simeq (\epsilon_{m',n})\text{-Hom}^{\mathcal{L}}(G(n), G_a) \quad \text{if } m' \geq m \geq N$$

Proof: Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \underline{\text{Hom}}_{G(m')} & \rightarrow & (\epsilon_{m',n})\text{-Hom}^{\mathcal{L}}(G(n), G_a) & \rightarrow & \underline{\text{Hom}}(G(n), G_a) \rightarrow 0 \\ & & \downarrow i & & \downarrow & & \parallel \\ 0 & \rightarrow & \underline{\text{Hom}}_{G(m)} & \rightarrow & (\epsilon_{m,n})\text{-Hom}^{\mathcal{L}}(G(n), G_a) & \rightarrow & \underline{\text{Hom}}(G(n), G_a) \rightarrow 0 \end{array}$$

and use the fact that i is an isomorphism if $m' \geq m \geq N$ [16, II. 3.3.20]

(2.3) Remark: The analogue of (2.2) remains true when G_a is replaced by any smooth group, and in particular by G_m

§3. EXTENSIONS OF TRUNCATED BARSOTTI-TATE GROUPS BY G_a

Assume now that S is affine. The following proposition tells us in particular that $\text{Ext}^1(G, G_a)$ is isomorphic to $\text{Ext}^1(G(N), G_a)$ via the map induced by $G(N) \hookrightarrow G$ and hence that $\text{Ext}^2(G, G_a) = (0)$. Undoubtedly this last fact can be obtained via Breen's method [4] for calculating Ext .

(3.1) Proposition: The coboundary map coming from the sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & G(N) & \rightarrow & G(2N) & \rightarrow & G(N) \rightarrow 0 \\ & & & & \downarrow \delta & & \downarrow \\ & & \text{Hom}(G(N), G_a) & \rightarrow & \text{Ext}^1(G(N), G_a) & & \end{array}$$

is an isomorphism.

Proof: The proof of (2.1) shows that the map is injective.

Surjectivity is equivalent to the assertion that

$\text{Ext}^1(G(N), G_a) \rightarrow \text{Ext}^1(G(2N), G_a)$ is the zero map. To see that this is true note that the groups in question are by the appendix (functorially) isomorphic to $\text{Ext}^1(\mathcal{L}^{G(N)*}, G_a)$ (resp. $\text{Ext}^1(\mathcal{L}^{G(2N)*}, G_a)$).

But by [16, II, 3.3.10] this map is zero.

(3.2) Corollary. The map $\text{Ext}^1(G, G_a) \rightarrow \text{Ext}^1(G(N), G_a)$ is an isomorphism.

Proof: Consider the commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & G(N) & \rightarrow & G(2N) & \rightarrow & G(N) \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & G(N) & \rightarrow & G & \xrightarrow{p^N} & G \rightarrow 0 \end{array}$$

Since the connecting homomorphism is functorial there is

a commutative diagram

$$\begin{array}{ccc}
 \text{Hom}(G(N), G_a) & \xrightarrow{\cong} & \text{Ext}^1(G, G_a) \\
 \parallel & & \downarrow \\
 \text{Hom}(G(N), G_a) & \xrightarrow{\cong} & \text{Ext}^1(G(N), G_a)
 \end{array}$$

Three sides being isomorphisms, the corollary is established.

§4. ON THE EXISTENCE OF \mathcal{H} -STRUCTURES

Let T be any scheme, H a commutative group scheme on T . Fix an extension of H by a smooth commutative group scheme L (in practice $L = G_m$ or G_a).

$$(4.1) \quad 0 \rightarrow L \rightarrow E \rightarrow H \rightarrow 0$$

Given a \mathcal{H} -structure on this extension we can modify it by adding an element of $\Gamma(T, \mathcal{O}_H \otimes \underline{\text{Lie}}(L))$ to obtain a new \mathcal{H} -structure on the extension. Conversely if we have two \mathcal{H} -structures on the extension then their difference is an element of $\Gamma(T, \mathcal{O}_H \otimes \underline{\text{Lie}}(L))$.

We denote by $\text{Hom}^\nabla(H, L)$ the subgroup of $\text{Hom}(H, L)$ consisting of the maps $\mathfrak{f}: H \rightarrow L$ with $d\mathfrak{f} = 0 \in \Gamma(\mathcal{O}_H \otimes \underline{\text{Lie}}(L))$. For an arbitrary $\mathfrak{f}: H \rightarrow L$ the automorphism of the trivial extension

$$0 \rightarrow L \rightarrow L \times H \rightarrow H \rightarrow 0$$

corresponding to \mathfrak{f} , transforms the trivial \mathcal{H} -structure into the \mathcal{H} -structure given by $d\mathfrak{f}$. This discussion explains why the following sequence is exact:

$$(4.2) \quad 0 \rightarrow \text{Hom}^\nabla(H, L) \rightarrow \text{Hom}(H, L) \rightarrow \Gamma(\mathcal{O}_H \otimes \underline{\text{Lie}}(L)) \rightarrow \text{Ext}^{\mathcal{H}}(H, L) \rightarrow \text{Ext}^1(H, L)$$

We can also pass to sheaves for the flat topology to obtain the sequence

$$(4.3) \quad 0 \rightarrow \underline{\text{Hom}}^\nabla(H, L) \rightarrow \underline{\text{Hom}}(H, L) \rightarrow \mathcal{O}_H \otimes \underline{\text{Lie}}(L) \rightarrow \underline{\text{Ext}}^{\mathcal{H}}(H, L) \rightarrow \underline{\text{Ext}}^1(H, L)$$

(4.4) Lemma: Assume T is affine and let

$$0 \rightarrow L \rightarrow E \rightarrow H \rightarrow 0$$

be an extension which defines the zero section of $\Gamma(T, \underline{\text{Ext}}^1(H, L))$.

Then this extension carries a \mathcal{G} -structure.

Proof: For variable T'/T consider the \mathcal{G} -structures on the restriction of the extension to T' . As noted above we obtain in this way a sheaf which is formally principal homogeneous under $\mathcal{U}_H \otimes \underline{\text{Lie}}(L)$. By assumption, locally this sheaf has sections, and hence the quasi-coherence of $\mathcal{U}_H \otimes \underline{\text{Lie}}(L)$ implies (since T is affine) that it has a global section.

(4.5) Remark: The lemma can be explained "geometrically" as follows: By assumption our extension is a torsieur under $\underline{\text{Hom}}(H, L)$. Let $\{U_i\}$ be a cover of T on which it is trivial and $\mathfrak{g}_{ij} \in \Gamma(U_i \cap U_j, \underline{\text{Hom}}(H, L))$ a corresponding cocycle. Since the cocycle $(d\mathfrak{g}_{ij})$ is a coboundary we can find \mathcal{G} -structures ρ_i on the trivial extension over U_i such that $\rho_i - \rho_j = d\mathfrak{g}_{ij}$. Thus \mathfrak{g}_{ij} is an isomorphism of \mathcal{G} -extensions over $U_i \cap U_j$ and by gluing we obtain a \mathcal{G} -structure on our original extension.

(4.6) Remark: Let H be finite and locally-free and $L = G_m$. Since $\underline{\text{Ext}}^1(H, G_m) = (0)$ it follows that (if T is affine) any extension of H by G_m has a \mathcal{G} -structure.

(4.7) The following discussion will be used in the proof of (4.12) below. Let T be a scheme, and X an arbitrary T -scheme. Let $T[\epsilon]$ be the scheme of dual numbers over T , $X[\epsilon] = \text{dfn. } X \times_T T[\epsilon], \pi_X: X[\epsilon] \rightarrow X$ the structural map. On X there is an exact sequence of sheaves:

$$0 \rightarrow G_a \rightarrow \pi_* G_m \rightarrow G_m \rightarrow 0$$

Corresponding to this sequence there is an "exact sequence" (of Picard categories) [7][15]:

$$0 \rightarrow \text{TORS}(X, G_a) \rightarrow \text{TORS}(X[\epsilon], G_m) \rightarrow \text{TORS}(X, G_m) \rightarrow 0$$

This means that we have an equivalence of categories, compatible with the "addition laws"

$$(4.8) \quad \text{TORS}(X, G_a) \xrightarrow{\sim} \text{category of pairs } (P, \mathfrak{g}) \text{ where } P \text{ is a } G_m\text{-torsieur on } X[\epsilon] \text{ and } \mathfrak{g}: P|_X \rightarrow G_m \text{ is an isomorphism of } G_m \text{ torsieurs on } X$$

This equivalence is functorial in the T -scheme X . Let us denote the above category of pairs by $\text{TORS}(T[\epsilon]/T; X[\epsilon], G_m)$. Because a \mathcal{G} -torsieur P on X under G_a is the torsieur P , plus the additional structure of an isomorphism of torsieurs $\nabla: \pi_1^*(P) \xrightarrow{\sim} \pi_2^*(P)$, satisfying the condition $\Delta^*(\nabla) = \text{id}_P$ (where $\pi_1, \pi_2: \Delta^1(X) \rightarrow X$ are the projections), it follows from the functorial nature of the above equivalence of categories that there is an induced equivalence:

$$(4.9) \quad \text{TORS}^{\mathcal{G}}(X, G_a) \xrightarrow{\sim} \text{TORS}^{\mathcal{G}}(T[\epsilon]/T; X[\epsilon], G_m)$$

where the category on the right has as objects those pairs (P, \mathfrak{g}) with P a \mathcal{G} -torsieur and \mathfrak{g} a horizontal isomorphism.

Let G be any T -group scheme. Extensions are torsieurs P , plus isomorphisms

$$s^*(P) \xrightarrow{\sim} p_1^*(P) \wedge p_2^*(P)$$

satisfying the commutative diagram (1.1.4.1) and (1.2.1) of [11, SGA 7, Exposé VII]. Thus the functorial nature of (4.8) implies that

it induces an equivalence:

$$(4.10) \quad \text{EXT}(G, G_a) \xrightarrow{\sim} \text{EXT}(T[\epsilon]/T; G[\epsilon], G_m)$$

Combining (4.9) and (4.10) there is an equivalence of categories

$$(4.11) \quad \text{EXT}^{\mathcal{L}}(G, G_a) \xrightarrow{\sim} \text{EXT}^{\mathcal{L}}(T[\epsilon]/T; G[\epsilon], G_m)$$

(4.12) Proposition: Assume T is affine, H a finite locally-free T -group. Any extension of H by G_a admits a \mathcal{L} -structure.

Proof: Fix an extension E . View E via (4.10) as an extension of $H[\epsilon]$ by G_m together with a trivialization, $\bar{\varphi}$, of the restriction of this extension to T . By (4.11), \mathcal{L} -structures on E are the same as \mathcal{L} -structures on E (thought of as an extension of $H[\epsilon]$ by G_m) which satisfy the additional property that $\bar{\varphi}$ is horizontal.

Via $\bar{\varphi}$ we transport the trivial \mathcal{L} -structure on $H \times G_m$ to $E|T$ to obtain a \mathcal{L} -structure ∇_0 . Because H is finite and locally free we can speak of the torseur (under $\underline{w}_{H[\epsilon]}$) of \mathcal{L} -structures on E . Denote it by \mathcal{L} and denote by $\mathcal{L}_0 (= \mathcal{L}/T)$ the torseur under \underline{w}_H of \mathcal{L} -structures on $E|T$. Since T is affine, we can choose an isomorphism $\underline{w}_{H[\epsilon]} \xrightarrow{\sim} \mathcal{L}$, whence an induced isomorphism $\underline{w}_H \xrightarrow{\sim} \mathcal{L}_0$. Viewing ∇_0 as an element in $\Gamma(\underline{w}_H)$, the (obvious) fact that $\Gamma(\underline{w}_{H[\epsilon]}) \rightarrow \Gamma(\underline{w}_H)$ is surjective shows that E has a \mathcal{L} -structure lifting ∇_0 and completes the proof of the proposition.

§5. RELATION BETWEEN $\text{EXT}^{\mathcal{L}}$ AND $\epsilon\text{-HOM}^{\mathcal{L}}$

(5.1) Proposition: Let $n \geq 2N$. The natural homomorphism $(\epsilon_{n,n})\text{-Hom}^{\mathcal{L}}(G(n), G_a) \rightarrow \text{Ext}^{\mathcal{L}}(G(n), G_a)$ is an isomorphism.

Proof: Consider the following commutative diagram:

$$(5.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & \underline{w}_{G(n)} & \rightarrow & (\epsilon_{n,n})\text{-Hom}^{\mathcal{L}}(G(n), G_a) & \rightarrow & \text{Hom}(G(n), G_a) \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \delta \\ 0 & \rightarrow & \underline{w}_{G(n)} & \rightarrow & \text{Ext}^{\mathcal{L}}(G(n), G_a) & \longrightarrow & \text{Ext}(G(n), G_a) \rightarrow 0 \end{array}$$

Here δ is the coboundary map which was shown above to be an isomorphism in (3.1). The result will follow once it is shown that $\underline{w}_{G(n)} \rightarrow \text{Ext}^{\mathcal{L}}(G(n), G_a)$ is injective. To do this we must show that the map

$$(5.4) \quad \text{Hom}(G(n), G_a) \rightarrow \underline{w}_{G(n)}$$

occurring in (4.2) is the zero map. Consider the sequence

$$0 \rightarrow G(n) \rightarrow G(2n) \rightarrow G(n) \rightarrow 0$$

It has been shown in the proof of (2.1) that

$\text{Hom}(G(2n), G_a) \rightarrow \text{Hom}(G(n), G_a)$ is the zero map, and has been shown in [16, II 3.3.20] that $\underline{w}_{G(2n)} \rightarrow \underline{w}_{G(n)}$ is an isomorphism.

Thus (5.4) is the zero map and the proposition is proved.

(5.5) Remark: The proposition probably remains true assuming only $n \geq N$. What must be shown is that (5.4) is the zero map under this weaker assumption. For $N = 1$, it is very easy to show this.

§6. CRYSTALLINE EXTENSIONS AND \mathcal{G} -EXTENSIONS

(6.1) Here we recall Grothendieck's definition of generalized extensions, and then we specialize the notion to arrive at the definition of crystalline extension.

Crystalline extensions will be used in showing that $\text{Lie}(E(G^*))$ is "crystalline in nature."

We shall constantly work with the following structure:

(6.2) Fix a scheme T , and G a commutative T -group. For each T -scheme T' let $\mathcal{J}_{T'} =$ category of $G_{T'}$ -torsors. The usual contracted product of $G_{T'}$ -torsors yields a functor

$$\mathcal{J}_{T'} \times \mathcal{J}_{T'} \rightarrow \mathcal{J}_{T'}$$

This structure is an example of a fibered category \mathcal{J} on $\mathcal{C} = \text{Sch}/T$ which is fibered in strictly commutative Picard categories [7,15].

