

CHAPTER ONE

EXPLICIT CONSTRUCTIONS OF UNIVERSAL EXTENSIONS

§1. GROUP SCHEMES AND THEIR RELATIONS TO VECTOR GROUPS

By group scheme over  $S$  we shall mean commutative flat separated group scheme locally of finite presentation over  $S$ .

If  $Q$  is any quasi-coherent  $\mathcal{O}_S$ -module, we may regard  $Q$  as a sheaf for the fppf site by the rule:

$$Q(S') = \Gamma(S', \varphi^*Q)$$

where  $\varphi: S' \rightarrow S$  is the structural morphism. If  $L$  is a locally free  $\mathcal{O}_S$ -module of finite rank, then  $L$ , regarded as a sheaf for the fppf site over  $S$ , is representable by a group scheme which is locally isomorphic to a finite product of  $G_a$ 's. Call such a group scheme  $L$  a vector group over  $S$ .

Fix a group scheme  $G$  over  $S$  and consider the following two universal problems:

Problem A (Universal homomorphism problem):

Vector group hull of  $G$ :

Find a mapping

$$\alpha: G \rightarrow V$$

to a vector group over  $S$ , which is universal for mappings of  $G$ , to vector groups, in the following sense:

The induced mapping

$$\alpha: \text{Hom}_{\mathcal{O}_S}(V, M) \rightarrow \text{Hom}_S(G, M)$$

is an isomorphism for all vector groups  $M$  over  $S$ . If such a  $V$  can be found, call it the vector-group hull of  $G$ .

Quasi-coherent hull of  $G$ :

Find a mapping

$$\alpha: G \rightarrow Q$$

where  $Q$  is a quasi-coherent sheaf, universal for mappings of  $G$  to quasi-coherent sheaves over  $S$ .

Problem B (Universal extension problem):

Assuming  $\text{Hom}(G, V) = (0)$  for all vector groups  $V$ ; find an extension of group schemes over  $S$ :

$$(e) \quad 0 \rightarrow V(G) \rightarrow E(G) \rightarrow G \rightarrow 0$$

such that  $V(G)$  is a vector group, and such that  $(e)$  is universal for all extensions of  $G$  by vector groups over  $S$ .

More precisely, we would like the mapping

$$\text{Hom}_{\mathcal{O}_S}(V(G), M) \rightarrow \text{Ext}_S^1(G, M)$$

induced by  $(e)$  to be an isomorphism. If such an  $(e)$  can be found call it the universal extension of  $G$ . Clearly  $(e)$  and  $E(G)$  and  $V(G)$  are determined up to canonical isomorphism by their role in problem B, and they are functors on the subcategory of group schemes admitting a solution to problem B.

Examples and discussion: (I. Existence of Solution to Problem A)

(1.1). Suppose  $\underline{\text{Hom}}(G, G_a)$  is a locally free  $\mathcal{O}_S$ -module of

finite rank. Set  $V = \underline{\text{Hom}}_{\mathcal{O}_S}(\underline{\text{Hom}}(G, G_a), \mathcal{O}_S)$ . Then

$$\begin{aligned} \underline{\text{Hom}}(G, M) &= \underline{\text{Hom}}(G, G_a) \otimes_{\mathcal{O}_S} M \\ &= \underline{\text{Hom}}_{\mathcal{O}_S}(V, M) \end{aligned}$$

and consequently  $V$  is the vector group hull of  $G$ .

(1.2). Suppose that the Cartier dual of  $G$  is representable by a group scheme. By the Cartier dual we mean the presheaf on  $\text{Sch}/S$  given by

$$G^* = \underline{\text{Hom}}_{\text{gr}}(G, G_m).$$

Then if

$$S \xleftarrow{e} G^*$$

denotes the zero-section of  $G^*/S$ , let

$$S \xleftarrow{e_1} G_1^* = \text{Inf}^1(G^*)$$

denote the first infinitesimal neighborhood of the zero-section.

The commutative diagram

$$\begin{array}{ccc} & & G_1^* \\ & \nearrow & \downarrow \\ S & & G^* \\ & \searrow & \end{array}$$

is a morphism of  $S$ -pointed  $S$ -schemes.

There is a natural isomorphism of functors on the category  $\text{Sch}/S$

$$\underline{\text{Hom}}_{S\text{-pointed } S\text{-schemes}}(G_1^*, G_m) \cong e^* \Omega_{G^*/S}^1$$

and we shall use the notation  $\underline{\omega}_{G^*}$  to denote the quasi-coherent

sheaf over  $S$  defined by either side of the above formula.

We have a natural isomorphism

$$G_1^* \cong \text{Spec}(\mathcal{O}_S \oplus \underline{u}_{G^*}).$$

Let  $\alpha: G \rightarrow \underline{u}_{G^*}$  denote the composition

$$\alpha: G \rightarrow \underline{\text{Hom}}(G^*, G_m) \rightarrow \underline{\text{Hom}}_{S\text{-pointed } S\text{-schemes}}(G_1^*, G_m) = \underline{u}_{G^*}$$

(1.3) Examples of groups  $G$  such that  $G^*$  is representable are the following:

- $G$  finite and locally-free
- $G$  locally constant for the f.p.q.c. topology  
[11, SGA<sub>3</sub> X 5.3]
- $G$  of multiplicative type and quasi-isotrivial  
[11, SGA<sub>3</sub> X 5.7]
- $G$  an abelian scheme (here  $G^* = (0)$  since denoting by  $\pi: G \rightarrow S$  the structural morphism,  $\pi_*(\mathcal{O}_G) = \mathcal{O}_S$  universally)
- $G = Z[T]$  where  $T$  is a finite, locally-free  $S$ -scheme. (i.e. for variable  $S'$  over  $S$ ,  $\Gamma(S', G)$  is the free- $Z$ -module on  $[\text{Hom}_S(S', T)]$ ).

The only example which requires (perhaps) any justification is e). But here  $\underline{\text{Hom}}_{\text{gr}}(G, G_m) \cong \text{Hom}(T, G_m)$  and hence its  $S'$ -valued points are simply the units in  $\Gamma(T \times_S S', \mathcal{O}_{T \times_S S'})$ . The representability follows now because we can (locally) choose a finite basis for the  $\mathcal{O}_S$ -module,  $\mathcal{O}_T$ , and a unit is a section such that multiplication by it defines an automorphism.

(1.4) Proposition: Let  $G$  be any abelian presheaf on  $\text{Sch}/S$  such that  $G^*$  is representable. The functor on quasi-coherent  $\mathcal{O}_S$ -modules  $M \rightarrow \underline{\text{Hom}}_{\text{gr}}(G, M)$  is representable by  $\underline{u}_{G^*}$  and the homomorphism  $\alpha: G \rightarrow \underline{u}_{G^*}$  above is the universal homomorphism from  $G$  to quasi-coherent  $\mathcal{O}_S$ -modules.

Proof: Let us first show that the functor is representable. For  $M$  a quasi-coherent  $\mathcal{O}_S$ -module let  $S_M$  be the affine  $S$ -scheme  $\text{Spec}(\mathcal{O}_S \oplus M)$ , where  $\mathcal{O}_S \oplus M$  is made into an algebra by requiring  $M^2 = (0)$  (i.e., it is the "dual numbers" on  $M$ ). Denote by  $\pi_M$  (resp.  $\eta_M$ ) the structural morphism (resp. unit section) of  $S_M$  which corresponds to the algebra homomorphisms  $\mathcal{O}_S \rightarrow \mathcal{O}_S \oplus M$  (resp.  $\mathcal{O}_S \oplus M \rightarrow \mathcal{O}_S$ ,  $M$  being mapped to zero).