If \mathcal{J} is any fibered category in strictly commutative Picard categories over \mathcal{C} (any category), and H any commutative group in \mathcal{C} , we may define the notion of \mathcal{J} -extension of H :

(6.3) Definition: An \mathcal{J} -extension of H is an object P of \mathcal{J}_H equipped with an isomorphism $s^*(P) \simeq p_1^*(P) \wedge p_2^*(P)$ such that the analogues of the usual diagrams (expressing the associativity and commutativity of the composition law) are commutative. If products do not exist in \mathcal{C} , the definition is modified by requiring that for every pair of points $p_1, p_2: X \rightarrow H$ we be given an isomorphism $(p_1 + p_2)^*(P) \simeq p_1^*(P) \wedge p_2^*(P)$ satisfying the usual conditions as discussed in [11, SGA₇VIII].

These extensions form a category $\text{EXT}(H, \mathcal{J})$ whose morphisms

are the morphism $\phi: P \rightarrow Q$ in \mathcal{J}_H such that the following diagram commutes

$$\begin{array}{ccc} s^*(P) & \xrightarrow{\simeq} & p_1^*(P) \wedge p_2^*(P) \\ s^*(\phi) \downarrow & & \downarrow p_1^*(\phi) \wedge p_2^*(\phi) \\ s^*(Q) & \xrightarrow{\simeq} & p_1^*(Q) \wedge p_2^*(Q) \end{array}$$

The functor $\wedge: \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ induces a composition law on the category $\text{EXT}(H, \mathcal{J})$. Passing to isomorphism classes of objects we obtain a commutative group $\text{Ext}(H, \mathcal{J})$. Finally the category $\text{EXT}(H, \mathcal{J})$ varies functorially with H and \mathcal{J} .

(6.4) We shall give several examples which illustrate the above.

(6.5) $\mathcal{C} = (\text{Sch}/T)$, $\mathcal{J} = G$ -torsors,

$\text{EXT}(H, \mathcal{J})$ is in a natural way equivalent to $\text{EXT}(H, G)$.

(6.6) $\mathcal{C} = (\text{Sch}/T)$, G a smooth T -group, $\mathcal{J} = \mathcal{G}$ -torsors under G . $\text{EXT}(H, \mathcal{J})$ is in a natural way equivalent to $\text{EXT}^{\mathcal{G}}(H, G)$.

(6.7) Let (T, I, γ) be a divided power scheme, i.e. $I \subset \mathcal{O}_T$ is endowed with divided powers. Let $(\text{Sch}/T)' = \mathcal{C}$ be the full sub-category of Sch/T consisting of those $X \rightarrow T$ such that the divided powers on I extend to X . Fix a smooth commutative T -group G (e.g. $G = G_a$ or $G = G_m$). For any X in $(\text{Sch}/T)'$ let G_X be the sheaf of groups on $\text{Crys}(X/T, I, \gamma)$, cf[3], defined by

$$\Gamma((U, T', \delta), G_X) = G(T') = \text{Hom}_T(T', G)$$

If $f: X' \rightarrow X$ is a morphism in $(\text{Sch}/T)'$, then there is an induced map $f_{\text{crys}}^*(G_X) \rightarrow G_{X'}$. This allows us to define the

fiber of \mathcal{J} at X , \mathcal{J}_X , to be $\text{TORS}(\text{Crys}(X/T, I, \gamma), G_X)$, the category of torseurs on the crystalline site of X with structural group G_X . The operation \wedge is just the usual contracted product of torseurs. Since morphisms between torseurs are necessarily isomorphism this category admits an alternative description: It is equivalent to the category of crystals in (small Zariski) G -torseurs, i.e. crystals for the stack $(U, T', \delta) \mapsto (G_X)(U, T', \delta)$ -torseurs. If H is a group in $(\text{Sch}/T)'$, we denote the category of extensions of H by \mathcal{J} by $\text{EXT}^{\text{crys}/T}(H, G)$ and refer to it as the category of crystalline extensions.

(6.8) Remarks: (i) When $G = G_m$, $\text{TORS}(\text{Crys}(X/T, I, \gamma), G_m)$ is equivalent to the category of invertible modules on $\text{Crys}(X/T, I, \gamma)$.

(ii) Where $G = G_a$, $\text{TORS}(\text{Crys}(X/T, I, \gamma), G_a)$ is equivalent to the category $\text{EXT}_{\mathcal{O}_{X, \text{crys}}}(\mathcal{O}_{X, \text{crys}}, \mathcal{O}_{X, \text{crys}})$

(iii) Although the localization allowed in $\text{Crys}(X/T, I, \gamma)$ is quite coarse this will not be bothersome since for the groups G_m and G_a Zariski torseurs are the same as (say) f.p.p.f. torseurs. When we do use G_m , the torseurs we'll consider will in fact have sections over closed sub-schemes defined by nilpotent ideals (c.f. §11). Because, previously, "torseur" was used with reference to one of the large sites (ZARISKI, ETALE, F.P.P.F.: for G_m and G_a the notions coincide) we recall how to pass from torseurs on the small site to torseurs on the large one. For simplicity let's work in the Zariski topology. For any

scheme Y there are two morphisms of topoi: $p : Y_{\text{ZAR}} \rightarrow Y_{\text{zar}}$, $r : Y_{\text{zar}} \rightarrow Y_{\text{ZAR}}$. The morphism p is defined by $\Gamma(Z, p^*(F)) = \Gamma(Z, g^*(F))$, if $g : Z \rightarrow Y$ and F is an ordinary Zariski sheaf on Y . The morphism r is defined simply by restricting a sheaf \mathcal{J} on the large Zariski site to the sub-category of opens of Y .

The functor $p \mapsto p^*(P) \wedge_{\mathcal{O}_Y}^{p^*(\mathcal{O}_Y^*)} G_m$ establishes an equivalence between G_m -torseurs on the small and large sites of Y (similarly for G_a -torseurs). The functoriality of this equivalence follows from that of the morphisms p in a straightforward manner.

(iv) Given X/T , there are functors

$$(6.9) \quad \text{TORS}^{\text{crys}/T}(X, G_m) \rightarrow \text{TORS}^{\mathcal{J}}(X, G_m)$$

$$(6.10) \quad \text{TORS}^{\text{crys}/T}(X, G_a) \rightarrow \text{TORS}^{\mathcal{J}}(X, G_a)$$

If X/T is smooth and $\text{Crys}(X/T)$ is replaced by the nilpotent crystalline site, then (6.9) is an equivalence of categories [2]. Using the fact that the "standard" connection of \mathcal{O}_X is nilpotent together with the interpretation of an object in $\text{TORS}^{\mathcal{J}}(X, G_a)$ as a short exact sequence of modules with integrable connection:

$$(6.11) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{M} \rightarrow \mathcal{O}_X \rightarrow 0$$

we see that (when X/T is smooth) (6.10) is an equivalence of categories.

(6.9) and (6.10) are functorial in X . Furthermore they are compatible with the "composition laws" with which both source and target are endowed. Let H be a T -group such that

$H, H \times_T H, H \times_T H \times_T H$ all belong to $(\text{Sch}/T)'$ (e.g. H/T flat, I principal). There are induced functors (compatible with the composition laws):

$$(6.12) \quad \text{EXT}^{\text{crys}/T}(H, G_m) \rightarrow \text{EXT}^{\mathcal{G}}(H, G_m)$$

$$(6.13) \quad \text{EXT}^{\text{crys}/T}(H, G_a) \rightarrow \text{EXT}^{\mathcal{G}}(H, G_a).$$

If H/T is smooth and we restrict to the nilpotent crystalline site (resp. no restriction) then (6.12) (resp. (6.13)) is an equivalence of categories.

(6.14) We shall need one last example of generalized extensions. Let (T, I, γ) be as above and let $T_0 = \text{Var}(I)$. Let $\mathcal{C} = (\text{Sch}/T_0)$. Let G be a smooth commutative T -group and define \mathcal{G} exactly as in (6.7), i.e. \mathcal{G}_X = category of G_X -torsors on $\text{Crys}(X/T, I, \gamma)$ for any T_0 -scheme X . If H is a group in \mathcal{C} we shall denote the category $\text{EXT}(H, \mathcal{G})$ by $\text{EXT}^{\text{crys}/T}(H/T_0, G)$, and if it is clear that H is a T_0 -group we shall drop the " T_0 " from the notation.

(6.15) Remarks: (i) The reason for distinguishing between (6.7) and (6.14) is that a T_0 -group scheme is never a T -group scheme.

(ii) If T' is a closed subscheme of T_0 and $\mathcal{C}' = (\text{Sch}/T')$, then with \mathcal{G} as in (6.14) there is the category $\text{EXT}^{\text{crys}/T}(H/T', G)$. This category differs from that of (6.14) since (because the ideal of T' in T need not have divided powers) even if H can be lifted to T , the category of

crystalline extension of a lifting can be different from this category.

(6.16) Let us indicate the functorial variation of examples (6.7) and (6.14) when (T, I, γ) varies. Let $(T', I', \gamma') \rightarrow (T, I, \gamma)$ be a divided power morphism. First assume X is a flat T -scheme, X' a flat T' -scheme, and assume we are given a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ (T', I', \gamma') & \longrightarrow & (T, I, \gamma) \end{array}$$

Let G be a smooth T -group, $G' = G \times_T T'$. Since crystalline torsors are crystals the general procedure for taking the inverse image of a crystal [3, IV 1.2; or 16, III, 3.8] permits us to define a functor

$$(6.17) \quad \text{TORS}^{\text{crys}/T}(X, G) \rightarrow \text{TORS}^{\text{crys}/T'}(X', G')$$

This functor varies functorially with (X', X) . In particular if H is a flat T -group, $H' = H \times_T T'$, there is an induced functor

$$(6.18) \quad \text{EXT}^{\text{crys}/T}(H, G) \rightarrow \text{EXT}^{\text{crys}/T'}(H', G')$$

If we assume X (resp X') is a T_0 (resp. T_0') scheme, the map (6.17) is still defined. Furthermore, if H is a T_0 group, $H' = H \times_{T_0} T_0'$, there is an induced functor

$$\text{EXT}^{\text{crys}/T}(H/T_0, G) \rightarrow \text{EXT}^{\text{crys}/T'}(H'/T_0', G').$$

§7. THE CRYSTALLINE NATURE OF $\text{EXT}^{\mathcal{L}}(-, G_a)$

Here we let S be a scheme on which p is locally nilpotent. and let (I, \mathcal{V}) be a divided power ideal of \mathcal{O}_S . Let G be a Barsotti-Tate group on S . The inclusions $G(n) \rightarrow G(n+1)$ induce functors

$$\begin{aligned} \text{EXT}^{\text{crys}/S}(G(n+1), G_a) &\rightarrow \text{EXT}^{\text{crys}/S}(G(n), G_a) \\ \text{EXT}^{\mathcal{L}}(G(n+1), G_a) &\rightarrow \text{EXT}^{\mathcal{L}}(G(n), G_a). \end{aligned}$$

By passage to \varprojlim , [9], we obtain from (6.10) a functor

$$(7.1) \quad \varprojlim \text{EXT}^{\text{crys}/S}(G(n), G_a) \rightarrow \varprojlim \text{EXT}^{\mathcal{L}}(G(n), G_a).$$

(7.2) Theorem: The functor (7.1) is an equivalence of categories.

Proof: Note that (7.1) is induced by the functor

$$(7.3) \quad \varprojlim \text{TORS}^{\text{crys}/S}(G(n), G_a) \rightarrow \varprojlim \text{TORS}^{\mathcal{L}}(G(n), G_a).$$

Since the category of crystalline extensions (resp. \mathcal{L} -extensions) is defined as consisting of crystalline (resp. \mathcal{L}) torsors, P , endowed with an isomorphism $s^*(P) \simeq p_1^*(P) \wedge p_2^*(P)$ (satisfying the associativity condition) and since the functor (7.3) is itself functorial with respect to the Barsotti-Tate group, G ; it suffices to show that (7.3) is an equivalence of categories. Assuming momentarily (7.7) below, we shall show that (7.3) is faithful, full and essentially surjective.

1) faithful. Let $(\mathfrak{f}_n), (\psi_n)$ be two morphisms between the object (P_n) and (Q_n) of $\varprojlim \text{TORS}^{\text{crys}/S}(G(n), G_a)$. Assuming $(\mathfrak{f}_n \xrightarrow{\text{id}}_{G(n)}) = (\psi_n \xrightarrow{\text{id}}_{G(n)})$, we must show $(\mathfrak{f}_n) = (\psi_n)$.

Fix an n and let $(U \hookrightarrow T, J, \mathcal{G})$ be an object of the crystalline site $\text{Crys}(G(n)/S, I, \mathcal{V})$. Obviously it is permissible to assume T is affine. By (7.7) below we can find for m sufficiently large a commutative diagram

$$(7.4) \quad \begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow \mathfrak{f}_n & & \swarrow f \\ G(n) & & \\ \downarrow & & \\ G(n+m) & & \end{array}$$

Let us use a vertical bar " $|$ " to denote restriction (or more properly inverse image). By hypothesis there are commutative diagrams:

$$(7.5) \quad \begin{array}{ccc} P_n & \xrightarrow{\sim} & P_{n+m} | G(n) \\ \mathfrak{f}_n \downarrow & & \downarrow \mathfrak{f}_{n+m} | G(n) \\ Q_n & \xrightarrow{\sim} & Q_{n+m} | G(n) \end{array}$$

$$\begin{array}{ccc} P_n & \xrightarrow{\sim} & P_{n+m} | G(n) \\ \psi_n \downarrow & & \downarrow \psi_{n+m} | G(n) \\ Q_n & \xrightarrow{\sim} & Q_{n+m} | G(n) \end{array}$$

But by definition of the inverse image of a crystal [3, IV 1.2, or 16, III, 3.8] we have

$$(\mathfrak{f}_{n+m} | G(n))_{U \hookrightarrow T} = f^*(\mathfrak{f}_{n+m} |_{G(n+m)} \text{id}_{G(n+m)})$$

and similarly for $(\psi_{n+m} | G(n))_{U \hookrightarrow T}$. Hence the commutativity of the diagrams (7.5) allow us to conclude $(\mathfrak{f}_n)_{U \hookrightarrow T} = (\psi_n)_{U \hookrightarrow T}$.

2) full: Here it will be convenient to denote the image of an object (P_n) (resp. on arrow $(\bar{\varphi}_n)$) of $\varprojlim \text{TORS}^{\text{crys}/S}(G(n), G_a)$ under (7.3) by (\bar{P}_n) (resp. $(\bar{\varphi}_n)$). Let $(\sigma_n): (\bar{P}_n) \rightarrow (\bar{Q}_n)$ be a morphism in $\varprojlim \text{TORS}^{\text{crys}/S}(G(n), G_a)$. We must show that there is a morphism $(\varphi_n): (P_n) \rightarrow (Q_n)$ in $\varprojlim \text{TORS}^{\text{crys}/S}(G(n), G_a)$ with $(\bar{\varphi}_n) = (\sigma_n)$. Just as in the proof of faithfulness above, we fix an n and an object $(U \hookrightarrow T, J, \mathfrak{g})$ of $\text{Crys}(G(n)/S, I, \gamma)$. Using diagram (7.4) we define $\varphi_{n, U \hookrightarrow T}$ to be the map obtained via transport of structure using the isomorphisms $P_n \xrightarrow{\sim} P_{n+m} | G(n)$ and $Q_n \xrightarrow{\sim} Q_{n+m} | G(n)$ from $f^*(\sigma_{n+m})$. It must be shown that this definition is independent of the choice of $f: T \rightarrow G(n+m)$, a lifting of $U \hookrightarrow G(n) \hookrightarrow G(n+m)$. Let f_1, f_2 be two liftings. By definition of the divided power neighborhood [3, I 4.32] of $\Delta: G(n+m) \rightarrow G(n+m) \times_S G(n+m)$, there is a map $\hat{f}: T \rightarrow D_{G(n+m)/S}^{(2)}$ with $p_1 \circ \hat{f} = f_1$, $p_2 \circ \hat{f} = f_2$. Augmenting m if necessary we can assume that $\Omega_{G(n+m)/S}^1$ is locally-free of finite rank [16, II 3.3.20]. Since a \mathcal{L} -torsieur under G_a can be interpreted as an exact sequence of modules with integrable connection, it follows from [3, II, 4.3.4, 4.3.10] that $p_1^*(\sigma_{n+m})$ is identifiable with $p_2^*(\sigma_{n+m})$, once we identify $p_i^*(P_{n+m})$ with $P_{n+m} |_{G(n+m) \hookrightarrow D^{(2)}(G(n+m), S)}$ (and similarly for $p_i^*(Q_{n+m})$). Hence we can identify $f_1^*(\sigma_{n+m})$ and $f_2^*(\sigma_{n+m})$ with $\hat{f}^*(p_1^*(\sigma_{n+m})) = \hat{f}^*(p_2^*(\sigma_{n+m}))$. This shows our definition of $\varphi_{n, U \hookrightarrow T}$ is independent of the choice of lifting and completes the proof that (7.3) is full.

3) essentially surjective: The proof here is quite similar to the proof of fullness above. Given an object (P'_n) in $\varprojlim \text{TORS}^{\text{crys}/S}(G(n), G_a)$, we obtain (P_n) in $\varprojlim \text{TORS}^{\text{crys}/S}(G(n), G_a)$

by defining $P_{n, U \hookrightarrow T}$ to be $f^*(P'_{n+m})$, where f is any morphism making (7.4) commutative. The fact that this definition makes sense and yields an object (P_n) such that $(\bar{P}_n) \cong (P'_n)$ follows by again invoking the above cited results in Berthelot's thesis [3, II 4.3.4, 4.3.10].

(7.6) Remark: The proof of faithfulness is valid if G_a is replaced by any smooth commutative S -group. For fullness and essential surjectivity the interpretation of G_a -torsieurs as extensions of \mathcal{O} by \mathcal{O} (and hence as modules with additional structure) was necessary in order to apply the results in [2, 3]. But if we modify the target by replacing $\text{TORS}^{\mathcal{L}}(G(n), G_m)$ by $\text{TORS}^{\text{nil } -\mathcal{L}}(G(n), G_m)$ (i.e. the category of line bundles endowed with a nilpotent integrable connection) or if we modify the source by using the nilpotent crystalline site (s) , then the above proof carries over to yield equivalences

$$(7.6.1) \quad \varprojlim \text{EXT}^{\text{crys}/S}(G(n), G_m) \xrightarrow{\sim} \varprojlim \text{EXT}^{\text{nil } -\mathcal{L}}(G(n), G_m)$$

$$(7.6.2) \quad \varprojlim \text{EXT}^{\text{Nil-crys}/S}(G(n), G_m) \xrightarrow{\sim} \varprojlim \text{EXT}^{\mathcal{L}}(G(n), G_m)$$

In the course of the above proof, use was made of:

(7.7) Lemma: Let G be a Barsotti-Tate group on S . G is formally smooth for nilimmersions (i.e. if X is an (absolutely) affine scheme over S , and X_0 is a closed sub-scheme defined by an ideal in which every element is nilpotent, then any morphism $X_0 \rightarrow G$ can be lifted to X).

Proof: Let (X, X_0) be as in the above explication. Write $X = \text{Spec}(A)$, $X_0 = \text{Spec}(A/I)$. For $\lambda \in L = (\text{set of finite subsets of } I)$, let I_λ be the finitely generated sub-ideal of I generated by λ , and let $X_\lambda = \text{Spec}(A/I_\lambda)$. Since X_0 is affine, the map $X_0 \rightarrow G$ factors through some $G(n)$. Because $G(n)$ is locally of finite presentation over S and $X_0 = \varprojlim X_\lambda$, it follows from [10, EGA_{IV} 8.13.1] that $X_0 \rightarrow G(n)$ can be lifted to $X_\lambda \hookrightarrow G(n)$ (for some λ). But $X_\lambda \hookrightarrow X$ is a nilpotent immersion. The result now follows since Barsotti-Tate groups are formally smooth [16, II, 3.3.13].

(7.8) Corollary (of 7.2): The category $\varprojlim \text{EXT}^{\text{crys}/S}(G(n), G_a)$ is rigid.

Proof: By (7.2) this category is equivalent to $\varprojlim \text{EXT}^{\mathcal{L}}(G(n), G_a)$. The automorphism group of the zero object $(G(n) \times_{G_a}$, trivial connection) of this category consists of compatible families of homomorphisms $g_n: G(n) \rightarrow G_a$ with $dg_n = 0$. But $(g_n) \in \varprojlim \text{Hom}(G(n), G_a) = \text{Hom}(G, G_a) = (0)$ and hence each g_n is zero.

(7.9) Let us denote by $\text{EXT}^{\text{crys}/S}(G, G_a)$ the category $\varprojlim \text{EXT}^{\text{crys}/S}(G(n), G_a)$. Similarly we write $\text{EXT}^{\mathcal{L}}(G, G_a)$ (resp. $\text{TORS}^{\mathcal{L}}(G, G_a)$, $\text{EXTRIG}(G, G_a)$, ... for the categories $\varprojlim \text{EXT}^{\mathcal{L}}(G(n), G_a)$ (resp. $\varprojlim \text{TORS}^{\mathcal{L}}(G(n), G_a)$, $\varprojlim \text{EXTRIG}(G(n), G_a)$, ...). Finally we write $\text{Ext}^{\text{crys}/S}(G, G_a)$, etc. for the abelian group of isomorphism classes of objects of $\text{EXT}^{\text{crys}/S}(G, G_a)$, etc. Observe that the action of $\Gamma(S, \mathcal{O}_S)$ on G_a gives $\text{Ext}^{\text{crys}/S}(G, G_a)$ a module structure.

Having introduced all this notation we can state the following immediate consequence of (7.8):

(7.10) Corollary: The (small) Zariski presheaf on S $U \mapsto \text{Ext}^{\text{crys}/U}(G|_U, G_a)$ is a sheaf of \mathcal{O}_S -modules.

(7.11) Let us denote this \mathcal{O}_S -module by $\text{Ext}^{\text{crys}/S}(G, G_a)$. The following proposition tells us that $\text{Ext}^{\text{crys}/S}(G, G_a)$ is canonically isomorphic with $\text{Lie}(E(G^*))$, the tangent space of the universal extension of the Cartier dual of G .

(7.12) Proposition: Assume S is affine and p^N kills S . The natural map $\text{Ext}^{\text{crys}/S}(G, G_a) \rightarrow \text{Ext}^{\mathcal{L}}(G(n), G_a)$ is an isomorphism provided $n \geq 2N$.

Proof: By (7.2) we may replace the source by $\text{Ext}^{\mathcal{L}}(G, G_a)$. Let (P_i) represent an element in $\text{Ext}^{\mathcal{L}}(G, G_a)$. To demonstrate injectivity we must show that

$$\begin{aligned} [P_n \cong \text{trivial } \mathcal{L}\text{-extension of } G(n) \text{ by } G_a] \\ \Downarrow \\ [(P_i) \cong \text{trivial } \mathcal{L}\text{-extension of } G \text{ by } G_a] \end{aligned}$$

Let O_i denote the trivial \mathcal{L} -extension of $G(i)$ by G_a . We are to produce for each $i \geq n$ an isomorphism $\theta_i: O_i \xrightarrow{\cong} P_i$ such that these form a compatible family.

Let (\bar{P}_i) be the object of $\text{EXT}(G, G_a)$ obtained by forgetting the \mathcal{L} -structure on each P_i . Since our definition of $\text{EXT}(G, G_a)$ as $\varprojlim \text{EXT}(G(i), G_a)$ coincides with the usual definition as the category of extensions of fppf sheaves, it follows from (3.2)

that there is a unique isomorphism $(\tau_i): (\overline{P}_i) \xrightarrow{\sim} (\overline{O}_i)$. By hypothesis there is an isomorphism $\sigma: P_n \xrightarrow{\sim} O_n$. But $\sigma \circ \tau_n^{-1}$ is (by the proof of (5.1), where it is shown that (5.4) is the zero map) a horizontal automorphism of O_n . Hence τ_n is actually an isomorphism between P_n and O_n (and not only between the underlying extensions). It remains to explain why each τ_i is horizontal. Using τ_i we obtain, via transport of structure, a \curvearrowright -structure on \overline{O}_i . This corresponds to an element η_i of $\Gamma(S, \underline{w}_G(i))$. By hypothesis $\eta_n = 0$ and since for $i \geq n$ the maps $\underline{w}_G(i) \rightarrow \underline{w}_G(n)$ are isomorphisms it follows that each $\eta_i = 0$. Thus for $i \geq n$, τ_i is horizontal and injectivity is established.

Let R be a \curvearrowright -extension of $G(n)$ by G_a . To prove surjectivity we must establish the existence of an object (P_i) in $\text{EXT}^{\curvearrowright}(G, G_a)$ with $P_n \simeq R$. By (3.2), there is an object (\overline{P}_i) in $\text{EXT}(G, G_a)$ with $\overline{P}_n \simeq R$, R being the underlying extension of R . Choosing an isomorphism ϕ between \overline{P}_n and R , we endow, via transport of structure, \overline{P}_n with a \curvearrowright -structure so that ϕ becomes a horizontal isomorphism. We must endow each $\overline{P}_i (i \geq n)$ with a \curvearrowright -structure so that the given maps $\overline{P}_n \simeq \overline{P}_i | G(n)$ are horizontal. Via transport of structure we put a \curvearrowright -structure on $\overline{P}_i | G(n)$. Since S is affine, (4.12) tells us that \overline{P}_i has at least one \curvearrowright -structure. But the set of \curvearrowright -structures on \overline{P}_i (resp. $\overline{P}_i | G(n)$) is principal homogeneous under $\Gamma(S, \underline{w}_G(i))$ (resp. $\Gamma(S, \underline{w}_G(n))$). Surjectivity now follows since the map $\Gamma(S, \underline{w}_G(i)) \rightarrow \Gamma(S, \underline{w}_G(n))$ is onto.

(7.13) Corollary: Let p be locally nilpotent on $S, (I, \gamma)$ be a divided power ideal in \mathcal{O}_S , G a Barsotti-Tate group on S . There is a (functorial in G) exact sequence

$$(7.14) \quad 0 \rightarrow \underline{w}_G \rightarrow \underline{\text{Ext}}^{\text{crys}/S}(G, G_a) \rightarrow \underline{\text{Ext}}(G, G_a) \rightarrow 0$$

which is canonically identified with the sequence obtained from the universal extension of G^* by taking tangent spaces. In particular $\underline{\text{Ext}}^{\text{crys}/S}(G, G_a)$ is a locally-free (of finite type) \mathcal{O}_S -module.

Proof: This follows immediately from (8.7), (3.2), (5.1) and (7.12).

(N.B.) The reader can check that our forward reference to (8.7) does not involve any logical circularity.

§8. PASSAGE TO LIE ALGEBRAS

To apply the results of §2-§7 to the universal extension we must relate $\text{Homrig}(-, G_m)$ to $\text{Homrig}(-, G_a)$ and $\text{Hom}^{\zeta}(-, G_m)$ to $\text{Hom}^{\zeta}(-, G_a)$.

Consider as usual an exact sequence of finite locally free S -groups

$$(8) \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

giving rise to the sequence

$$(8.1) \quad 0 \rightarrow \underline{w}_C \rightarrow (\epsilon)\text{-Homrig}(A, G_m) \rightarrow A^* \rightarrow 0$$

For affine S , the sequence of S -valued points is exact. Thus the snake lemma together with a previously noted fact (passage to Lie algebra commutes with passage to associated Zariski sheaf) tells us that the corresponding sequence

$$(8.2) \quad 0 \rightarrow \underline{w}_C \rightarrow \text{Lie}((\epsilon)\text{-Homrig}(A, G_m)) \rightarrow \text{Lie}(A^*) \rightarrow 0$$

is also exact.

If we replace G_m by G_a we have the analogue of (8.1):

$$(8.3) \quad 0 \rightarrow \underline{w}_C \rightarrow (\epsilon)\text{-Homrig}(A, G_a) \rightarrow \text{Hom}(A, G_a) \rightarrow 0$$

Let $\pi: S[\epsilon] \rightarrow S$ be the structural map so that there is an exact sequence on S

$$0 \rightarrow G_a \rightarrow \pi_*(G_m) \rightarrow G_m \rightarrow 0$$

Let $\bar{\pi}: A \rightarrow G_a$ be a homomorphism and σ be a rigidification on the resulting extension

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\ & & \downarrow \bar{\pi} & & \downarrow & & \parallel \\ 0 & \rightarrow & G_a & \rightarrow & E & \rightarrow & C \rightarrow 0 \\ & & & & \swarrow \sigma & & \downarrow \\ & & & & & & \text{Inf}(C) \end{array}$$

Applying π^* to the whole diagram and "pushing out" along the map $\pi^*(G_a) \rightarrow \pi^*(\pi_*G_m) \rightarrow G_m$, we obtain an element of $\text{Lie}((\epsilon)\text{-Homrig}(A, G_m))$. This procedure defines a homomorphism from the extension (8.3) to (8.2), which is an isomorphism on end-groups. Hence

$$\text{Lie}((\epsilon)\text{-Homrig}(A, G_m)) \cong (\epsilon)\text{-Homrig}(A, G_a)$$

(8.4) Remark: The above discussion is valid also when "Homrig" is replaced by "Hom $^{\zeta}$ ", and hence $\text{Lie}((\epsilon)\text{-Hom}^{\zeta}(A, G_m)) \cong (\epsilon)\text{-Hom}^{\zeta}(A, G_a)$.