There is an obvious homomorphism  $\pi_{M^*}(G_{m_{S_M}}) \rightarrow G_{m_S}$  which arises functorially because  $\text{id}_{(\widehat{\text{Sch}}/S)} = \eta_M^* \circ \pi_M^*$  and because there is a map  $\pi_{M^*} \rightarrow \eta_M^*$  since  $\pi_M \circ \eta_M = \text{id}_S$ . The kernel of the map is  $M$  and using the definition of  $G_m$  we see that there is an exact sequence:

$$0 \rightarrow M \rightarrow \pi_{M^*}(G_{m_{S_M}}) \rightarrow G_{m_S} \rightarrow 0$$

$$\text{Thus } \underline{\text{Hom}}_{\text{gr}}(G, M) \cong \text{Ker}[\underline{\text{Hom}}_{\text{gr}}(G, \pi_{M^*}(G_{m_{S_M}})) \rightarrow \underline{\text{Hom}}_{\text{gr}}(G, G_{m_S})]$$

$$\cong \text{Ker}[\underline{\text{Hom}}_{\text{gr}}(\pi_M^*(G), G_{m_{S_M}}) \rightarrow \underline{\text{Hom}}_{\text{gr}}(G, G_{m_S})]$$

$$= \text{dfn. Ker}[\Gamma(S_M, G^*) \rightarrow \Gamma(S, G^*)]$$

$$\cong \text{Ker}[\Gamma(S_M, G^*_{S_M}) \rightarrow \Gamma(S, \eta_{M^*}(G^*))]$$

$$\cong \underline{\text{Hom}}_{\mathcal{O}_{S_M}}(\underline{u}_{G^*} \otimes \mathcal{O}_{S_M}, M), \text{ by}$$

[8, II §4, 3.5] since  $G^*$  is representable. Finally, by adjointness we have

$$\text{Hom}_{\mathcal{O}_{S_M}}(\underline{w}_{G^*} \otimes_{\mathcal{O}_{S_M}} M) \cong \text{Hom}_{\mathcal{O}_S}(\underline{w}_{G^*}, M).$$

Because all of the above isomorphisms are functorial in the quasi-coherent module  $M$ , it follows that  $\underline{w}_{G^*}$  represents the functor  $M \mapsto \text{Hom}_{\text{gr}}(G, M)$ .

To calculate what the universal map  $G \rightarrow \underline{w}_{G^*}$  is, let us first observe that for  $M = \underline{w}_{G^*}$ ,  $S_M$  is the first infinitesimal neighborhood of the unit section of  $G^*, \text{Inf}^1(G^*)$ . From the explicit definition of the mapping

$$\text{Hom}_{\mathcal{O}_{\text{Inf}^1(G^*)}}(\underline{w}_{G^*} \otimes_{\mathcal{O}_{\text{Inf}^1(G^*)}} \underline{w}_{G^*}) \rightarrow \text{Ker}[\Gamma(\text{Inf}^1 G^*, G^*) \rightarrow \Gamma(S, G^*)]$$

given in [8, II 4, 3.2] it follows that  $\text{id}_{\underline{w}_{G^*}} \in \text{Hom}_{\mathcal{O}_S}(\underline{w}_{G^*}, \underline{w}_{G^*})$  corresponds to the inclusion  $\text{Inf}^1(G^*) \hookrightarrow G^*$ .

From this point on, the remainder of the proof of the proposition is entirely formal. Recall that  $G^* = \text{dfn Hom}_{\text{gr}}(G, G_m)$  and hence there is a **tautological** pairing  $G \times G^* \rightarrow G_m$  which defines two group homomorphisms  $G_{G^*} \rightarrow G_{m_{G^*}}$  and  $G^*_G \rightarrow G_{m_G}$ , the knowledge of which allows us to reconstruct the pairing. The homomorphism  $G_{G^*} \rightarrow G_{m_{G^*}}$  is (by very definition of  $G^*$ ) universal in an obvious sense. Thus the morphism  $\text{Inf}^1(G^*) \rightarrow G^*$  defines a homomorphism  $G_{\text{Inf}^1(G^*)} \rightarrow G_{m_{\text{Inf}^1(G^*)}}$  as well as a morphism  $\text{Inf}^1 G^*_G \rightarrow G_{m_G}$ . In particular for any  $S$ -scheme  $T$  and point  $\xi \in G(T)$  we obtain a morphism  $\text{Inf}^1 G^*_T \rightarrow G_{m_T}$  which is simply the restriction of the map  $G^*_T \rightarrow G_{m_T}$  to  $\text{Inf}^1 G^*_T$ . This element is  $\alpha(\xi)$  and hence the proposition is proved since the two ways of obtaining a map  $T \times \text{Inf}^1(G^*) \rightarrow T \times G_m$ :

- a) viewing  $G_{\text{Inf}^1(G^*)} \rightarrow G_{m_{\text{Inf}^1(G^*)}}$  as giving for  $\xi \in G(T)$  a map  $\text{Inf}^1(G^*)_T \rightarrow G_{m_T}$

- b)  $\text{Inf}^1 G^* \rightarrow G_{m_T}$  as the restriction of  $G^*_T \rightarrow G_{m_T}$  both come from restricting the map  $G \times G^* \rightarrow G_m$  to  $T \times \text{Inf}^1(G^*) \rightarrow G \times G^*$ .

(1.5) Corollary: For  $G$  an abelian scheme and  $M$  quasi-coherent,  $\text{Hom}_{\text{gr}}(G, M) = (0)$ .

Proof: In this case  $G^* = (0)$  by 1.2(d) and hence  $\underline{w}_{G^*} = (0)$ .

(1.6) A given group scheme  $G/S$  may have a vector group hull and a quasi-coherent hull which differ. Consider  $S = \text{Spec}(\mathbb{Z}_p)$  and  $G = \mathbb{Z}/p$ . Its vector group hull is zero, whereas its quasi-coherent hull, by the previous proposition, is  $\underline{w}_{\mathbb{Z}/p}$ .

A related issue is the question of commutation with base change. The quasi-coherent hull, constructed by the previous proposition commutes with all base changes, whereas the vector group hull constructed in (1.1) does not.

## 2. EXISTENCE OF SOLUTIONS TO PROBLEM B

(1.7) Suppose that

- (a)  $\underline{\text{Hom}}(G, G_a) = 0$   
 (b)  $\underline{\text{Ext}}(G, G_a)$  is a locally free  $\mathcal{O}_S$ -module of finite rank

as sheaves for the Zariski topology over  $S$ . Set

$$V(G) = \underline{\text{Hom}}_{\mathcal{O}_S}(\underline{\text{Ext}}(G, G_a), \mathcal{O}_S)$$

Then a universal extension of  $G$  exists with the above  $V(G)$  as vector group.

This assertion follows easily from the evident

$$\underline{\text{Ext}}(G, M) = \underline{\text{Ext}}(G, G_a) \otimes_{\mathcal{O}_S} M$$

where  $M$  is any locally free  $\mathcal{O}_S$ -module of finite rank.

There are three important cases where hypotheses (a) and (b) hold:

(1.8) Barsotti-Tate groups over bases  $S$  such that  $p$  is nilpotent on  $S$ .

If  $G$  is a Barsotti-Tate group (i.e. a  $p$ -divisible group) over such an  $S$ , let  $G^*$  denote the Cartier dual of  $G$ , and let  $G(n)$  be the kernel of multiplication by  $p^n$ . If  $n$  is sufficiently large so that  $p^n = 0$  on  $S$ , then  $\underline{w}_{G^*} = \underline{w}_{G^*(n)}$  is locally free of finite rank over  $S$  and the argument of (16 IV, 1) shows that  $\underline{\text{Ext}}(G, G_a)$  is  $\underline{\text{Hom}}_{\mathcal{O}_S}(\underline{w}_{G^*}, \mathcal{O}_S)$ . Therefore the hypotheses (a) and (b) above hold. The construction given shows more. Namely, there is the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & G(n) & \rightarrow & G & \xrightarrow{p^n} & G \rightarrow 0 \\ & & \alpha \downarrow & & \downarrow & & \downarrow = \\ 0 & \rightarrow & \underline{w}_{G^*} & \rightarrow & E(G) & \rightarrow & G \rightarrow 0 \end{array}$$

where the vertical map  $\alpha$  is the vector group hull of  $G(n)$ . This construction clearly commutes with all base changes.

(1.9) Abelian schemes over any base  $S$ .

If  $G$  is an abelian scheme over  $S$  of dimension  $d$ , it satisfies the following hypotheses for all  $S'/S$ :

a) Any morphism of sheaves of sets over  $S'$

$$\varphi: G_{S'} \rightarrow Q_{S'}$$

to any quasi-coherent sheaf  $Q$  over  $S'$  is a constant map.