(8.5) Let S be a scheme with $p^N \cdot 1_S = 0$ and let G be a Barsotti-Tate group on S . The universal extension of G^* by a vector group is

$$(8.6) \quad 0 \rightarrow \underline{w}_{G(N)} \rightarrow \varinjlim (\epsilon_{N,n})\text{-Hom}^{\zeta}(G(n), G_m) \rightarrow G^* \rightarrow 0$$

Because " \varinjlim " is exact and Lie is defined as a kernel, it follows from the preceding discussion that

$$\begin{aligned} & \text{Lie}(\varinjlim (\epsilon_{N,n})\text{-Hom}^{\zeta}(G(n), G_m)) \\ &= \varinjlim \text{Lie}((\epsilon_{N,n})\text{-Hom}^{\zeta}(G(n), G_m)) \\ &= \varinjlim (\epsilon_{N,n})\text{-Hom}^{\zeta}(G(n), G_a) \\ &= (\epsilon_{N,N})\text{-Hom}^{\zeta}(G(N), G_a) \quad (\text{by 2.1}) \end{aligned}$$

Summarizing, we state

(8.7) Proposition: If p^N kills S and $n \geq N$, then the tangent space $\underline{\text{Lie}}(E(G^*))$ is $(\epsilon_{N,n})\text{-Hom}^{\mathbb{Z}}(G(n), G_a)$.

§9. THE CRYSTALLINE NATURE OF THE LIE ALGEBRA OF THE UNIVERSAL EXTENSION

Fix a scheme S on which p is locally nilpotent and let $G \in \text{B.T.}(S)$, the category of Barsotti-Tate groups on S . Let us explain how to endow $\underline{\text{Lie}}(E(G))$ with a crystalline structure. More precisely we'll define a contravariant functor

$$D^*: \text{B.T.}(S)^\circ \rightarrow (\text{Crystals in locally-free modules on } S).$$

Let U be open in S and let $U \rightarrow (T, I, \gamma)$ be a divided power thickening and assume p is locally nilpotent on T . Let G be a Barsotti-Tate group over S and let G (again) denote its restriction to U . Let G' be any lifting of G to T . Using the abuse of notation indicated in (6.14), we know $\text{EXT}^{\text{crys}/T}(G, G_a) \cong \text{EXT}^{\text{crys}/T}(G', G_a)$ since reduction modulo a divided power ideal induces a functorial equivalence between crystals (of any species whatsoever) on $G'(n)/T$ and crystals on $G(n)/T$. As a consequence of the work of Grothendieck and Illusie [13, 14] we know that, locally on T , we can find such a G' . If H' is a Barsotti-Tate group on T and $H = H' \times_T U$ then a homomorphism $u: G \rightarrow H$ induces a map $\underline{\text{Ext}}^{\text{crys}/T}(H, G_a) \rightarrow \underline{\text{Ext}}^{\text{crys}/T}(G, G_a)$. Thus we obtain a map $f_u: \underline{\text{Lie}}(E(H'^*)) \rightarrow \underline{\text{Lie}}(E(G'^*))$. If u is an isomorphism, then f_u is an isomorphism. In particular, it follows that whenever G' and G'' are liftings of G to a divided power neighborhood, $\underline{\text{Lie}}(E(G'^*))$ and $\underline{\text{Lie}}(E(G''^*))$ are canonically isomorphic.

Let $V \hookrightarrow (T', I', \gamma) \xrightarrow{\mathbb{Q}} U \hookrightarrow (T, I, \gamma)$ be a morphism in the crystalline site of S . If G' is a lifting of G to T

then $\phi^*(G')$ is a lifting of $G|V$ to T' . Thus we obtain a commutative diagram of isomorphisms

$$(9.1) \quad \begin{array}{ccc} \phi^*(\underline{\text{Lie}}(E(G^*))) & \xrightarrow{\sim} & \underline{\text{Lie}}(E(\phi^*(G'^*))) \\ \downarrow \cong & & \downarrow \cong \\ \phi^*(\underline{\text{Ext}}^{\text{crys}/T}(G, G_a)) & \xrightarrow{\sim} & \underline{\text{Ext}}^{\text{crys}/T'}(G|V, G_a) \end{array}$$

Thus the functor \mathbb{D}^* can be explicitly defined by

$$(9.2) \quad \mathbb{D}^*(G)_{U \hookrightarrow (T, I, \gamma)} = \underline{\text{Ext}}^{\text{crys}/T}(G, G_a)$$

(9.3) Remark: The above definition of \mathbb{D}^* is intrinsic, i.e. it is defined entirely in terms of S (without using liftings of Barsotti-Tate groups). Liftings are used to show that $\mathbb{D}^*(G)_{U \hookrightarrow (T, I, \gamma)}$ is locally-free and to show that $\mathbb{D}^*(G)$ is a crystal rather than just a sheaf on the crystalline site.

§10. A DEFORMATIONAL DUALITY THEOREM FOR BARSOTTI-TATE GROUPS:
AN EASY CONSEQUENCE OF THE THEORY OF ILLUSIE

Let S be an affine scheme on which p^N is zero. Let $S_0 \hookrightarrow S$ be a closed subscheme defined by the vanishing of an ideal $I \subset \mathcal{O}_S$ with $I^{k+1} = (0)$. Let G be a Barsotti-Tate group on S . Denote by $G(S/S_0)$ the kernel of $G(S) \rightarrow G(S_0)$ and denote by $\text{EXT}(S/S_0; G, G_m)$ the category of extensions of G by G_m trivialized over S_0 . We write $\text{Ext}(S/S_0; G, G_m)$ for the group of isomorphism classes of objects of $\text{EXT}(S/S_0; G, G_m)$.

In [16, appendix, 2.5] under the additional assumptions

- 1) $S = \text{Spec}(R)$, R an artin local ring
- 2) $S_0 = \text{Spec}(k)$, $k = \text{residue field of } R$, k perfect
- 3) $G = \mathbb{Q}_p/\mathbb{Z}_p$

it was proved that there is a canonical isomorphism

$$(10.1) \quad \mu(R) \simeq \text{Ext}_S(\mathbb{Q}_p/\mathbb{Z}_p, G_m)$$

Since $\mu = G^*$ is a formal group and since $S_0 = \text{Spec}(k)$, and k is a field; $\mu(S) = G^*(S/S_0)$. On the other hand the fact that k is perfect implies $\text{Ext}(S/S_0, G, G_m) \simeq \text{Ext}(G, G_m)$. Thus the isomorphism can be written as

$$G^*(S/S_0) \simeq \text{Ext}(S/S_0; G, G_m)$$

Making extensive use of L. Illusie's deformation theory [14, VII], we prove the following generalization:

(10.2) Deformational duality Theorem: If S, S_0, G satisfy the initial conditions above then there is a canonical (functorial)

isomorphism

$$G^*(S/S_0) \xrightarrow{\cong} \text{Ext}(S/S_0; G, G_m).$$

We shall give two constructions of a map

$$G^*(S/S_0) \rightarrow \text{EXT}(S/S_0; G, G_m).$$

(10.3) Let $\phi: G(n) \rightarrow G_m$ be an element of $G^*(S)$. By pushing out the Kummer sequence we obtain an extension of G by G_m

$$\begin{array}{ccccccc} 0 & \rightarrow & G(n) & \rightarrow & G & \rightarrow & G \rightarrow 0 \\ & & \downarrow \phi & & \downarrow & & \parallel \\ 0 & \rightarrow & G_m & \rightarrow & E & \rightarrow & G \rightarrow 0 \end{array}$$

If $\phi \in G^*(S/S_0)$, then the restriction of the extension E to S_0 has a canonical trivialization.

(10.4) Let us write $\text{TORS}(S, T_p(G^*))$ for the category $\varprojlim \text{TORS}(S, G^*(n))$ (i.e. the category whose objects are compatible families of torseurs, $P(n)$ a torseur under $G^*(n)$, where the transition morphism $G^*(n+m) \rightarrow G^*(n)$ is p^m). Similarly we write $\text{TORS}(S/S_0, T_p(G^*))$ for the category of torseurs under $T_p(G^*)$ equipped with a trivialization over S_0 .

Because the $G(n)$'s are finite and locally-free $\text{Ext}^1(G(n), G_m) = (0)$ and hence $\text{TORS}(S, G^*(n)) \xrightarrow{\cong} \text{EXT}(G(n), G_m)$. Explicitly an equivalence is given as follows:

Given a $G^*(n)$ torseur P we twist $G_m \times G(n)$ by the torseur- P . In down to earth terms this means we take the sheaf-theoretic quotient of $P \times (G_m \times G(n))$ by the action of $G^*(n)$ given by

$$(p, g, x) + \phi = (p - \phi, g - \phi(x), x)$$

where $p \in P(S')$, $g \in G_m(S')$, $x \in G(n)(S')$, $\phi: G(n)_{S'} \rightarrow G_m_{S'}$, S' an S -scheme.

A quasi-inverse to this functor is given by assigning to an extension

$$0 \rightarrow G_m \xrightarrow{i} E \xrightarrow{\pi} G(n) \rightarrow 0$$

the $G^*(n)$ -torseur of splittings of this extension, i.e. the torseur P with

$$P(S') = \{ \sigma: E \rightarrow G_m \mid \sigma \circ i = \text{id}_{G_m} \}$$

where $\sigma + \phi = \text{def } \sigma + \phi \circ \pi$, for $\phi: G(n)_{S'} \rightarrow G_m_{S'}$. Since $\text{TORS}(S, T_p(G^*)) \xrightarrow{\cong} \varprojlim \text{EXT}(G(n), G_m) \xrightarrow{\cong} \text{EXT}(G, G_m)$

we define a map $G^*(S) \rightarrow \text{EXT}(G, G_m)$ by composing the above equivalence with the map $G^*(S) \xrightarrow{\alpha} \text{TORS}(S, T_p(G^*))$ whose definition is as follows: if $g^* \in G^*(S)$, let $\alpha(g^*)$ be the family $(P(n))$ where $P(n)$ is the $G^*(n)$ -torseur $(p^n)^{-1}(g^*)$ arising from the exact sequence

$$0 \rightarrow G^*(n) \rightarrow G^* \xrightarrow{p^n} G^* \rightarrow 0$$

Clearly this induces a map $G^*(S/S_0) \rightarrow \text{TORS}(S/S_0, T_p(G))$.

Remark: The fact that the two definitions in (10.3) and (10.4) are equivalent is a trivial exercise in the use of the definition of the Cartier dual. For the proof of (10.2) it is more convenient to work with (10.4) while for the eventual application to the construction of crystals (10.3) is more convenient.

(10.5) Let us observe that the category $\text{EXT}(S/S_0; G, G_m)$ is rigid. For if we identify an automorphism of the trivial extension

$$0 \rightarrow G_m \rightarrow G_m \times G \rightarrow G \rightarrow 0$$

with an element f , of $\text{Hom}(G, G_m)$ then to say this automorphism defines a map in $\text{EXT}(S/S_0; G, G_m)$ is equivalent to saying $f|_{S_0} = 0$. But from [16, II 3.3.17 + proof of 3.3.21] we know this implies $f = 0$.

(10.6) Let us prove the map $G^*(S/S_0) \rightarrow \text{Tors}(S/S_0, T_p(G^*))$ is injective. Given $g^* \in G^*(S/S_0)$, to say the corresponding torseur $P(g^*)$ is trivial means that there is a sequence of elements (g_n) , $g_n \in G^*(S)$ such that

- 1) $p \cdot g_{n+1} = g_n$
- 2) $p^n g_n = g^*$
- 3) $g_n|_{S_0} = 0$ for all n

But $G^*(S/S_0) \subseteq \Gamma(S, \text{Inf}^k(G^*)) \subseteq G(Nk)$ [16, II 3.3.16].

Hence p^{Nk} kills each g_n . It follows that $g^* = 0$.

(10.7) The proof of the surjectivity of the map $G^*(S/S_0) \rightarrow \text{Tors}(S/S_0, T_p(G^*))$ seems to be more difficult. Since this is an assertion about any Barsotti-Tate group, we shall drop the "*".

(10.8) Let P_0 be a torseur under $T_p(G_0)$ ($G_0 = G \times_S S_0$). Denote by $D(P_0)$ (resp. $D(P_0(n))$) the set of isomorphism classes of deformations of P_0 (resp. $P_0(n)$) to a $T_p(G)$ (resp. $G(n)$) torseur on S .

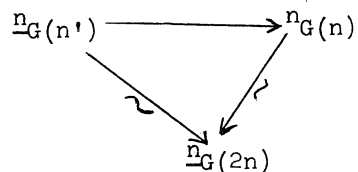
(10.9) Proposition (using Illusie):

- (i) For each n , $D(P_0(n)) \neq \emptyset$
- (ii) For $n \geq n' \geq N$, $D(P_0(n)) \rightarrow D(P_0(n'))$ is surjective
- (iii) $D(P_0) \rightarrow \varprojlim D(P_0(n))$ is onto and hence by (i) and (ii), $D(P_0) \neq \emptyset$.
- (iv) For $n \geq n' \geq kN$, the map $D(P_0(n)) \rightarrow D(P_0(n'))$ is bijective.
- (v) If $n \geq kN$, the map $D(P_0) \rightarrow D(P_0(n))$ is bijective.

Proof: (i) By using an induction on k , we can assume $k = 1$. Then from the theory of deformations of torseurs [14, VII:2.4.4, 2.4.4.1, 4.1.1.3] we know that the obstruction to lifting $P_0(n)$ lies in $H^2(S, \mathcal{L}_G^V \otimes I)$. Using the notation of [16, II 3.3.9], $\mathcal{L}_G^V \otimes I$ "is" the complex $L_0^V \otimes I \rightarrow L_{-1}^V \otimes I$, a complex of quasi-coherent sheaves on S . Since S is affine the H^2 is zero and $P_0(n)$ can be lifted.

(ii) Once again using induction on k , leads us to the case $k = 1$.

From [14, VII 2.4.4, 2.4.4.1, 4.1.1.3] we know that $D(P_0(n))$ is principal homogeneous under $H^1(S, \mathcal{L}_G^V \otimes I)$. Since $n, n' \geq N$, it follows from [16, II 3.3.6, 3.3.20] that this H^1 is $\Gamma(\underline{n}_G^V \otimes I)$ (resp. $\Gamma(\underline{n}'_G \otimes I)$). But by [16, II 3.3.4, 3.3.7, 3.3.16] we know the projection $G(2n) \rightarrow G(n)$ (resp. $G(2n) \rightarrow G(n')$) induces an isomorphism $\underline{n}_G(n) \xrightarrow{\sim} \underline{n}_G(2n)$ (resp. $\underline{n}_G(n') \xrightarrow{\sim} \underline{n}_G(2n)$). From the functorial nature of the co-Lie complex follows a commutative diagram



Thus the map $\underline{n}_G(n) \otimes I \rightarrow \underline{n}_G(n') \otimes I$ is an isomorphism and

(ii) follows now from the fact that $D(P_0(n)) \neq \emptyset$.

(iii) Let $(\zeta_n) \in \varprojlim D(P_0(n))$ and choose for each n a representative P_n of ζ_n . Then $P_n \wedge^{G(n)} G(n-1) \simeq P_{n-1}$ and we can successively choose these isomorphisms so that (P_n) is a

"torseur" under $T_p(G)$ which lifts P_0 . The remainder of

(iii) is clear.

(iv) For $k=1$, the assertion follows from the fact noted in

(ii) that $\underline{n}_G(n) \simeq \underline{n}_G(n')$. Let us filter S by the closed sub-schemes defined by powers of I : $S_0 \subseteq S_1 \subseteq \dots \subseteq S_{k-1} \subseteq S_k = S$.

By induction on k , we can assume (iv) true for the pair

$S_0 \hookrightarrow S_{k-1}$. Let $P(n), P'(n)$ be two deformations of $P_0(n)$

such that the induced $G(n')$ -torseurs $P(n'), P'(n')$ are isomorphic deformations of $P_0(n')$. We are to prove that $P(n) \simeq P'(n)$.

Let $u(n'): P(n') \xrightarrow{\sim} P'(n')$ be an isomorphism of deformations.

By the induction hypothesis we can find an isomorphism

$v(n): P_{k-1}(n) \rightarrow P'_{k-1}(n)$ (where the subscript "k-1" indicates restriction to S_{k-1}).

$v(n')$ and $u_{k-1}(n')$ are two isomorphisms between the deformations $P_{k-1}(n')$ and $P'_{k-1}(n')$. Their "difference" is thus an element of $G_{k-1}(n')(S_{k-1}/S_0)$. But from [16, II, 3.3.16] we know $\text{Inf}^{k-1}(G_{k-1}) = \text{Inf}^{k-1}(G_{k-1}((k-1)N))$. Thus multiplication by $p^{(k-1)N}$

kills this difference. Since $n' \geq kN$, this tells us that $v(N) = u_{k-1}(N)$. Observe that via $v(n)$, $P'(n)$ becomes a deformation of $P_{k-1}(n)$, while $P(n)$ is via $\text{id}_{P_{k-1}(n)}$ a deformation of $P_{k-1}(n)$. The equality $v(N) = u_{k-1}(N)$ says precisely that $P(N)$ and $P'(N)$ are, via $u(N)$, isomorphic as deformations of $P_{k-1}(N)$. Thus we may apply the result known to be true for the case $k=1$, to the pair $S_{k-1} \hookrightarrow S$ and the integers n, N (after all, $S_{k-1} \hookrightarrow S$ is a first order thickening). Thus there is an isomorphism $v': P(n) \rightarrow P'(n)$ which lifts v . This completes the proof.

(v) Let $P, P' \in D(P_0)$ and assume $P(kN) \simeq P'(kN)$. We are to show P is isomorphic to P' . From (iv) we know that for $n \geq kN$, $P(n)$ is isomorphic to $P'(n)$. For any n and any i let ϕ and ψ be two isomorphisms between $P(n+kN+i)$ and $P'(n+kN+i)$. Their "difference" is an element of $G(n+kN+i)(S/S_0)$. As noted already in the proof of (iv), this group is killed under multiplication by p^{kN} . Thus ϕ and ψ induce the same isomorphism between $P(n)$ and $P'(n)$: call it σ_n . It is clear that the σ_n 's fit together to give an isomorphism between P and P' . This completes the proof of the proposition.

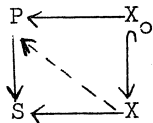
(10.10) To complete the proof of (10.2) we must establish surjectivity. From 10.9 (v), it suffices to establish surjectivity for the composite map

$$(10.11) \quad G(S/S_0) \rightarrow \text{Tors}(S/S_0, T_p(G)) \rightarrow \text{Tors}(S/S_0, G(kN))$$

(10.12) Lemma. Let S be a scheme on which p is locally nil-

potent, G be a Barsotti-Tate group on S , and P be a torseur on S under G . Then, P is formally smooth.

Proof: We must show that there is an arrow rendering the following diagram commutative (where X is affine and X_0 is defined by the vanishing of an ideal of square-zero).



By making the base change $X \rightarrow S$, we can assume S is affine (hence killed by a power of p). We are given a section of P over S_0 and our problem is to lift it. Since G is formally smooth, and P is a G -torseur, it suffices to show that P is trivial (i.e. has a section). Since S is affine, [11, SGA₄VI (5.2)] tells us that $H^1(S, G) \simeq \varinjlim H^1(S, G(n))$. Hence we can assume that for some n , P' is a $G(n)$ -torseur on S which has a section over S_0 and that $P' \wedge^{G(n)} G \simeq P$. Viewing P' as a deformation of the trivial $G_0(n)$ -torseur on S_0 it defines an element in $\text{Ext}^1(\mathcal{L}_{G_0(n)}, I)$. From [16, II 3.3.9] we know that if n, m are taken sufficiently large, the map $\text{Ext}^1(\mathcal{L}_{G_0(n)}, I) \rightarrow \text{Ext}^1(\mathcal{L}_{G_0(n+m)}, I)$ is zero. This tells us in particular that $P' \wedge^{G(n)} G(n+m)$ is a trivial torseur. Hence P has a section.

(10.13) We consider the exact sequence

$$0 \rightarrow G(n) \rightarrow G \xrightarrow{p^n} G \rightarrow 0$$

where n is an integer $\geq N$. The functor $\text{TORS}(S/S_0, G(n)) \rightarrow \text{TORS}(S/S_0, G)$ induces an equivalence of

categories between $\text{TORS}(S/S_0, G(n))$ and the category of pairs (Q, s) , where (Q, \mathfrak{s}) is an object of $\text{TORS}(S/S_0, G)$ and s is a section of $Q \wedge^{G} \xrightarrow{p^n} G$ such that $s|_{S_0} = \mathfrak{s} \wedge^{G_0} \xrightarrow{p^n} G_0$ (\mathfrak{s} being an element in $\Gamma(S_0, Q)$). This follows immediately from a momentary perusal of the proof of the corresponding fact when S_0 is suppressed [9, III, 3.2.1]. The point is that the quasi-inverse functor is given by $((Q, \mathfrak{s}), s) \mapsto \pi^{-1}(s)$, where π is the obvious map $Q \rightarrow Q \wedge^{G} \xrightarrow{p^n} G$.

(10.14) It is now standard [12, p. 17-18] that from the exact sequence

$$0 \rightarrow G(n) \rightarrow G \xrightarrow{p^n} G \rightarrow 0$$

we obtain a long exact sequence:

$$(10.15) \quad 0 \rightarrow G(n)(S/S_0) \rightarrow G(S/S_0) \xrightarrow{\partial} \text{Tors}(S/S_0, G(n)) \rightarrow \text{Tors}(S/S_0, G)$$

where ∂ is the map (10.11).

From this sequence the surjectivity of ∂ follows immediately since (10.12) tells us in particular that the map $\text{Tors}(S/S_0, G(n)) \rightarrow \text{Tors}(S/S_0, G)$ is the zero map. Hence (10.2) has been proved.

(10.16) Corollary: Assume the extension

$$0 \rightarrow G_{mS_0} \rightarrow E_0 \rightarrow G_0 \rightarrow 0$$

arises from pushing out along $g_0 \in \Gamma(S_0, G_0^*)$. The set of isomorphism classes of extensions lifting E_0 is in bijective correspondance with $\{g \in \Gamma(S, G^*) \mid g \text{ lifts } g_0\}$.

Proof: One checks immediately that the set of extensions lifting E_0 is principal homogeneous under $\text{Ext}(S/S_0; G, G_m)$, and hence the assertion follows immediately from (10.2).

(10.17) It is quite simple to globalize the above result. Let S be a scheme on which p is locally nilpotent and let G be a Barsotti-Tate group on S . Since G is locally of finite presentation we know

$$G(S/S_{\text{red}}) = \bigcup_{S_\lambda \text{ defined by a nilpotent ideal}} G(S/S_\lambda)$$

whenever S is affine. By abuse of notation we shall continue to write this even if S is not affine. On the other hand if $S_0 \subset S_1 \subset S$ and S is an infinitesimal neighborhood of S_0 , then there is a natural functor $\text{EXT}(S/S_1; G, G_m) \rightarrow \text{EXT}(S/S_0; G, G_m)$ which is easily seen to be fully-faithful. By abuse of notation we shall write $\text{EXT}(S/S_{\text{red}}; G, G_m)$ for the category $\varinjlim \text{EXT}(S/S; G, G_m)$ where the limit is taken over the index set of sub-schemes S_λ defined by a nilpotent ideal. Notational consistency dictates that we further abuse notation by writing $\text{Ext}(S/S_{\text{red}}; G, G_m) = \varinjlim \text{Ext}(S/S_\lambda; G, G_m)$. It is easy to show that we are guilty of a genuine abuse of notation for even if S consists of one point and is of characteristic p , there are extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by G_m which split over S_{red} but do not split where pulled back via a nilpotent immersion.

We've defined above a homomorphism of presheaves on (Sch/S)

$$(10.18) \quad T \mapsto G^*(T/T_{\text{red}}) \rightarrow T \mapsto \text{Ext}(T/T_{\text{red}}; G_T, G_m).$$

Furthermore (10.2) tells us that this is an isomorphism whenever T is affine.

(10.19) If F is an abelian presheaf on Sch/S we denote by \overline{F} the presheaf on Sch/S defined by $T \mapsto \text{UF}(T/T_\lambda)$, T_λ running through subschemes of T defined by the vanishing of a nilpotent ideal. As an exception, if G is a Barsotti-Tate group on S , " \overline{G} " will be used to denote the formal Lie group associated to G . Passing, in (10.18) to associated sheaves for the Zariski topology on Sch/S we obtain an isomorphism

$$(10.20) \quad \overline{G^*} \xrightarrow{\sim} \overline{\text{EXT}}(G, G_m)$$

where $\overline{\text{Ext}}$ is the presheaf $T \mapsto \text{Ext}(T/T_{\text{red}}; G_T, G_m)$. (N.B. Since G^* is ind-representable by affine schemes (relative to S) sheafification for the Zariski topology gives us an f.p.p.f. sheaf whose sections over an arbitrary S -scheme T can be explicitly described: $\overline{G^*}(T) = \{x \in G^*(T) \mid x \text{ restricted to any affine open } U \text{ of } T, \text{ dies when further restricted to a closed sub-scheme } U_0 \subset U \text{ defined by a nilpotent ideal}\}$).

§11. THE CRYSTALLINE NATURE OF THE FORMAL COMPLETION OF THE
UNIVERSAL EXTENSION

Let $S_0 \hookrightarrow (S, I, \gamma)$ be a nilpotent immersion defined by a divided power ideal I . Let G_0 be a Barsotti-Tate group on S_0 . We wish to assign to G_0 a formal group $\overline{E}^*(G_0)_{S_0} \hookrightarrow S$ which will be canonically isomorphic to the formal group associated to $E(G^*)$, $E(G^*)$, wherever G is a lifting of G_0 to S . We shall give an explicit description of the points of this functor with values in a flat S -scheme S' .

(11.1) Let $S'_0 = S' \times_S S_0$ and let $G'_0 = G_0 \times_{S_0} S'_0$. As explained in (6.14) we can consider the category $\text{EXT}^{\text{crys}/S'}(G'_0, G_m)$
= $\text{dfn} \leftarrow \text{Lim} \text{EXT}^{\text{crys}/S'}(G'_0(n), G_m)$.

For any closed subscheme $\overline{S}'_0 \hookrightarrow S'_0$ defined by a nilpotent ideal, we have the notion of a crystalline extension of $G'_0 \times_{\overline{S}'_0} \overline{S}'_0$ by G_m (relative to S') as given in (6.15). This allows us to speak of the category whose objects are pairs (P, η) where P is an object of $\text{EXT}^{\text{crys}/S'}(G'_0, G_m)$ and η is a trivialization of the underlying \mathcal{L} -extension of P restricted to \overline{S}'_0 . When \overline{S}'_0 is allowed to vary we obtain a direct system of categories and taking the direct limit we obtain a category which we denote by $\text{EXT}^{\text{crys}/S'}(S'_0/S'_0, \text{rel} G'_0, G_m)$. We write $\overline{\text{EXT}}^{\text{crys}/S'}(G'_0, G_m)$ for the group of isomorphism classes of objects of this category.

(11.2) Let G' be a Barsotti-Tate group on S' which lifts G'_0 . For any closed sub-scheme \overline{S}' , of S' which is defined by a nilpotent ideal there is the category of \overline{S}' -trivialized

\mathcal{L} -extensions of G' by G_m . Passing to the limit over such closed sub-schemes and then taking isomorphism classes of objects we obtain a group $\overline{\text{EXT}}^{\mathcal{L}}(G', G_m)$.

(11.3) Proposition: The natural functor

$$\overline{\text{EXT}}^{\text{crys}/S'}(G', G_m) \rightarrow \overline{\text{EXT}}^{\mathcal{L}}(G', G_m)$$

is an equivalence of categories.

Proof: The fact that the functor is fully-faithful is proved exactly as was done in the proof of (7.2). In fact it follows immediately from (7.6.1) since $\text{EXT}^{\text{nil-}\mathcal{L}}(G', G_m)$ is a full sub-category of $\text{EXT}^{\mathcal{L}}(G', G_m)$.

Let E be an object in $\overline{\text{EXT}}^{\mathcal{L}}(G', G_m)$. Since E becomes the trivial \mathcal{L} -extension when we pass to a closed sub-scheme $\overline{S}' \hookrightarrow S'$ defined by a nilpotent ideal, if we view E as a family of line bundles with integrable connection, \mathcal{L}_n on $G'(n)$, each of these line bundles becomes trivial on \overline{S}' . Fix an n and let D be a nilpotent S' derivation of $\mathcal{O}_{G'(n)}$ to itself. For $N \gg 0$ $\nabla(D)^N(\mathcal{L}_n) \subseteq (\text{ideal of } \overline{S}' \text{ in } S')$. \mathcal{L}_n (since $\mathcal{L}_n|_{G'(n)} \times_{S'} \overline{S}' \simeq (\mathcal{O}, \text{standard connection})$. Since the ideal of \overline{S}' in S' is nilpotent, $\nabla(D)$ is a nilpotent endomorphism of \mathcal{L}_n . Thus the connection on each \mathcal{L}_n is nilpotent [3, II, 4.3.6] (N.B. Berthelot defines this notion only when Ω^1 is locally-free of finite rank so a more correct assertion would be for $n \gg 0$ the connection on each \mathcal{L}_n is nilpotent). Thus our \mathcal{L} -extension E is isomorphic to a crystalline extension and the proof is complete.

(11.4) Corollary: The natural functor

$$\overline{\text{EXT}}^{\text{crys}/S'}(G'_0, G_m) \rightarrow \overline{\text{EXT}}^{\zeta}(G', G_m)$$

is an equivalence of categories.

Proof. Since the closed sub-schemes of S'_0 defined by a nilpotent ideal define by composition with $S'_0 \hookrightarrow S'$ a co-final system of closed sub-schemes of S' (defined by a nilpotent ideal), and since the ideal of S'_0 in S' has divided powers, (11.4) follows immediately from (11.3) plus the usual equivalence $\overline{\text{EXT}}^{\text{crys}/S'}(G'_0, G_m) \cong \overline{\text{EXT}}^{\text{crys}/S'}(G', G_m)$.