Explicitly,  $\varphi$  admits a factorization:

$$\begin{array}{ccc} G_{S'} & \xrightarrow{\quad} & Q_{S'} \\ f_{S'} \searrow & & \nearrow \\ & S' & \end{array} \quad \text{section}$$

$$(b) \quad f_* \mathcal{O}_{G_{S'}} = \mathcal{O}_{S'}$$

$$(c) \quad R^1 f_* \mathcal{O}_{G_{S'}} = R^1 f_* \mathcal{O}_G \otimes \mathcal{O}_{S'}, \text{ is locally free of rank } d.$$

Here is a proof of a):

Lemma: Let  $f: A \rightarrow S$  be an abelian scheme and  $M$  a quasi-coherent  $\mathcal{O}_S$ -module. Any map  $A \rightarrow M$  is constant.

Proof: A map  $A \rightarrow M$ , may be viewed as an element of  $\Gamma(A, f^*(M)) = \Gamma(S, f_* f^*(M))$ . The map  $\Gamma(S, M) \rightarrow \Gamma(S, f_* f^*(M))$  corresponds then to  $\eta: S \rightarrow M \mapsto \eta \circ f: A \rightarrow M$ . Thus to conclude it suffices to show the map  $\Gamma(S, M) \rightarrow \Gamma(S, f_* f^*(M))$  is bijective. Let us form the cartesian square:

$$\begin{array}{ccc} A & \xleftarrow{\quad} & A_{S[M]} \\ f \downarrow & & \downarrow f_M \\ S & \xleftarrow{\quad} & S[M] \end{array}$$

$$\begin{aligned} \text{Then } \Gamma(S, \mathcal{O}_S) \oplus \Gamma(S, M) &= \Gamma(S[M], \mathcal{O}_{S[M]}) \simeq \Gamma(S[M], f_{M*}(\mathcal{O}_{A_{S[M]}})) \\ &= \Gamma(A_{S[M]}, \mathcal{O}_{A_{S[M]}}) = \Gamma(A, \mathcal{O}_A) \oplus \Gamma(A, f^*(M)) \\ &= \Gamma(A, \mathcal{O}_A) \oplus \Gamma(S, f_* f^*(M)) \end{aligned}$$

since (b)  $f_*(\mathcal{O}_A) = \mathcal{O}_S$ , universally.

Let

$$w = \underline{\text{Hom}}_{\mathcal{O}_S}(R^1 f_* \mathcal{O}_G, \mathcal{O}_S).$$

(1.10) Proposition: If  $G$  satisfies the above hypotheses

(a), (b), (c), then  $G$  possesses a universal extension,

$$0 \rightarrow \omega \rightarrow E(G) \rightarrow G \rightarrow 0$$

which is indeed universal for all extensions of  $G$  by quasi-coherent sheaves. (We assume  $G \rightarrow S$  is quasi-compact).

Proof. (After Rosenlicht, and Serre, [22,25])

Let  $M$  denote a quasi-coherent sheaf. By our assumptions (notably a) the category of extensions,  $\text{EXT}(G, M)$  is rigid.

Thus, the presheaf for the flat topology

$$S' \mapsto \text{Ext}^1(G_{S'}, M_{S'})$$

is a sheaf.

We shall show that the composition

$$\lambda: \text{Ext}^1(G, M) \rightarrow H^1(G, f^*M) \rightarrow \Gamma(S, R^1f_*f^*M).$$

is an isomorphism. But by the above remark, we may assume  $S$  affine.

$\lambda$  is injective:

For let  $E$  be an extension of  $G$  by  $M$  and assume  $\varphi: G \rightarrow E$  is a section (as sheaves of sets). By subtracting  $\varphi(0)$  we may suppose that  $\varphi(0) = 0$ . The map  $G \times G \rightarrow E$  which expresses the obstruction to  $\varphi$  being a homomorphism actually maps  $G \times G$  into  $M$  and brings  $0$  to  $0$ . After hypothesis (a), one may see that this obstruction is zero.

$\lambda$  is surjective:

Let  $E$  be a principal homogeneous space for  $M$  over the base  $G$ . Since  $S$  is affine,  $E$  admits a section  $e$  lying over the zero-section of  $G$ . We now follow Serre's prescription for imposing a group structure on  $E$  with zero-section  $e$ , which establishes  $E$  as a group extension

$$0 \rightarrow M \rightarrow E \rightarrow G \rightarrow 0$$

[25, VII, 15]. To follow out this prescription one need only know that the cohomology class in  $H^1(G, f^*M)$  representing the principal homogeneous space  $E$  is primitive. But  $H^1(G, f^*M)$  consists entirely of primitive elements as follows from the Kunneth formula if  $G$  is an abelian scheme and [21 bis, III, 4.2] in general.

Our plan is to establish the isomorphism

$$\text{Hom}_{\mathcal{O}_S}(\omega, M) \cong \text{Ext}^1(G, M)$$

and, consequently, representability of the functor

$$M \mapsto \text{Ext}^1(G, M)$$

We do this by demonstrating these isomorphisms:

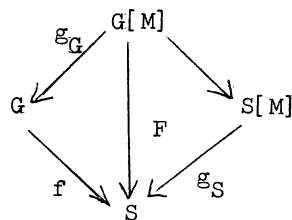
$$\Gamma(S, R^1f_*f^*M) \cong \Gamma(S, R^1f_*\mathcal{O}_G \otimes M) \cong \text{Hom}_{\mathcal{O}_S}(\omega, M)$$

To establish the first isomorphism above, we need a lemma:

(1.11) Lemma:  $R^1f_*\mathcal{O}_G \otimes M \rightarrow R^1f_*f^*M$  is an isomorphism, for  $M$  any quasi-coherent  $\mathcal{O}_S$ -module.

(N.B. This follows from (c) but the following proof is valid whenever  $R^1f_*\mathcal{O}_G$  is a flat  $\mathcal{O}_S$ -module).

Proof. We shall force the Kunnetth theorem (10, EGA<sub>III</sub>, 6.7.8) to yield this result, resorting to a technical trick. Let  $S[M]$  denote the scheme, affine over  $S$ , whose underlying space is  $S$ , and whose structural sheaf is  $\mathcal{O}_S \oplus M$ , taken to be a ring in the obvious say. Form the diagram,



and note that  $R^1 f_* (\mathcal{O}_{G[M]}) = R^1 f_* \mathcal{O}_G \oplus R^1 f_* M$ , using that  $g_G$  is affine.

But, by Kunnetth,

$$R^1 f_* (\mathcal{O}_{G[M]}) = R^1 f_* \mathcal{O}_G \oplus (\mathcal{O}_S \oplus M)$$

using that  $g_S$  is affine.

The lemma follows, and so does (1.10).

(1.12) If  $A$  is an abelian scheme over the base  $S$ , where  $p$  is nilpotent on  $S$ , let  $G$  denote the  $p$ -divisible group associated to  $A$  over  $S$ . It is an easy exercise to see the pullback to  $G$  of the universal extension of  $A$  over  $S$  is the universal extension of  $G$  over  $S$ . More explicitly, consider the map

$$\lambda: \omega_{G*} \rightarrow \omega_{A*}$$

which determines the pullback to  $G$  of the universal extension of  $A$ . This map  $\lambda$  is easily seen to be the natural isomorphism.

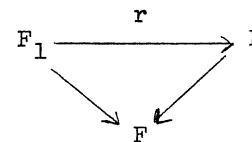
§2. RIGIDIFICATION OF HOM AND EXT

(2.1) Fix an  $S$ -group scheme  $G$  and an exact sequence (of fppf sheaves of abelian groups over  $S$ )

$$(\epsilon) \quad 0 \rightarrow G \rightarrow E \rightarrow F \rightarrow 0$$

Let  $F_1 = \text{Inf}_S^1(F) \subset F$  denote the first infinitesimal neighborhood of the zero section of  $F$  over  $S$ . Regard  $F_1$  as an  $S$ -pointed sheaf.

By definition a rigidification  $r$  of the extension  $(\epsilon)$  is a homomorphism of  $S$ -pointed  $S$ -schemes making the following commutative diagram:



A rigidified extension of  $F$  by  $G$  is a pair consisting in an extension  $(\epsilon)$  together with a rigidification of it,  $r$ .

If  $H$  is an  $S$ -group scheme, an  $(\epsilon)$ -rigidified homomorphism from  $G$  to  $H$  consists in a homomorphism of  $S$ -groups

$$\varphi: G \rightarrow H$$

together with a rigidification  $r$  of the induced (pushout) exact sequence  $(\varphi_* \epsilon)$ .

If

$$(\epsilon) \quad 0 \rightarrow G \rightarrow E \rightarrow F \rightarrow 0$$

$$(\epsilon') \quad 0 \rightarrow G \rightarrow E' \rightarrow F \rightarrow 0$$

are two extensions, provided with rigidifications  $r, r'$

respectively, then on the Baer sum  $(\bar{\epsilon})$  of  $(\epsilon)$  and  $(\epsilon')$  there is a natural rigidification  $\bar{r}$ , which we shall call the Baer sum of the rigidifications  $r$  and  $r'$ . This is obtained from the natural rigidification on the external product:

$$\begin{array}{ccccccc}
 (\epsilon \times \epsilon'): & 0 & \rightarrow & G \times G & \rightarrow & E \times E' & \xrightarrow{\quad} & F \times F' & \rightarrow & 0 \\
 & & & & & \uparrow & \swarrow & \nearrow & & \\
 & & & & & \bar{r} & & r \times r' & & \\
 & & & & & (F \times F)_1 & \xrightarrow{\quad} & F_1 \times F_1 & & 
 \end{array}$$

Denote by  $\text{Extrig}(F,G)$  the set of isomorphism classes of rigidified extensions of  $F$  by  $G$ . Denote by  $(\epsilon)\text{-Homrig}(G,H)$  the set of isomorphism classes of  $(\epsilon)$ -rigidified homomorphisms from  $G$  to  $H$ . One checks easily that Baer sum induces an abelian group structure on  $\text{Extrig}(F,G)$  and on  $(\epsilon)\text{-Homrig}(G,H)$ .  $\text{Extrig}(F,G)$  is bifunctorial in  $F$  and  $G$ . As for  $(\epsilon)\text{-Homrig}(G,H)$ , it is functorial in  $H$ , and if  $\varphi: G \rightarrow G'$  is a homomorphism of  $S$ -groups, one gets a natural homomorphism

$$(\varphi_* \epsilon)\text{-Homrig}(G',H) \rightarrow (\epsilon)\text{-Homrig}(G,H).$$

There are two objects of this section:

To express the universal extension of a Barsotti-Tate group (over a base  $S$  on which  $p$  is locally nilpotent) as a direct limit of  $\epsilon\text{-Homrig}$ 's (2.5.7).

To express the universal extension of an abelian scheme as an  $\text{Extrig}$  (2.6.7).

(2.2) Let us consider the special case where  $(\epsilon)$  is an exact sequence of finite locally-free groups and where  $H = G_m$ . Furthermore let us assume that the base scheme,  $S$ , is affine.

(2.2.1) Proposition There is an exact sequence of abelian groups:

$$(2.2.2) \quad 0 \rightarrow \Gamma(S, \underline{w}_F) \rightarrow (\epsilon)\text{-Homrig}(G, G_m) \rightarrow \Gamma(S, G^*) \rightarrow 0$$

Proof: The map  $(\epsilon)\text{-Homrig}(G, G_m) \rightarrow \Gamma(S, G^*)$  is defined by forgetting the rigidification  $r$  of the rigidified homomorphism  $(\varphi, r)$ . Given  $\varphi: G \rightarrow G_m$ , consider the corresponding extension

$$(\epsilon_\varphi) \quad 0 \rightarrow G_m \rightarrow G_m \overset{G}{\parallel} E \rightarrow F \rightarrow 0$$

It makes  $G_m \overset{G}{\parallel} E$  a principal homogeneous space over  $F$  under the group  $G_m$ . Thus by descent [11,10; S.G.A<sub>1</sub> XI 4.3, EGA<sub>IV</sub> 17.7.3]  $G_m \overset{G}{\parallel} E$  is a smooth  $F$ -scheme. Viewing  $F_1$  as an  $F$ -scheme via the inclusion  $F_1 \subset F$  we view  $S$  as an  $F$ -scheme defined by the vanishing of an ideal of square zero: namely  $\underline{w}_F$ . Because  $G_m \overset{G}{\parallel} E$  is smooth over  $F$ , the identity section can be lifted so as to obtain a commutative diagram:

$$\begin{array}{ccccccc}
 0 & \rightarrow & G_m & \rightarrow & G_m \overset{G}{\parallel} E & \rightarrow & F & \rightarrow & 0 \\
 & & & & \uparrow & \nearrow & \uparrow & & \\
 & & & & & & F_1 & & \\
 & & & & & & \uparrow & & \\
 & & & & & & S & & 
 \end{array}$$

(2.2.3)

This shows the map  $(\epsilon)\text{-Homrig}(G, G_m) \rightarrow \Gamma(S, G^*)$  is surjective.

By definition the kernel of this map consists of pairs  $(0, \tau)$  where  $\tau$  is a rigidification on the trivial extension:

$$0 \rightarrow G_m \rightarrow G_m \times F \rightarrow F \rightarrow 0.$$

But to give a morphism of  $S$ -pointed  $S$ -schemes,  $F_1 \rightarrow G_m \times F$ , which projects to the inclusion  $F_1 \subset F$ , is equivalent to giving a

morphism of  $S$ -pointed  $S$ -scheme,  $F_1 \rightarrow G_m$ , which is the same as giving an element in  $\Gamma(S, \underline{w}_F)$ .

Since it is clear that the map  $\Gamma(S, \underline{w}_F) \rightarrow (\epsilon)\text{-Homrig}(G, G_m)$  defined by the above is additive, the proof of the proposition is complete.

(2.3) Let  $(\epsilon)\text{-Homrig}(G, G_m)$  denote the sheaf associated to the ZARISKI presheaf whose value on an  $S$ -scheme  $S'$  is  $(\epsilon_{S'})\text{-Homrig}(G_{S'}, G_{m_{S'}})$ . Then without any hypothesis on the scheme  $S$  we have the following corollary:

(2.3.1) Corollary: There is an exact sequence of ZARISKI (resp. f.p.p.f.,...) sheaves on  $S$ :

$$0 \rightarrow \underline{w}_F \rightarrow (\epsilon)\text{-Homrig}(G, G_m) \rightarrow G^* \rightarrow 0$$

In particular  $(\epsilon)\text{-Homrig}(G, G_m)$  is a commutative flat  $S$ -group, provided  $\underline{w}_F$  is finite, locally-free.

(2.4) Let  $(\epsilon) 0 \rightarrow G \rightarrow E \rightarrow F \rightarrow 0$  be an exact sequence of finite, locally-free  $S$ -groups. The next proposition is the basic result relating  $(\epsilon)$ -rigidified homomorphisms to the construction given in (1.4) above. It and its analogue for abelian schemes given below in (2.6) are the basic results which will allow us to obtain an explicit description of the universal extension of a Barsotti-Tate group (resp. an abelian scheme).

(2.4.1) Proposition. There is a canonical and functorial homomorphism of groups  $E^* \rightarrow (\epsilon)\text{-Homrig}(G, G_m)$ , which will be explicitly constructed in the proof, rendering the following

diagram commutative:

$$(2.4.2) \quad \begin{array}{ccccccc} 0 & \rightarrow & F^* & \rightarrow & E^* & \longrightarrow & G^* \rightarrow 0 \\ & & \downarrow -\alpha & & \downarrow & & \parallel \\ 0 & \rightarrow & \underline{w}_F & \rightarrow & (\epsilon)\text{-Homrig}(G, G_m) & \rightarrow & G^* \rightarrow 0 \end{array}$$

Proof:  $(\epsilon)\text{-Homrig}(G, G_m)$  is the sheaf associated to the pre-sheaf  $S' \mapsto (\epsilon_{S'})\text{-Homrig}(G_{S'}, G_{m_{S'}})$ . Thus it suffices to construct a mapping on the level of presheaves, and since every "object" occurring in (2.4.2) commutes with base change it suffices to construct the map  $\Gamma(S, E^*) \rightarrow (\epsilon)\text{-Homrig}(G, G_m)$ . Let  $\phi : E \rightarrow G_m$  be an element in  $\Gamma(S, E^*)$ . Because we require the right hand square of (2.4.2) to commute we must assign to  $\phi$  a pair  $(\phi|G, r)$  where  $r$  is a rigidification of the extension  $(\phi|G)_*(\epsilon)$ . That is we must define  $r$ , a morphism of pointed  $S$ -schemes, which renders the following diagram commutative:

$$(2.4.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & G & \rightarrow & E & \xrightarrow{\pi} & F \rightarrow 0 \\ & & \downarrow \phi|G & & \downarrow \phi & & \parallel \\ & & G_m & \rightarrow & G_m \amalg E & \xrightarrow{\tilde{\pi}} & F \rightarrow 0 \\ & & & & & & \downarrow J \\ & & & & & & F_1 \end{array}$$

$r$  (dashed arrow from  $F_1$  to  $G_m \amalg E$ )

Using  $\phi$  we obtain a splitting,  $G_m \amalg E \xrightarrow{\tilde{\pi}} G_m$ , of the lower horizontal line of (2.4.3). Composing the "trivial" rigidification  $r_0 : F_1 \hookrightarrow F \hookrightarrow G_m \times F$  with  $(\tilde{\phi}, \tilde{\pi})^{-1} : G_m \times F \rightarrow G_m \amalg E$  we obtain the desired rigidification  $r$ .