(11.5) Proposition: Let S be affine. There is a natural exact sequence

$$(11.6) \quad 0 \rightarrow \underline{\omega}_G(S/S_{\text{red}}) \rightarrow \overline{\text{Ext}}^{\zeta}(G, G_m) \rightarrow \overline{\text{Ext}}(G, G_m) \rightarrow 0$$

Proof: Given $\tau \in \underline{\omega}_G(S/S_{\text{red}})$ let τ' denote the ζ -structure on $G_m \times G$ defined by τ . Assume $(G_m \times G, \tau')$ is isomorphic to the trivial ζ -extension $(G_m \times G, \text{trivial})$ via an isomorphism ϕ which reduces to $\text{id}_{G_m \times G}$ modulo some nilpotent ideal. Then ϕ is necessarily equal to $\text{id}_{G_m \times G}$ and hence τ must be 0.

Let E be a trivialized ζ -extension whose underlying extension is isomorphic to $G_m \times G$ via an isomorphism, ϕ , respecting the trivializations (all trivializations over some $S'_0 \hookrightarrow S$ defined by a nilpotent ideal). Using ϕ let us equip $G_m \times G$ with a ζ -structure, τ' , by transport of structure. Since

τ' comes from a unique $\tau \in \Gamma(S, \underline{\omega}_G)$ and since the restriction of ϕ to S'_0 is compatible with trivializations it follows that $\tau \in \Gamma(S/S_{\text{red}}, \underline{\omega}_G)$ and exactness at $\overline{\text{Ext}}^{\zeta}(G, G_m)$ has been established.

It remains to check the surjectivity of $\overline{\text{Ext}}^{\zeta}(G, G_m) \rightarrow \overline{\text{Ext}}(G, G_m)$. Let E be an extension of G by G_m , and ϕ_0 a trivialization of $E \times_S S_0$. From (4.4) it follows that each of the induced extensions

$$0 \rightarrow G_m \rightarrow E \times_S G(n) \rightarrow G(n) \rightarrow 0$$

has a ζ -structure. Since for n large the maps $\Gamma(S, \underline{\omega}_{G(n+1)}) \rightarrow \Gamma(S, \underline{\omega}_{G(n)})$ are onto it follows that E itself carries at least one ζ -structure, ρ . The "difference" between ρ_0 and the ζ -structures on E_0 obtained via ϕ is an element of $\Gamma(S_0, \underline{\omega}_{G_0})$. Since the map $\Gamma(S, \underline{\omega}_G) \rightarrow \Gamma(S_0, \underline{\omega}_{G_0})$ is onto we can modify ρ to obtain a new ζ -structure on E so that ϕ_0 is horizontal. This completes the proof.

(11.7) Corollary: Assume p is locally nilpotent on S , G a Barsotti-Tate group on S . Sheafifying the sequence (11.6) we obtain an exact sequence

$$(11.8) \quad 0 \rightarrow \overline{\omega}_G \rightarrow \overline{\text{Ext}}^{\zeta}(G, G_m) \rightarrow \overline{\text{Ext}}(G, G_m) \rightarrow 0,$$

This sequence is canonically isomorphic to the exact sequence of formal groups obtained by completing the universal extension of G^* along the identity section:

$$(11.9) \quad 0 \rightarrow \overline{\omega}_G \rightarrow \overline{E(G^*)} \rightarrow \overline{G^*} \rightarrow 0.$$

Proof: The exactness of (11.9) is proved in [16, IV (1.2.1)]. From (4.6) we know that for S affine $\Gamma(S, E(G^*))$ is equal to $\varinjlim (\epsilon_n) - \text{Hom}^{\mathcal{L}}(G(n), G_m)$ where (ϵ_n) is the exact sequence

$$(\epsilon_n) \quad 0 \rightarrow G(n) \rightarrow G \xrightarrow{p^n} G \rightarrow 0$$

and where $(\epsilon_n) - \text{Hom}^{\mathcal{L}}(G(n), G_m)$ is the group whose elements are pairs $(\phi: G(n) \rightarrow G_m, \rho)$ a \mathcal{L} -structure on the extension $\phi_*(\epsilon_n)$.

Thus using (10.3) we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \overline{u}_G & \rightarrow & \overline{\text{Hom}}^{\mathcal{L}}(\epsilon_n, G_m) & \rightarrow & \overline{G}^* & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \overline{u}_G & \rightarrow & \overline{\text{Ext}}^{\mathcal{L}}(G, G_m) & \longrightarrow & \overline{\text{Ext}}(G, G_m) & \longrightarrow & 0 \end{array}$$

The corollary now follows from (10.20) and the five lemma.

(11.10) Let $S_0 \rightarrow (S, I, \gamma)$ be as in the beginning of this section. Assume given two Barsotti-Tate groups G, H on S and a homomorphism $u_0: G_0 \rightarrow H_0$ between their restrictions to S_0 . We shall associate to u_0 a homomorphism $v: \overline{E}(H^*) \rightarrow \overline{E}(G^*)$ which lifts $E(u_0^*)$.

If T is flat over S , the isomorphism (11.4): $\overline{\text{Ext}}^{\mathcal{L}}(G_T, G_m) \simeq \overline{\text{Ext}}^{\text{crys}/T}(G_{0_T}, G_m)$, together with the corresponding isomorphism with H replacing G , gives us an arrow v_T rendering the following diagram commutative:

$$\begin{array}{ccc} \overline{\text{Ext}}^{\text{crys}/T}(G_{0_T}, G_m) & \xrightarrow{\sim} & \overline{\text{Ext}}^{\mathcal{L}}(G_T, G_m) \\ \uparrow \overline{\text{Ext}}^{\text{crys}/T}(u_{0_T}, G_m) & & \uparrow v_T \\ \overline{\text{Ext}}^{\text{crys}/T}(H_{0_T}, G_m) & \xrightarrow{\sim} & \overline{\text{Ext}}^{\mathcal{L}}(H_T, G_m) \end{array}$$

Sheafifying and using (11.7) we find for T flat over S a morphism $\overline{E}(H^*)(T) \rightarrow \overline{E}(G^*)(T)$.

The existence of the homomorphism $\overline{E}(H^*) \rightarrow \overline{E}(G^*)$ now follows since $\overline{E}(H^*) = \varinjlim \text{Inf}^k(\overline{E}(H^*))$ and each Inf^k is flat over S .

(11.11) It follows immediately from (11.10) that if G and H are two liftings of the Barsotti-Tate group G_0 on S_0 , then $\overline{E}(G)$ is canonically isomorphic to $\overline{E}(H)$. Exactly as in (7.17), (7.18), the functor \overline{E}^* is explicitly defined by setting for S' an S -scheme

$$\Gamma(S', \overline{E}^*(G_0)_{S_0 \hookrightarrow S'}) = \Gamma(S', \overline{\text{Ext}}^{\text{crys}/S}(G_0, G_m))$$

where $\overline{\text{Ext}}^{\text{crys}/S}(G_0, G_m)$ denotes the prolongation to (Sch/S) of the sheaf on the small flat site of S associated to the pre-sheaf:

$$T \mapsto \overline{\text{Ext}}^{\text{crys}/T}(G_{0_T}, G_m)$$

(11.12) Remarks:

(i) In order to know that $\overline{E}^*(G_0)_{S_0 \hookrightarrow S}$ is a formal group we have made use of a lifting G of G_0 . In order to know that if

$$\begin{array}{ccc} S_0 & \hookrightarrow & (S', J, \gamma') \\ & \searrow & \downarrow f \\ & & (S, I, \gamma) \end{array}$$

is a commutative diagram where f is a divided power morphism, then $f^*(\overline{E}^*(G_0)_{S_0 \hookrightarrow S}) \rightarrow \overline{E}^*(G_0)_{S_0 \hookrightarrow S'}$ is an isomorphism; we make use of a lifting of G_0 . (If we don't assume the existence of a lifting then there doesn't appear to be any standard terminology which describes what $\overline{E}^*(G_0)$ is).

(ii) $\overline{\mathbb{E}^*}(G_o)$ is a crystal relative to a crystalline site which sits in-between the nilpotent site and the full Berthelot site: objects are divided power thickenings $S_o \hookrightarrow (S, I, \delta)$ where I is a nilpotent ideal, but the divided powers are not necessarily nilpotent. The reason for this was alluded to in (10.16).

(11.13) Let us check that $\text{Lie}(\overline{\mathbb{E}^*}(G_o))$ is canonically isomorphic to $\mathbb{D}^*(G_o)$ on their common domain of definition: Let $S_o \hookrightarrow (S, I, \gamma)$ be a divided power thickening of S_o by a nilpotent ideal.

Assume S is affine. We shall define a map

$$\text{Ext}^{\text{crys}/S}(G_o, G_a) \rightarrow \text{Ker}[\overline{\text{Ext}}^{\text{crys}/S[\epsilon]}(G_o, G_m) \rightarrow \overline{\text{Ext}}^{\text{crys}/S}(G_o, G_m)],$$

For any S_o -scheme X , there is a commutative diagram

$$\begin{array}{ccc} X[\epsilon] & \longrightarrow & S[\epsilon] \\ \downarrow & & \downarrow \pi \\ X & \longrightarrow & S \end{array}$$

which gives rise to a morphism of topoi

$$\pi: (X[\epsilon]/S[\epsilon])_{\text{crys}} \longrightarrow (X/S)_{\text{crys}}$$

Using the definition of π [3, III, 2.2.3] one checks easily that for any object $(U \hookrightarrow T, J, \delta)$ of the crystalline site of X , $\pi^{-1}(U \hookrightarrow T) = U[\epsilon] \hookrightarrow T[\epsilon]$. Thus $\pi_*(G_m)_{U \hookrightarrow T} = G_{m, T[\epsilon]}$ and there is an exact sequence of sheaves of groups in $(X/S)_{\text{crys}}$

$$0 \rightarrow G_a \rightarrow \pi_*(G_m) \rightarrow G_m \rightarrow 0$$

Thus we obtain an equivalence of categories

$$(11.14) \text{TORS}^{\text{crys}/S}(X, G_a) \cong \text{Ker}[\text{TORS}^{\text{crys}/S[\epsilon]}(X[\epsilon], G_m) \rightarrow \text{TORS}^{\text{crys}/S}(X, G_m)].$$

This equivalence is functorial in the S_o -scheme X and hence we obtain

$$(11.15) \text{EXT}^{\text{crys}/S}(G_o, G_a) \cong \text{Ker}[\text{Ext}^{\text{crys}/S[\epsilon]}(G_o[\epsilon], G_m) \rightarrow \text{Ext}^{\text{crys}/S}(G_o, G_m)].$$

This permits us to define the map

$$(11.16) \text{Ext}^{\text{crys}/S}(G_o, G_a) \rightarrow \text{Ker}(\overline{\text{Ext}}^{\text{crys}/S[\epsilon]}(G_o[\epsilon], G_m) \rightarrow \overline{\text{Ext}}^{\text{crys}/S}(G_o, G_m)).$$

Before we prove the bijectivity of this map, let us note that the category $\overline{\text{EXT}}^{\text{crys}/S}(G_o, G_m)$ is rigid. This follows immediately from (11.4) (and hence we use once again the fact that Barsotti-Tate groups can be lifted).

Let P, Q be representatives of elements of $\text{Ext}^{\text{crys}/S}(G_o, G_a)$. To say they define the same element in Ker is equivalent to asserting that there is an isomorphism of crystalline extensions

$$\rho: P \wedge^{G_a} \pi_*(G_m) \cong Q \wedge^{G_a} \pi_*(G_m)$$

such that $\rho \wedge^{G_m}$ induces the identity automorphism of the \mathcal{L} -extension $G_m \times G_o$ (once we identify $P \wedge^{G_a} G_m$ and $Q \wedge^{G_a} G_m$ with $G_m \times G_o$). But using the rigidity of $\overline{\text{EXT}}^{\text{crys}/S}(G_o, G_m)$ noted above, it follows that $\rho \wedge^{G_m}$ is actually the identity automorphism of the crystalline extension $G_m \times G_o$. It now follows from (11.15) that $P \cong Q$.

On the other hand the surjectivity of (11.16) is clear since a crystalline extension, P , of $G_o[\epsilon]$ by G_m trivialized as \mathcal{L} -extension over some closed subscheme $T \subseteq S[\epsilon]$, and which is trivialized over S as crystalline extension (in a

compatible fashion over $S \cap T \subset S[\epsilon]$) defines a crystalline extension of G_0 by G_a , Q , which is isomorphic to P as a crystalline extension (an isomorphism certainly compatible with the trivialization over $S \cap T$).

§12. THE CRYSTALLINE NATURE OF THE UNIVERSAL EXTENSION (ON THE NILPOTENT CRYSTALLINE SITE)

In this section we shall show that the universal extension of a Barsotti-Tate group can be extended to a crystal on the nilpotent crystalline site.

Let S_0 be a scheme, (S, I, γ) a nilpotent divided power thickening. Fix a Barsotti Tate group G_0 on S_0 . Following the procedure(s) used in previous sections we shall define for S' a flat S -scheme, a group $\mathbb{E}(G_0)_{S_0 \hookrightarrow S}(S')$, such that sheafification gives us the value of our crystal on (S, I, γ) .

(12.1) Consider the category whose objects are triples:

- (12.2) (i) an element $g_0 \in \Gamma(S_0, G_0^*)$
 (ii) a nilpotent crystalline extension of G_0 by G_m (relative to S), $E \in \text{EXT}^{\text{nil crys}/S}(G_0, G_m)$
 (iii) an isomorphism ρ between the extension P_{g_0} , associated to g_0 , and the ordinary extension underlying E .

Morphisms between (g_0, E, ρ) and (g'_0, E', ρ') are defined only if $g_0 = g'_0$ and then a morphism is a morphism of crystalline extensions $E \rightarrow E'$ which is compatible with ρ and ρ' .

(12.3) Definition: Let $\mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}(S) =$ group of isomorphism classes of objects of the above category.

(12.4) Let G be a lifting of G_0 to S which we assume to be affine. We construct a map $\Gamma(S, E(G^*)) \rightarrow \mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}(S)$ by

interpreting an element of $\Gamma(S, E(G^*))$ as an element, \mathfrak{g} , of $\varinjlim(\epsilon_n)\text{-Hom}^{\mathcal{L}}(G(n), G_n)$ (as in the proof of (11.7)) and assigning to \mathfrak{g} the isomorphism class of the triple:

(i) g_0 = restriction to S_0 of the element of $\Gamma(S, G^*)$ which is the image of \mathfrak{g} under $E(G^*) \rightarrow G^*$

(ii) E , the object of $\text{EXT}^{\text{Nil-crys}/S}(G_0, G_m)$ corresponding to \mathfrak{g} via the equivalence (7.6.2) plus the equivalence $\text{EXT}^{\text{Nil-crys}/S}(G_0, G_m) \cong \text{EXT}^{\text{Nil-crys}/S}(G, G_m)$

(iii) the canonical isomorphism $P_{g_0} \cong E$ (i.e. the identity map).

(12.5) Proposition: The map defined in (12.4) is an isomorphism.