It remains to show that the left hand square of (2.4.2) is

commutative. Thus let  $\psi: F \rightarrow G_m$  be given so that the diagram corresponding to (2.4.3) is:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \longrightarrow & E & \xrightarrow{\pi} & F \longrightarrow 0 \\ & & \downarrow \psi \circ \pi & & \downarrow \pi & & \parallel \\ 0 & \longrightarrow & G_m & \longrightarrow & G_m & \longrightarrow & E \longrightarrow F \longrightarrow 0 \end{array}$$

Identifying  $G_m \underset{G}{\parallel} E$  with  $G_m \times F$ , then  $(\psi \circ \pi, \pi)$  is identified with the automorphism of  $G_m \times F$  taking  $(x, f)$  to  $(x + \psi(f), f)$ . This shows that to  $\psi \circ \pi$  the pair  $(0, -\psi|_{F_1})$  is assigned. By definition of  $\alpha$  and of the map  $\underline{w}_F \rightarrow (\epsilon)\text{-Homrig}(G, G_m)$  it follows that the diagram commutes. Finally the fact that the map  $E^* \rightarrow (\epsilon)\text{-Homrig}(G, G_m)$ , which has been defined above is a homomorphism of groups, follows directly from the definitions.

(2.5) (The Universal Extension of a Barsotti-Tate group)

Assume that our base  $S$  is killed by  $p^n$  and fix a Barsotti-Tate group  $G$  on  $S$ . For any  $i \geq 1$  let  $(\epsilon_{n,i})$  be the extension:

$$(\epsilon_{n,i}) \quad 0 \rightarrow G(i) \rightarrow G(n+i) \xrightarrow{p^i} G(n) \rightarrow 0$$

By (2.4.1) we obtain a commutative diagrams:

$$(2.5.1_i) \quad \begin{array}{ccccccc} 0 & \longrightarrow & G^*(n) & \longrightarrow & G^*(n+i) & \xrightarrow{p^n} & G^*(i) \longrightarrow 0 \\ & & \downarrow -\alpha & & \downarrow & & \parallel \\ 0 & \longrightarrow & \underline{w}_G(n) & \longrightarrow & (\epsilon_{n,i})\text{-Homrig}(G(i), G_m) & \longrightarrow & G^*(i) \longrightarrow 0 \end{array}$$

From the proof of (2.4.1) and the explicit definition of (2.7) it follows that the following diagrams

$$(2.5.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & G(i+1) & \longrightarrow & G(n+i+1) & \xrightarrow{p^{i+1}} & G(n) \longrightarrow 0 \\ & & \downarrow p & & \downarrow p & & \parallel \\ 0 & \longrightarrow & G(i) & \longrightarrow & G(n+1) & \xrightarrow{p_i} & G(n) \longrightarrow 0 \end{array}$$

give rise to commutative diagrams:

$$(2.5.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & G^*(n) & \longrightarrow & G^*(n+1) & \xrightarrow{p^i} & G^*(i) \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G^*(n) & \longrightarrow & G^*(n+1) & \xrightarrow{p^{i+1}} & G^*(i+1) \longrightarrow 0 \\ & & \downarrow -\alpha & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{w}_G(n) & \longrightarrow & (\epsilon_{n,i})\text{-Homrig} & \longrightarrow & G^*(i) \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{w}_G(n) & \longrightarrow & (\epsilon_{n,i+1})\text{-Homrig} & \longrightarrow & G^*(i+1) \longrightarrow 0 \end{array}$$

Hence passing to the direct limit we find a commutative diagram:

$$(2.5.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & G^*(n) & \longrightarrow & G^* & \xrightarrow{p^n} & G^* \longrightarrow 0 \\ & & \downarrow -\alpha & & \downarrow & & \parallel \\ 0 & \longrightarrow & \underline{w}_G(n) & \longrightarrow & \varinjlim (\epsilon_{n,i})\text{-Homrig} & \longrightarrow & G^* \longrightarrow 0 \end{array}$$

But we know that pushing out the extension  $0 \rightarrow G^*(n) \rightarrow G^* \xrightarrow{p^n} G^* \rightarrow 0$  via  $\alpha$  gives the universal extension of  $G^*$ . Hence there is a canonical isomorphism  $E(G^*) \xrightarrow{\sim} \varinjlim (\epsilon_{n,i})\text{-Homrig}(G(i), G_m)$  which makes the following diagram commute:

$$(2.5.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \underline{w}_G(n) & \longrightarrow & E(G^*) & \longrightarrow & G^* \longrightarrow 0 \\ & & \downarrow -\text{id} & & \downarrow & & \parallel \\ 0 & \longrightarrow & \underline{w}_G(n) & \longrightarrow & \varinjlim (\epsilon_{n,i})\text{-Homrig} & \longrightarrow & G^* \longrightarrow 0 \end{array}$$

Also it follows that the hypotheses that  $p^n$  kills  $S$  can be replaced by the assumption that  $p$  is locally-nilpotent on  $S$ .

To be more precise consider the exact sequences:

$$(\epsilon_i) \quad 0 \rightarrow G(i) \rightarrow G \xrightarrow{p^i} G \rightarrow 0$$

The map of sequence  $(\epsilon_{n,i})$  to  $(\epsilon_i)$  defines the homomorphism  $(\epsilon_i)\text{-Homrig}(G(i), G_m) \rightarrow (\epsilon_{n,i})\text{-Homrig}(G(i), G_m)$ . If  $p^n$  kills  $S$ , then this map is an isomorphism because  $G_1 = \text{Inf}^1(G) \subseteq G(n)$  [16, II 3.3.16]. Thus the map  $(\epsilon_i)\text{-Homrig}(G(i), G_m) \rightarrow G^*(i)$  is an epimorphism since this is a local property on  $S$ . Also the fact that  $G_1$  is affine on  $S$  insures that the map  $\underline{w}_G \rightarrow (\epsilon_i)\text{-Homrig}(G(i), G_m)$  is well-defined and that the sequence:

$$0 \rightarrow \underline{w}_G \rightarrow (\epsilon_i)\text{-Homrig}(G(i), G_m) \rightarrow G^*(i) \rightarrow 0$$

is exact.

Passing to the direct limit we obtain an exact sequence:

$$(2.5.6) \quad 0 \rightarrow \underline{w}_G \rightarrow \varinjlim (\epsilon_i)\text{-Homrig}(G(i), G_m) \rightarrow G^* \rightarrow 0$$

Let  $0 \rightarrow \underline{w}_G \rightarrow E(G^*) \rightarrow G^* \rightarrow 0$  be the universal extension of  $G^*$  by a vector group. Then there is a unique linear map  $\underline{w}_G \rightarrow \underline{w}_G$  giving the extension (2.5.6) by pushing out. By (2.5.5) this map is  $-\text{id}$  locally and hence is  $-\text{id}$ . Finally because of the functoriality of Homrig discussed in (2.1) we can state:

(2.5.7) Proposition. Let  $S$  be a scheme on which  $p$  is locally nilpotent. The two contravariant functors from the category of Barsotti-Tate groups to the category of abelian (f.p.p.f) sheaves on  $S$ :

- a)  $G \mapsto E(G^*)$   
 b)  $G \mapsto \varinjlim (\epsilon_i)\text{-Homrig}(G(i), G_m)$

are canonically isomorphic. Furthermore the natural exact sequence

$$0 \rightarrow \underline{w}_G \rightarrow \varinjlim (\epsilon_i)\text{-Homrig}(G(i), G_m) \rightarrow G^* \rightarrow 0$$

is "the" universal extension of  $G^*$  by a vector group.