Proof: To show injectivity let $\mathfrak{g} \in \Gamma(S, E(G^*))$ be given, let $g = \text{image of } \mathfrak{g} \text{ under } E(G^*) \rightarrow G^*$. Assume the triple defined by \mathfrak{g} is isomorphic to the triple $(0, \text{trivial crystalline extension, identity})$, i.e. there is a map $E \cong G_0 \times G_m$ of crystalline extensions and the map on underlying extensions is the identity.

Since we can interpret the crystalline extensions E and $G_0 \times G_m$ as \mathcal{L} -extensions of G by G_m , it follows from (10.3) that $g = 0$. Hence \mathfrak{g} is given by an element of $\Gamma(S, \underline{u}_G)$. The rigidity of the category $\text{EXT}(S/S_0; G, G_m)$ insures that the isomorphism $E \cong G_0 \times G_m$, when interpreted as a map of \mathcal{L} -extension of G by G_m , is the identity. This forces the element of $\Gamma(S, \underline{u}_G)$, and hence \mathfrak{g} , to be zero.

To prove surjectivity, let (g_0, E, ρ) be a triple. We interpret E (as explained in 12.4(ii)) as an object of

$\text{EXT}^{\mathcal{L}}(G, G_m)$ whose underlying structure of extension we denote by E' . From (10.16) the pair (E', ρ) determines an element g , of $\Gamma'(S, G^*)$ which lifts g_0 . Let ∇ be the \mathcal{L} -structure on P_g obtained via transport of structure from E using the isomorphism $P_g \cong E'$. If $\mathfrak{g} = (g, \nabla)$ then, by construction, the image of \mathfrak{g} is the class of the triple (g_0, E, ρ) .

(12.6) Corollary: Sheafifying the map

$$\Gamma(S, E(G^*)) \rightarrow \mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}(S)$$

we obtain an isomorphism (of sheaves of groups on the small flat site of S)

$$E(G^*) \rightarrow \mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}$$

(12.7) Let G_1, G_2 be two liftings of G_0 to S . Just as in (11.10), (11.11) there is a canonical isomorphism $E(G_1^*) \cong E(G_2^*)$.

In fact more generally we can state

(12.8) Corollary: There is a functor

$\text{B.T.}(S_0)^\circ \rightarrow \text{Crystals in groups on the nilpotent site of } S_0$
given by $G_0 \mapsto \mathbb{E}^*(G_0)$ (where $\mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}$ has been explicitly defined via (12.4)).

(12.9) We now wish to show that "completing along the identity element" the crystal $\mathbb{E}^*(G_0)$ gives us a crystal in formal groups which is canonically isomorphic to the crystal $\overline{\mathbb{E}^*}(G_0)$ (of §11), when the latter is restricted to the nilpotent crystalline site.

Let $S_0 \hookrightarrow (S, I, \psi)$ be a thickening of the nilpotent site of S_0 . In order to show the formal groups on $S \left(\overline{\mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}} \right)$ and $\mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}$ are isomorphic, it suffices to show that their values on flat S -schemes are functorially isomorphic. Thus by localization it suffices to treat the case when S is affine. Since $\mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}$ is ind-representable by affine groups, it follows that

$$\Gamma(S, \overline{\mathbb{E}^*(G_0)_{S_0 \hookrightarrow S}}) = \text{group of classes of triples } (g_0, E, \rho)$$

such that for some nilpotent immersion $T \hookrightarrow S_0$, the inverse image to T of the \mathcal{L} -extension underlying E becomes isomorphic via ρ_T to the trivial \mathcal{L} -extension of G_{0T} by G_m .

To check that this description is correct we use the fact that $\mathbb{E}^*(G_0)_{S_0 \xrightarrow{\text{id}} S_0} = i^*(\mathbb{E}^*(G_0)_{S_0 \hookrightarrow S})$, $i: S_0 \hookrightarrow S$ being the inclusion, and the fact that the crystalline extensions of G_0 by G_m (relative to S_0) are simply the \mathcal{L} -extensions.

Consider now the map

$$(g_0, E, \rho) \mapsto_{\text{class of } (E, \rho|_T)} \text{in } \overline{\text{Ext}}^{\text{crys}/S}(G_0, G_m)$$

The injectivity of this map follows from the injectivity of map $G_0^*(S_0/T) \rightarrow \text{Ext}(S_0/T; G_0, G_m)$. For if (g'_0, E', ρ') is a second triple and $(E', \rho'|_T) \simeq (E, \rho|_T)$, then there is an isomorphism of crystalline extensions $\eta: E \rightarrow E'$ such that $\eta|_T \circ \rho|_T = \rho'|_T$. But $\eta \circ \rho$ and ρ' are then equal by (10.5).

N.B. We view E as an object of $\overline{\text{Ext}}^{\text{crys}/S}(G_0, G_m)$ using (11.3). The surjectivity of the map follows immediately from the assertion of surjectivity implicit in (10.3).

§13. RELATION BETWEEN THE UNIVERSAL EXTENSION CRYSTAL OF AN ABELIAN VARIETY AND THAT OF ITS ASSOCIATED BARSOTTI-TATE GROUP

We shall now show that our construction of the crystals (of various sorts) associated to a Barsotti-Tate group is compatible with our earlier construction of the crystals associated to an abelian scheme.

Let S_0 be a scheme (with p locally nilpotent), A_0/S_0 an abelian scheme, $G_0 = \varinjlim A_0(n)$ the associated Barsotti-Tate group. Fix a nilpotent divided power thickening $S_0 \hookrightarrow (S, I, \psi)$ and assume S_0 is affine.

(13.1) Lemma: Let the triple (g_0, E, ρ) define an element of $\Gamma(S, \mathbb{E}^*(G_0)_{S_0 \hookrightarrow S})$. Then up to isomorphism there is a unique crystalline extension E' in $\overline{\text{Ext}}^{\text{nil-crys}/S}(A_0, G_m)$ such that there is an isomorphism ρ' between the extension of A_0 by G_m defined by g_0 and the extension underlying E' , such that $(g_0, E'|_{G_0}, \rho'|_{G_0})$ is isomorphic to (g_0, E, ρ) . (N.B. ρ' is necessarily unique).

Proof: Let A/S be any abelian scheme lifting A_0 , let G be the associated Barsotti-Tate group. Corresponding to the triple (g_0, E, ρ) , there is a pair $g \in \Gamma(S, G^*)$, ∇ a \mathcal{L} -structure on the extension

$$(13.2) \quad 0 \rightarrow G_m \rightarrow \mathcal{G} \rightarrow G \rightarrow 0$$

obtained by pushing out the "Kummer sequence" along g . This \mathcal{L} -structure defines a rigidification on (13.2). But

(13.2) is obtained by restricting to G an extension

$$0 \rightarrow G_m \rightarrow \mathcal{E}' \rightarrow A \rightarrow 0$$

Since $\text{Inf}^1(G) = \text{Inf}^1(A)$, this extension has a canonical rigidification, i.e. a canonical \mathcal{L} -structure. It follows immediately from [I,(3.2.3)] that this \mathcal{L} -structure extends the given

\mathcal{L} -structure on (13.2). Via the equivalence of categories $\text{EXT}^{\mathcal{L}}(A, G_m) \cong \text{EXT}^{\text{nil-crys}/S}(A_0, G_m)$ \mathcal{E}' defines an object E' of $\text{EXT}^{\text{nil-crys}/S}(A_0, G_m)$ such that $E'|_{A_0} \cong E$, and E' clearly satisfies the conditions with $\rho' = \text{"id"}$.

Let E'' be a second object of $\text{EXT}^{\text{nil-crys}/S}(A_0, G_m)$ which satisfies the conditions, i.e. so that there is a ρ'' . By hypothesis there is an isomorphism $\mathfrak{E}: E'|_{G_0} \xrightarrow{\sim} E''|_{G_0}$ of crystalline extensions such that the following diagram commutes

$$\begin{array}{ccc} E'|_{G_0} & \xrightarrow{\mathfrak{E}} & E''|_{G_0} \\ \rho'|_{G_0} \swarrow & & \searrow \rho''|_{G_0} \\ & P_{g_0} & \end{array}$$

We must show that E' and E'' are isomorphic crystalline extensions. Corresponding to E'' is a \mathcal{L} -extension \mathcal{E}'' of A by G_m . Since \mathfrak{E} is a map of crystalline extensions there is a map $F: \mathcal{E}'|_G \rightarrow \mathcal{E}''|_G$ which lifts \mathfrak{E} . As the extension underlying the \mathcal{L} -extension $\mathcal{E}'' - \mathcal{E}'$ is trivialized over S_0 , this extension is obtained via pushing out a "Kummer sequence" along an element, g' , of $\Gamma(S, G^*)$, such that $g'|_{S_0} = 0$ [19, (19.1)]. But g and $g+g'$ are two sections in $\Gamma(S, G^*)$ lifting g_0

with the corresponding extensions, yielding via $F|_{G_0}$ isomorphic deformations of P_{g_0} . Hence from (10.16) it follows that $g' = 0$ and hence that the extensions underlying \mathcal{E}' and \mathcal{E}'' are isomorphic via a unique isomorphism τ . By the rigidity of $\text{EXT}(A_0, G_m)$, $\tau|_{S_0} \circ \rho = \rho''$, and hence by the rigidity of the category of deformations of P_{g_0} , $\tau|_G = F$. Since $\text{Inf}^1(A) \subseteq G$ τ induces an isomorphism of the rigidified extensions \mathcal{E}' and \mathcal{E}'' . But from [I,(3.2.3)] we know this means τ is an isomorphism of \mathcal{L} -extensions. Via the equivalence $\text{EXT}^{\mathcal{L}}(A, G_m) \cong \text{EXT}^{\text{nil-crys}/S}(A_0, G_m)$, we see τ induces an isomorphism between E' and E'' . This completes the proof.

(13.3) Remark: Although we have used a lifting in the proof of (13.1) the result is clearly independent of any such choice.

(13.4) Let A and B be abelian schemes on S , G, H the corresponding Barsotti-Tate groups. Assume $u_0: A_0 \rightarrow B_0$ is a homomorphism inducing $\tilde{u}_0: G_0 \rightarrow H_0$. In §1 (resp. §12) there is associated a homomorphism $E(B^*) \rightarrow E(A^*)$ (resp. $E(H^*) \rightarrow E(G^*)$). It is an immediate consequence of (13.1) that the following diagram commutes:

$$(13.5) \quad \begin{array}{ccc} E(H^*) & \longrightarrow & E(G^*) \\ \downarrow & & \downarrow \\ E(B^*) & \longrightarrow & E(A^*) \end{array}$$

Passing to tangent spaces we find that the map

$$D^*(H_0)_{S_0 \hookrightarrow S} \longrightarrow D^*(G_0)_{S_0 \hookrightarrow S}$$

coincides with the map $H^1(B, \mathcal{O}_{B_{\text{crys}}}) \rightarrow H^1(A, \mathcal{O}_{A_{\text{crys}}})$ induced (from u_0) by crystalline cohomology.

§14. GROTHENDIECK'S DUALITY FORMULA FOR THE LIE COMPLEX

Let S be a scheme, G a finite, locally-free (commutative) S -group. In the course of the proof given below we shall recall a construction of the co-Lie complex, \mathcal{L}^G , associated to G . Let M be a quasi-coherent \mathcal{O}_S -module. From [14, VII, 1.1] we know it is entirely harmless to identify \mathcal{L}^G and M with the corresponding objects that they define on the flat site of S . With this understanding the formula is:

$$(14.1) \quad R \underline{\text{Hom}}_{\mathcal{O}_S}(\mathcal{L}^G, M) \simeq_{\tau \leq 1} R \underline{\text{Hom}}_{\mathbb{Z}}(G^*, M)$$

This isomorphism is functorial in both arguments and when S is affine there is a similar isomorphism with "Hom" replacing "Hom".

Taking $M = \mathcal{O}_S$ we find a formula for the Lie complex:

$$(14.2) \quad \mathcal{L}^G \simeq_{\tau \leq 1} R \underline{\text{Hom}}_{\mathbb{Z}}(G^*, G_a)$$

If S is affine applying H^1 (to the formula involving $R \text{Hom}$) yields

$$(14.3) \quad \text{Ext}^1(\mathcal{L}^G, M) \simeq \text{Ext}_{\mathbb{Z}}^1(G^*, M)$$

(a formula used above in (3.1))

If instead we took H^0 the formula becomes

$$(14.4) \quad \text{Hom}_{\mathcal{O}_S}(\mathcal{L}^G, M) \simeq \text{Hom}(G^*, M)$$

Proof (Grothendieck): From [11, SGA₄, VII, 3.5] we know there is a partial resolution of G .

$$(14.5) \quad L_2 \rightarrow L_1 \rightarrow L_0 \xrightarrow{\epsilon} G$$

Each L_i is a sum of sheaves of the form $\mathbb{Z}[T_i]$ where T_i is a finite product of copies of G , L_0 is simply $\mathbb{Z}[G]$. This resolution is functorial in G . From (I,(1.3)) it follows that $G^1 = \text{dfn. } L_1^*$ is a smooth commutative group scheme. Because $\text{Ext}^i(\mathbb{Z}[T], G_m) = R^1 f_{T*}(G_{m_T}) = 0$ ($f_T: T \rightarrow S$ being the structural map for a finite locally-free S -scheme), the complex

$$G^* = G^0 \rightarrow G^1 \rightarrow G^2$$

has

$$(14.6) \quad \begin{cases} H^0(G^*) = G^* \\ H^1(G^*) = \text{Ext}^1(G, G_m) = (0) \text{ since } G \text{ is finite,} \\ \text{locally-free} \end{cases}$$

Thus if $\bar{G} = \text{Ker}(G^1 \rightarrow G^2)$ we obtain an exact sequence

$$(14.7) \quad 0 \rightarrow G^* \rightarrow G^0 \rightarrow \bar{G} \rightarrow 0$$

It follows from [8, II, 5.22] that \bar{G} is a smooth S -group. We define the co-Lie complex of G^* by:

$$(14.8) \quad \mathcal{L}_G^* = \text{dfn. } \underline{\mathcal{L}}_G \rightarrow \underline{\mathcal{L}}_{G^0}$$

(where $\underline{\mathcal{L}}_G$ is placed in degree -1)

In (I,(1.2)) we've defined a map

$$L_i \rightarrow \underline{\mathcal{L}}_{G^i}$$

Applying $\underline{\text{Hom}}(\ , M)$ (resp. $\text{Hom}(\ , M)$) we obtain a morphism of complexes

$$(14.9) \quad \begin{array}{ccccc} \underline{\text{Hom}}(\underline{\mathcal{L}}_{G^0}, M) & \longrightarrow & \underline{\text{Hom}}(\underline{\mathcal{L}}_{G^1}, M) & \longrightarrow & \underline{\text{Hom}}(\underline{\mathcal{L}}_{G^2}, M) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\text{Hom}}(L_0, M) & \longrightarrow & \underline{\text{Hom}}(L_1, M) & \longrightarrow & \underline{\text{Hom}}(L_2, M) \end{array}$$

(I,(1.4)) tells us that (14.9) is an isomorphism of complexes.