(2.6) (The Universal extension of an abelian scheme)

Let  $S$  be a scheme and  $A$  an abelian scheme on  $S$ . Let  $0 \rightarrow G_m \rightarrow E \rightarrow A \rightarrow 0$  be an extension of  $A$  by  $G_m$ . Then  $E$  is representable and the morphism  $E \rightarrow A$  is smooth, so that if  $S$  is affine this extension admits a rigidification. Thus if we denote by Extrig $(A, G_m)$  the ZARISKI sheaf associated to the presheaf  $S' \mapsto \text{Extrig}(A_{S'}, G_{m_{S'}})$  we find (just as in (2.2.1)) an exact sequence:

$$(2.6.1) \quad 0 \rightarrow \underline{w}_A \rightarrow \text{Extrig}(A, G_m) \rightarrow \text{Ext}^1(A, G_m) \rightarrow 0$$

But the dual abelian scheme,  $A^*$ , exists and is isomorphic to  $\text{Ext}^1(A, G_m)$  [21, 19]. From descent it follows that Extrig $(A, G_m)$  is representable and is a smooth  $S$ -group.

(2.6.2) We shall see below that the extension (2.6.1) is the universal extension of  $A^*$  by a vector group. Let us begin with a special case where an explicit isomorphism between the universal extension and the extension (2.6.1) can be given. Thus assume  $p^n$  is zero on  $S$ . Recall then that  $\underline{w}_{A(n)} = \underline{w}_A$  and

and that the universal extension of  $A^*$  by a vector group is obtained as a "pushout" as in the following diagram:

$$(2.6.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & A^*(n) & \rightarrow & A^* & \xrightarrow{p^n} & A^* \rightarrow 0 \\ & & \alpha \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \underline{\psi}_A(n) & \rightarrow & \mathcal{E} & \longrightarrow & A^* \rightarrow 0 \end{array}$$

Our isomorphism is obtained from a homomorphism  $A^* \rightarrow \underline{\text{Extrig}}(A, G_m)$  which renders the diagram obtained by replacing  $\mathcal{E}$  by  $\underline{\text{Extrig}}$  commutative. To define the map it suffices to do so on the level of presheaves, and hence, because everything is compatible with base change, to define a map  $\Gamma(S, A^*) \rightarrow \underline{\text{Extrig}}(A, G_m)$ .

If  $0 \rightarrow G_m \rightarrow E \rightarrow A \rightarrow 0$  is an extension we can pull it back via the homomorphism  $A \xrightarrow{p^n} A$  and obtain a commutative diagram:

$$(2.6.4) \quad \begin{array}{ccccccc} & & & & \text{Inf}^1(A(n)) = A(n)_1 & & \\ & & & & \downarrow & & \\ & & \text{Ker} & \xrightarrow{\sim} & A(n) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & G_m & \rightarrow & E \times_A A & \rightarrow & A \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow p^n \\ 0 & \rightarrow & G_m & \rightarrow & E & \rightarrow & A \rightarrow 0 \end{array}$$

The kernel of the map  $E \times_A A \xrightarrow{\text{pr}_1} E$  is mapped isomorphically under the projection  $\text{pr}_2: E \times_A A \rightarrow A$  to  $A(n)$ . This allows us to find a unique arrow  $A(n)_1 \rightarrow \text{Ker}$  making the diagram commute. Because  $p^n$  kills  $S$ ,  $\text{Inf}^1(A(n)) = \text{Inf}^1(A) = A_1$  and hence composing this arrow with the inclusion  $\text{Ker} \hookrightarrow E \times_A A$  we obtain a rigidified extension of  $A$  by  $G_m$ . This defines the desired homomorphism. It remains to show that the diagram:

$$(2.6.5) \quad \begin{array}{ccccccc} 0 & \rightarrow & A^*(n) & \rightarrow & A^* & \xrightarrow{p^n} & A^* \rightarrow 0 \\ & & \alpha \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \underline{\psi}_A(n) & \rightarrow & \underline{\text{Extrig}} & \longrightarrow & A^* \rightarrow 0 \end{array}$$

is commutative. The right hand square commutes by definition of the morphism  $A^* \rightarrow \underline{\text{Extrig}}(A, G_m)$ . To check the commutativity of the left hand square let the extension

$$(e) \quad 0 \rightarrow G_m \rightarrow E \rightarrow A \rightarrow 0$$

represent an element in  $A^*(n)$ . Then there exists a unique homomorphism  $\phi: E \times_A A \rightarrow G_m$  which splits the extension in the upper row of (2.6.4). It follows from the explicit form of Cartier duality given for example in [18 bis] that the identification of  $A^*(n)$  with  $A(n)^* = \underline{\text{Hom}}(A(n), G_m)$  makes correspond to (e) the homomorphism  $\psi: A(n) \rightarrow G_m$  which is the following composition:

$$(2.6.6) \quad A(n) \xrightarrow{i} E \times_A A \xrightarrow{\phi} G_m$$

Thus going around the left hand square:

$A^*(n) \xrightarrow{\alpha} \underline{\psi}_A(n) \rightarrow \underline{\text{Extrig}}(A, G_m)$  assigns to (e) the trivial extension  $G_m \times A$  together with the rigidification whose components are  $\psi|_{A(n)_1}: A(n)_1 \rightarrow G_m$  and the canonical inclusion  $A(n)_1 \hookrightarrow A$ .

We must check that this extension is isomorphic to the extension given by the upper row of (2.6.4), via an isomorphism respecting the rigidified structures. The unique isomorphism between these two extensions is given by the map  $\tau: E \times_A A \rightarrow G_m \times_A A$

whose components are  $\phi$  and  $\text{pr}_2$ .

For  $x$ , an  $S'$ -valued point of  $A_1$ , the rigidification on  $E \times_A A$  assigns to it the  $S'$ -valued point  $(0, x)$ . Certainly the second component of  $\tau((0, x))$  is  $x$  while the first component is  $\phi((0, x)) = \phi \circ i|_{A_1}(x) = \psi|_{A_1}(x)$ . Thus  $\tau$  is an isomorphism of rigidified extensions and the diagram (2.6.5) commutes as asserted.

(2.6.7) Proposition: Let  $S$  be a scheme,  $A$  an abelian scheme on  $S$  and let  $E(A^*)$  denote the universal extension of  $A^*$  by a vector group. The canonical morphism  $E(A^*) \rightarrow \text{Extrig}(A, G_m)$  (arising from the definition of the universal extension and the extension (2.6.1)) is an isomorphism, which is functorial in  $A$ .

Proof: Observe first that both the universal extension and  $\text{Extrig}(A, G_m)$  are compatible with arbitrary base change. For  $E(A^*)$  this follows from the fact that all objects (= group or map between groups) entering into the proof of its existence in (1.9) are compatible with base change. To show the map  $E(A^*) \rightarrow \text{Extrig}(A, G_m)$  is an isomorphism is equivalent to showing that the map  $\omega_A \rightarrow \underline{\omega}_A$  giving rise to it is an isomorphism. This problem is local on  $S$  and hence  $S$  can be assumed to be affine. Because  $A$  is proper and smooth on  $S$  (hence of finite presentation on  $S$ ) we can assume that  $S = \text{Spec}(R)$  where  $R$  is a ring of finite type over  $\mathbb{Z}$  [10, EGA<sub>IV</sub> 8.9.1, 8.10.5, ...]. From (2.6) it follows that for any maximal ideal  $\underline{m} \subset R$ , the corresponding map  $\omega_A / \underline{m}^n \omega_A \rightarrow \underline{\omega}_A / \underline{m}^n \underline{\omega}_A$  is an isomorphism ( $n \geq 1$ ). Hence the determinant of the corresponding endomorphism of

$\omega_A \otimes \hat{R}_{\underline{m}}$  is a unit in  $\hat{R}_{\underline{m}}$ . This implies that this determinant is actually invertible in  $\hat{R}_{\underline{m}}$ . Because this holds for all maximal ideals  $\underline{m}$ , the endomorphism of  $\omega_A$  is an automorphism.

To check the functoriality of this isomorphism, consider two abelian schemes  $A, B$  on  $S$  and a homomorphism  $u: A \rightarrow B$ . The assertion means that the following diagram is commutative:

(2.6.8)

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \omega_B & \longrightarrow & E(B^*) & \longrightarrow & B^* & \longrightarrow & 0 \\
 & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} & & \\
 0 & \longrightarrow & \omega_B & \longrightarrow & \text{Extrig}(B, G_m) & \longrightarrow & B^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow u^* & & \\
 0 & \longrightarrow & \omega_A & \longrightarrow & E(A^*) & \longrightarrow & A^* & \longrightarrow & 0 \\
 & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} & & \\
 0 & \longrightarrow & \omega_A & \longrightarrow & \text{Extrig}(A, G_m) & \longrightarrow & A^* & \longrightarrow & 0
 \end{array}$$

To check that the two ways of going from  $E(B^*)$  to  $\text{Extrig}(A, G_m)$  coincide, observe that their difference is a map  $E(B^*) \rightarrow \omega_A$  which vanishes on  $\omega_B$  and hence gives a map  $B^* \rightarrow \omega_A$ , necessarily zero by (1.5).

### §3. RIGIDIFIED EXTENSIONS AND $G$ -EXTENSIONS

Let

$$(*) \quad 0 \rightarrow G_m \rightarrow E \rightarrow A \rightarrow 0$$

be an extension over an affine base  $S$ , where  $A/S$  is an abelian scheme.

In this section we will show, in detail, how the following two additional structures on  $(*)$  are equivalent:

- (a) A rigidification of  $(*)$
- (b) An integrable connection on  $E$  regarded as a  $G_m$ -torsor over  $A$  (this connection being required to be compatible with the group structure of the extension  $E$ ).

In this way we shall obtain yet another explicit description of the universal extension of an abelian scheme.

#### (3.1) The definitions.

By torsor for  $G$  over  $S$  we shall mean principal homogeneous space, locally trivial for the étale topology. There are many equivalent ways to define connection and we shall take the definition using the fewest words:

Definition: Let  $X$  be an  $S$ -scheme,  $G$  a commutative smooth  $S$ -group, and  $P$  a torsor on  $X$  under the group  $G_X$ . Let  $\Delta^1(X) = \Delta^1(X/S)$  denote the first infinitesimal neighborhood of the diagonal map  $X \rightarrow X \times_S X$ . The two projections

$p_j: X \times_S X \rightarrow X$  ( $j=1,2$ ) induce morphisms  $p_j: \Delta^1(X) \rightarrow X$ . A connection  $\nabla$  on the  $G_X$ -torsor  $P$  is an isomorphism of  $G_{\Delta^1(X)}$ -torsors:

$$\nabla: p_1^*(P) \rightarrow p_2^*(P)$$

which restricts to the identity on  $X$ . (That is,  $\Delta^*(\nabla) = \text{id}_P$ ).

Given an  $\mathcal{O}_X$ -module  $E$  a connection on  $E$  is an  $\mathcal{O}_{\Delta^1(X)}$  isomorphism  $\nabla: p_1^*(E) \rightarrow p_2^*(E)$  restricting to the identity on  $X$ . Given  $(E, \nabla)$ , an  $\mathcal{O}_X$ -module with connection, we may obtain an  $\mathcal{O}_S$ -linear homomorphism

$$\nabla': E \rightarrow E \otimes \Omega_{X/S}^1$$

(satisfying the Leibniz product rule) as follows:

Denote by  $j_1, j_2$  the two ring homomorphisms  $\mathcal{O}_X \rightarrow \mathcal{O}_{\Delta^1(X)}$  corresponding to the two projections  $p_1, p_2$ . One obtains the corresponding morphisms  $j_1(E): E \rightarrow p_1^*(E)$ ,  $j_2(E): E \rightarrow p_2^*(E)$ .

Define:

$$\nabla' = \nabla^{-1} \circ j_2(E) - j_1(E).$$

#### (3.1.2) Examples

- a) If  $G = G_m$ , then connections on the  $G_m$ -torsor  $P$  are in one-one correspondence with connections on the line bundle,  $\mathcal{L}$ , which is associated to  $P$ .
- b) If  $G = G_a$ , then  $G_a$ -torsors  $P$  correspond to extensions  $(e)$  of  $\mathcal{O}_X$  by  $\mathcal{O}_X$ :

$$(e) \quad 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$$

and connections on  $P$  correspond to isomorphisms of extensions  $p_1^*(e) \xrightarrow{\sim} p_2^*(e)$  which restrict to  $\text{id}_e$  on  $X$ .

(3.1.3) The  $G$ -torsors with connection  $(P, \nabla)$  are the objects of a category in which the morphisms,  $\text{Hom}(P, \nabla), (Q, \bar{\nabla})$  are precisely those morphisms  $\eta: P \rightarrow Q$  of  $G$ -torsors such that the following diagram commutes:

$$\begin{array}{ccc} p_1^*(P) & \xrightarrow{p_1^*(\eta)} & p_1^*(Q) \\ \nabla \downarrow & & \downarrow \bar{\nabla} \\ p_2^*(P) & \xrightarrow{p_2^*(\eta)} & p_2^*(Q) \end{array}$$

Such an  $\eta: P \rightarrow Q$  is said to be horizontal when the connections on  $P$  and  $Q$  are understood as being given.

(3.1.4) (The curvature of a connection). The curvature tensor will be an element in  $\Gamma(X, \Omega_{X/S}^2 \otimes \text{Lie}(G))$ . First we define the curvature of a connection on the trivial bundle  $G_X$  and then show that these tensors can be patched together to give a definition for an arbitrary torsor  $P$ .

A connection on  $G_X$  is simply an automorphism of  $G_{\Delta^1(X)}$  which restricts to the identity. It is completely determined by telling what it does to the unit section and hence is determined by giving an arbitrary element  $\xi$  in  $\text{Ker}(\Gamma(\Delta^1(X), G) \rightarrow \Gamma(X, G)) = \text{Hom}_{\mathcal{O}_X}(\omega_G \otimes \mathcal{O}_X, \Omega_{X/S}^1) = \Gamma(X, \Omega_{X/S}^1 \otimes \text{Lie}(G))$ . The image of  $\xi$  in  $\Gamma(X, \Omega_{X/S}^2 \otimes \text{Lie}(G))$  under  $d \otimes \text{id}: \Omega_{X/S}^1 \otimes \text{Lie}(G) \rightarrow \Omega_{X/S}^2 \otimes \text{Lie}(G)$  is by definition the curvature form of the connection.

Now if  $P$  is an arbitrary  $G$ -torsor on  $X$ , endowed with a connection, then after an étale base change  $X' \rightarrow X$ , by our

definition of torsor,  $P$  becomes trivial. There is an induced connection on  $P_{X'}$ . Choosing a trivialization of  $P_{X'}$ , construct the curvature of the induced connection which lies in  $\Gamma(X', \Omega_{X'/S}^2 \otimes \text{Lie}(G)) = \Gamma(X', f^* \Omega_{X/S}^2 \otimes \text{Lie}(G))$ . (We obtain the above equality because  $X' \rightarrow X$  is étale.) In order to show that this local construction descends to define a section of  $\Omega_X^2 \otimes \text{Lie}(G)$  over  $X$ , which will be by definition the curvature, it suffices to show that the curvature of  $P_X$  is independent of the choice of trivialization, since then the application of  $p_1^*$  and  $p_2^*$  to our section in  $\Gamma(X', \Omega_{X'/S}^1 \otimes \text{Lie}(G))$  yields the same section of  $\Gamma(X' \times_X X', \Omega_{X' \times_X X'}^1 \otimes \text{Lie}(G))$  and we can apply descent. To do this take two trivializations

$$\phi: P \rightarrow G, \quad \psi: P \rightarrow G$$

and express the comparison  $\psi \circ \phi^{-1}$  as an  $S$ -morphism

$$g: X \rightarrow G.$$

One checks readily that the difference between the two curvatures is given by  $da \in \Gamma(X, \Omega_{X/S}^2 \otimes \text{Lie}(G))$  where  $\alpha = p_2^*(g) - p_1^*(g)$  is interpreted as an element in

$$(*) \quad \begin{array}{ccc} \text{Ker}(\text{Hom}(\Delta^1(X), G) \rightarrow \text{Hom}(X, G)) & & \\ \searrow \sim & & \\ & \Gamma(X, \Omega_{X/S}^1 \otimes \text{Lie}(G)) & \end{array}$$

and  $d$  is induced from exterior differentiation

$$d: \Omega_{X/S}^1 \rightarrow \Omega_{X/S}^2.$$

We must now show

(3.1.5) Lemma:  $da = 0$

Proof: Let the same letter  $\pi$  denote the structural morphisms

$\pi: G \rightarrow S, \pi: X \rightarrow S$  for no confusion will result.