Observe that $\text{Ext}^i(\mathbb{Z}[T], M) = R^i f_{T*}(M_T) = (0)$ for $i > 0$ since the map f_T is affine and M is quasi-coherent. Furthermore if S is affine, $\text{Ext}^i(\mathbb{Z}[T], M) = (0)$ for $i > 0$. Since each $\underline{\mathcal{L}}_{G^i}$ is locally-free, it is also true that $\text{Ext}^j(\underline{\mathcal{L}}_{G^i}, M) = (0)$ for $j > 0$ (resp. $\text{Ext}^j(\underline{\mathcal{L}}_{G^i}, M) = 0$ if S is affine).

Since L is a partial resolution of G , the complex $\underline{\text{Hom}}(L_0, M) \rightarrow \underline{\text{Hom}}(L_1, M) \rightarrow \underline{\text{Hom}}(L_2, M)$ has $H^0 = \underline{\text{Hom}}(G, M)$, $H^1 = \underline{\text{Ext}}^1(G, M)$ (resp. without underlining if S is affine). In fact "killing" the H^2 of this complex we obtain the complex $\tau_{\leq 1} R \underline{\text{Hom}}_{\mathbb{Z}}(G, M)$. On the other hand by applying $\tau_{\leq 1}(\)$ to (14.9) we obtain

$$\begin{array}{ccc} \underline{\text{Hom}}(\underline{\mathcal{L}}_{G^0}, M) & \longrightarrow & \underline{\text{Hom}}(\underline{\mathcal{L}}_G, M) \\ \downarrow & & \downarrow \\ \tau_{\leq 1}(R \underline{\text{Hom}}_{\mathbb{Z}}(G, M)) & & \end{array}$$

Since the source of this arrow is $R \underline{\text{Hom}}(\mathcal{L}_G^*, M)$ (14.1) is established.

§15. COMPARISON WITH CLASSICAL DIEUDONNÉ THEORY

(15.1) Denote by $W_n = \text{Spec}(\mathbb{Z}[W_0, \dots, W_{n-1}])$ the group scheme of Witt vectors of length n and by $\mathfrak{s}_n: W_n \rightarrow (\mathbb{G}_a)^n$ the homomorphism given by ghost components. Let $T: W_n \rightarrow W_{n+m}$ be the homomorphism defined on S -valued points (S any scheme) by

$$T(w_0, \dots, w_{n-1}) = (0, \dots, 0, w_0, \dots, w_{n-1})$$

and let $R: W_{n+1} \rightarrow W_n$ be the homomorphism defined on S -valued points by

$$R(w_0, \dots, w_n) = (w_0, \dots, w_{n-1})$$

Using the mappings T , the W_n 's form an inductive system and we denote by W the direct limit.

(15.2) Let k be a perfect field of characteristic p . Classically [18 bis, 3.12], one defines the Dieudonné module of a unipotent p -divisible group, G , as

$$D^*(G) = \text{Hom}_{k\text{-gr}}(G, W_k)$$

This definition can be extended to a toroidal p -divisible group, G , by setting

$$D^*(G) = D^*(G^*)^V$$

In (9.2) we defined for G a Barsotti-Tate group over an arbitrary base S (with p locally nilpotent) a crystal on S in locally-free modules, $\mathcal{D}^*(G)$. The category of crystals in locally-free modules on $S_0 = \text{Spec}(k)$ (relative to \mathbb{Z}_p) is equivalent to the category of free $W(k)$ -modules. Explicitly the

equivalence is given by

$$M \mapsto \varprojlim M_{W_n}(k)$$

where M is a crystal and $M_{W_n}(k)$ denotes its value on the thickening $S_0 \hookrightarrow S_n = \text{Spec}(W_n(k))$

Regarding $\mathcal{D}^*(G)$ as a free $W(k)$ -module we can ask about its relation to $D^*(G)$. The answer is provided by the following theorem of Grothendieck.

(15.3) Theorem: There is a canonical isomorphism of functors $D^* \rightarrow \mathcal{D}^*$ (which will be explicitly constructed below).

(15.4) Because of the decomposition of the category of p -divisible groups/ k into the product of the category of toroidal p -divisible groups and the category of unipotent p -divisible groups, it suffices to consider only unipotent groups.

The key to proving (15.3) is Grothendieck's observation that, over \mathbb{Z} , the extension

$$(15.5) \quad 0 \rightarrow G_a \rightarrow W \xrightarrow{V} W \rightarrow 0$$

is endowed with a canonical structure of \mathcal{G} -extensions.

To see this one first considers the extensions

$$(15.6_n) \quad 0 \rightarrow G_a \xrightarrow{T} W_{n+1} \xrightarrow{R} W_n \rightarrow 0.$$

Let $s: W_n \rightarrow W_{n+1}$ be the set-theoretic section given by:

$$s(w_0, \dots, w_{n-1}) = (w_0, \dots, w_{n-1}, 0)$$

The section s determines a trivialization of the G_{a, W_n} -torseur W_{n+1} . Using this trivialization, endow W_{n+1} with a structure of \mathcal{G} -torseur, ∇_0 .

We modify ∇_0 by defining a new \mathcal{H} -structure

$$(15.7_n) \quad \nabla_n = \nabla_0 - \omega_n$$

where

$$(15.8_n) \quad \omega_n = w_0^{p^n-1} dw_0 + \dots + w_{n-1}^{p-1} dw_{n-1} \in \Gamma(\mathbb{Z}, \Omega_{W_n}^1)$$

As will be shown in (15.10) ∇_n makes the extension (15.6_n) into a \mathcal{H} -extension. From the explicit construction of ∇_n it is immediate that the following is compatible with \mathcal{H} -structures

$$(15.9) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbb{G}_a & \xrightarrow{T} & W_{n+1} & \xrightarrow{R} & W_n \rightarrow 0 \\ & & \parallel & & \downarrow T & & \downarrow T \\ 0 & \rightarrow & \mathbb{G}_a & \xrightarrow{T} & W_{n+2} & \xrightarrow{R} & W_{n+1} \rightarrow 0 \end{array}$$

Passing to the limit we obtain ∇_∞ , the desired structure of \mathcal{H} -extension on (15.5).

Let us stop here to check

(15.10) Proposition:

(i) Let $t: (\mathbb{G}_a)^n \rightarrow (\mathbb{G}_a)^{n+1}$ be the map $(x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{n-1}, 0)$. View t as a splitting of the extension

$$0 \rightarrow \mathbb{G}_a \rightarrow \mathbb{G}_a^{n+1} \rightarrow \mathbb{G}_a^n \rightarrow 0$$

and endow this extension with its trivial \mathcal{H} -structure.

Over $\mathbb{Z}[\frac{1}{p}]$ the diagram

$$(15.11) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbb{G}_a & \xrightarrow{T} & W_{n+1} & \xrightarrow{R} & W_n \rightarrow 0 \\ & & \downarrow p^n x & & \downarrow \xi_{n+1} & & \downarrow \xi_n \\ 0 & \rightarrow & \mathbb{G}_a & \rightarrow & \mathbb{G}_a^{n+1} & \rightarrow & \mathbb{G}_a^n \rightarrow 0 \end{array}$$

allows us to transport the just-described \mathcal{H} -structure on the lower row to (15.6_n).

Assertion: This \mathcal{H} -structure coincides with ∇_n .

(ii) ∇_n makes (15.6_n) into a \mathcal{H} -extension.

Proof: (ii) is an immediate consequence of (i) since the obstruction to the isomorphism

$$s^*(W_{n+1}) \xrightarrow{\sim} p_1^*(W_{n+1}) \wedge p_2^*(W_{n+1})$$

being horizontal is an element of the free abelian group

$\Gamma(\Omega_{W_n \times W_n}^1)$ which dies when we tensor this group with $\mathbb{Z}[\frac{1}{p}]$.

Let $t' = \xi_{n+1}^{-1} \circ t \circ \xi_n$ be the splitting obtained by transport of structure. It suffices to show

$$(15.12) \quad d(t'-s) = -\omega_n$$

But $t'-s(w_0, \dots, w_{n-1}) = (0, \dots, 0, w_n)$ where $p^n w_n + p^{n-1} w_{n-1}^p + \dots + w_0^{p^n} = 0$.

That is $w_n = \frac{-1}{p^n} (w_0^{p^n} + \dots + p^{n-1} w_{n-1}^p)$

Thus $d(t'-s) = -(w_0^{p^n-1} dw_0 + \dots + w_{n-1}^{p-1} dw_{n-1}) = -\omega_n$.

(15.13) Remark: It follows from (15.10) (i) that the rigidification on (15.6_n) associated to ∇_n is the restriction of s to $\text{Inf}^1(W_n)$.

(15.14) We can now define the map $D^*(G) \rightarrow \mathbb{D}^*(G)$. For each n interpret the restriction to $W_n(k)$ of the \mathcal{H} -extension (15.5) as being an object in $\text{EXT}^{\text{crys}}/W_n(k)(W_k, \mathbb{G}_a, W_n(k))$. Pulling back

this object by a homomorphism $\varphi: G \rightarrow \mathbb{W}_{\rightarrow, k}$ gives us an object in $\text{EXT}^{\text{crys}/W_n(k)}(G, \mathbb{G}_a)$. These objects, for variable n , piece together and we obtain the desired map

$$\begin{array}{ccc} D^*(G) & \xrightarrow{\quad} & \mathbb{D}^*(G) \\ \parallel & & \parallel \\ \text{Hom}_{k\text{-gr}}(G, \mathbb{W}_{\rightarrow}) & \xrightarrow{\quad} & \varprojlim \text{Ext}^{\text{crys}/W_n(k)}(G, \mathbb{G}_a) \end{array}$$

Proof of (15.3): By Nakayama's lemma it suffices to prove that the reduction modulo p of this map is injective since the source and target are free $W(k)$ -modules of the same rank (= height (G)).

Let $\varphi: G \rightarrow \mathbb{W}_{\rightarrow, k}$ be such that $\varphi^*(\nabla_{\infty})$ is the trivial structure of \mathbb{L} -extension. We are to show that φ admits a factorization as $\varphi = G \xrightarrow{P} G \xrightarrow{\tilde{\varphi}} \mathbb{W}_{\rightarrow, k}$.

By assumption there is a unique arrow $\psi: G \rightarrow W$ which makes the following diagram commutative.

$$(15.15) \quad \begin{array}{ccccccc} 0 \rightarrow & \mathbb{G}_a & \rightarrow & \mathbb{W}_{\rightarrow} & \times & G & \rightarrow & G & \rightarrow & 0 \\ & \parallel & & \downarrow & \downarrow & \searrow & & \downarrow & & \\ & \mathbb{G}_a & \rightarrow & \mathbb{W}_{\rightarrow} & & \psi & & \varphi & & \\ & & & \downarrow & & \swarrow & & & & \\ 0 \rightarrow & \mathbb{G}_a & \rightarrow & \mathbb{W}_{\rightarrow} & & V & \rightarrow & \mathbb{W}_{\rightarrow} & \rightarrow & 0 \end{array}$$

We want to show that ψ can be written as $\psi = G \xrightarrow{\tau} \mathbb{W}_{\rightarrow, k} \xrightarrow{F} \mathbb{W}_{\rightarrow, k}$, for some τ . To show this it suffices to show $\psi|_{\text{Ker}(F_G)} = 0$. Continue to denote this restricted map by ψ . For $n \gg 0$ (15.15) induces a diagram

$$\begin{array}{ccccccc} 0 \rightarrow & \mathbb{G}_a & \rightarrow & \mathbb{W}_{n+1} & \times & \text{Ker}(F_G) & \rightarrow & \text{Ker}(F_G) & \rightarrow & 0 \\ & & & \downarrow & \downarrow & \searrow & & \downarrow & & \\ 0 \rightarrow & \mathbb{G}_a & \xrightarrow{T} & \mathbb{W}_{n+1} & & \psi & & \varphi & & \\ & & & \downarrow & & \swarrow & & & & \\ & & & \mathbb{W}_n & & R & \rightarrow & \mathbb{W}_n & \rightarrow & 0 \end{array}$$

From our assumption that $\varphi^*(\nabla_{\infty}) = \text{trivial structure of } \mathbb{L}\text{-extension}$ and (15.13) it follows that

$$(15.16) \quad s \circ R \circ \psi|_{\text{Inf}^1(\text{Ker}(F_G))} = \psi|_{\text{Inf}^1(\text{Ker}(F_G))}.$$

The following lemma shows that this implies $\psi = 0$ and completes the proof of (15.3).

(15.17) Lemma: Let H be a finite commutative k -group satisfying

- a) $F_H = 0$
- b) V_H is nilpotent

Let $\psi: H \rightarrow \mathbb{W}_{n+1}$ have components ψ_0, \dots, ψ_n and assume $\psi_n|_{\text{Inf}^1(H)} = 0$. Then $\psi = 0$.

Proof. We use induction on the index of nilpotency of V_H . If $V_H = 0$, then ψ factors through $\mathbb{G}_a \xrightarrow{T} \mathbb{W}_{n+1}$ and we may view ψ as a homomorphism $H \xrightarrow{\psi} \mathbb{G}_a$. The assumption $\psi|_{\text{Inf}^1(H)} = 0$ implies $\text{Lie}(\psi) = 0$ and the result follows from [8, II, §7, 4.3(b)].

Assume the result for groups killed by V^m and that V^{m+1} kills H . Consider the exact sequence (which defines K):

$$0 \rightarrow K \rightarrow H \xrightarrow{V^m} H(p^{-m}).$$

Because V^{m+1} kills H , $V_{H/K} = 0$. The induction assumption tells us that ψ factors as

$$\begin{array}{ccc} H & \xrightarrow{\quad} & H/K \\ \downarrow \psi & & \searrow \psi \\ \mathbb{W}_{n+1} & & \end{array}$$

To conclude we must show $\underline{\text{Lie}}(\tilde{\psi}) = 0$. Because $F_H = 0$,

$V_{K^*}, V_{H^*}, V_{(H/K)^*}$ are all zero. The exact sequence

$$0 \rightarrow (H/K)^* \rightarrow H^* \rightarrow K^* \rightarrow 0$$

gives rise to an exact sequence of Dieudonné modules

$$\begin{array}{ccccccc} 0 & \rightarrow & D^*(K^*) & \longrightarrow & D^*(H^*) & \longrightarrow & D^*((H/K)^*) \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ 0 & \rightarrow & \underline{\text{Lie}}(K) & \longrightarrow & \underline{\text{Lie}}(H) & \longrightarrow & \underline{\text{Lie}}(H/K) \longrightarrow 0 \end{array}$$

But $\psi|_{\text{Inf}^1(H)} = 0$ implies $\underline{\text{Lie}}(\psi) = 0$ and since $\underline{\text{Lie}}(H)$ maps onto $\underline{\text{Lie}}(H/K)$ it follows that $\underline{\text{Lie}}(\tilde{\psi}) = 0$.

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