The element  $\alpha$  may be viewed as a homomorphism:

$$\alpha: \omega_G \rightarrow \pi_* \Omega_{X/S}^1$$

by means of the isomorphism

$$(**) \quad \Gamma(X, \Omega_{X/S}^1 \otimes \underline{\text{Lie}}(G)) \cong \text{Hom}_{\mathcal{O}_S}(\omega_G, \pi_* \Omega_{X/S}^1)$$

Using the diagram:

$$(3.1.6) \quad \begin{array}{ccccc} & & G \times G & \xrightarrow{\text{subtraction}} & (G) \\ & \nearrow (p_1 \circ g, p_2 \circ g) & \updownarrow & & \updownarrow \\ \Delta^1(X) & \xrightarrow{\Delta^1(g)} & \Delta^1(G) & \longrightarrow & \text{Inf}_S^1(G) \\ \uparrow & & \updownarrow & & \updownarrow \\ X & \xrightarrow{g} & G & \longrightarrow & S \end{array}$$

and the isomorphisms (\*), (\*\*) above one can see that  $\alpha$  is the composition of the two top horizontal arrows in the following diagram:

$$\begin{array}{ccccc} \omega_G & \longrightarrow & \pi_* \Omega_{G/S}^1 & \xrightarrow{dg} & \pi_* \Omega_{X/S}^1 \\ & & \downarrow d & & \downarrow d \\ & & \pi_* \Omega_{G/S}^2 & \longrightarrow & \pi_* \Omega_{X/S}^2 \end{array}$$

Since the image of  $\omega_G$  in  $\pi_* \Omega_{G/S}^1$  is killed by  $d$ , the

lemma follows, and our construction of the global curvature of the  $G$ -torseur  $P$  is concluded.

If the curvature associated to  $(P, \nabla)$  is zero we say that the connection  $\nabla$  is integrable.

(3.1.7) (The multiplicative de Rham complex)

Consider the map of sheaves for  $X_{\text{et}}$ , the small étale site of  $X$ :

$$\begin{array}{ccc} G_X & \xrightarrow{d} & \Omega_{X/S}^1 \otimes \underline{\text{Lie}} G \\ g \mapsto & \longrightarrow & \alpha = p_2^*(g) - p_1^*(g) . \end{array}$$

(3.1.5) implies that

$$\Omega_{X/S}^*(G) = \text{defn } G_X \xrightarrow{d} \Omega_{X/S}^1 \otimes \underline{\text{Lie}} G \xrightarrow{d} \Omega_{X/S}^2 \otimes \underline{\text{Lie}} G \dots \rightarrow \Omega_{X/S}^n \otimes \underline{\text{Lie}} G \rightarrow \dots$$

may be viewed as a complex of sheaves on  $X_{\text{et}}$ .

If  $G = G_a$  we obtain the ordinary de Rham complex

$$\mathcal{O}_X \rightarrow \Omega_{X/S}^1 \rightarrow \dots$$

If  $G = G_m$ , we obtain a complex called the multiplicative de Rham complex:

$$\mathcal{O}_X^* \xrightarrow{d \log} \Omega_{X/S}^1 \xrightarrow{d} \Omega_{X/S}^2 \rightarrow \dots$$

(3.1.8) A  $G$ -torseur endowed with an integrable connection is what Grothendieck calls a  $\zeta$ -torseur. The  $\zeta$ -torseurs form a full sub-category of the category introduced in (3.1.3) Denote this category by  $\text{TORS } \zeta(X, G)$ .

(3.1.9) Because  $G$  is commutative we can define the contracted product  $P \overset{G}{\wedge} Q$  of two  $G$ -torseurs. It is by definition the

associated sheaf of the presheaf which is the quotient of  $P \times Q$  by the action of  $G: g \cdot (p, q) = (gp, g^{-1}q)$ .  $P \overset{G}{\wedge} Q$  is made into a  $G$ -torseur by letting  $G$  act on either of the factors. If  $P$  and  $Q$  are endowed with connections  $\nabla_P$  and  $\nabla_Q$ , then

$$\nabla_P \overset{G}{\wedge} \nabla_Q: p_1^*(P) \overset{G}{\wedge} p_1^*(Q) \xrightarrow{\sim} p_2^*(P) \overset{G}{\wedge} p_2^*(Q)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ p_1^*(P \overset{G}{\wedge} Q) & & p_2^*(P \overset{G}{\wedge} Q) \end{array}$$

defines a connection on  $P \overset{G}{\wedge} Q$ . Furthermore the curvature tensor associated to  $\nabla_P \overset{G}{\wedge} \nabla_Q$  is the sum of that associated to  $\nabla_P$  and that associated to  $\nabla_Q$ . In particular, the contracted product of  $\mathcal{L}$ -torseurs is a  $\mathcal{L}$ -torseur.

If  $X$  is an  $S$ -group, then it is possible to impose additional structures on a  $G_X$ -torseur  $P$ : namely to require that  $P$  has the structure of an  $S$ -group so that we obtain a (central) extension

$$0 \rightarrow G \rightarrow P \rightarrow X \rightarrow 0$$

In our context (i.e. given that  $P$  is a torseur) the most convenient way to express this is by giving an isomorphism:

$$\beta: \pi_1^*(P) \overset{G}{\wedge} \pi_2^*(P) \xrightarrow{\sim} s^*(P)$$

(where  $\pi_1, \pi_2: X \times X \rightarrow X$  are the projections and  $s: X \times X \rightarrow X$  is the addition law) and requiring the appropriate diagrams, (expressing the associativity and commutativity) to commute.

(3.1.10) By combining the notion of torseur endowed with an integrable connection, with the notion of a group extension of  $G$  by  $X$  we are led to the following definition (following

again Grothendieck's terminology).

**Definition.** A  $\mathcal{L}$ -extension of the smooth group  $G$  by the commutative group  $X$  is a triple  $(P, \nabla, \beta)$ , where  $(P, \nabla)$  is a  $\mathcal{L}$ -torseur on  $X$  under  $G$ ,  $(P, \beta)$  defines a group structure on  $P$ , making it an extension of  $X$  by  $G$ , and where  $\beta: \pi_1^*(P) \overset{G}{\wedge} \pi_2^*(P) \xrightarrow{\sim} s^*(P)$  is a horizontal morphism.

We denote by  $\text{EXT}^{\mathcal{L}}(X, G)$  the category whose objects are the  $\mathcal{L}$ -extensions and whose morphisms are the horizontal morphisms between extensions. Because  $G$  is commutative, the category of extensions of  $X$  by  $G$ ,  $\text{EXT}(X, G)$  is endowed with a "composition law" which corresponds to taking the contracted product of the underlying torseurs. Upon passing to the set of isomorphism classes of objects the induced composition law gives the standard group structure to  $\text{Ext}^1(X, G)$ . From the description of the composition law in terms of contracted product of torseurs it is clear that we can define the "Baer sum" of two  $\mathcal{L}$ -extensions and that by passing to isomorphism classes we obtain a group  $\text{Ext}^{\mathcal{L}}(X, G)$ .

Let

$$(e) \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be an exact sequence of finite locally free (commutative)  $S$ -groups. An  $(e)$ - $\mathcal{L}$  homomorphism  $A \rightarrow G$  is by definition a pair  $(\phi, \nabla)$  where  $\phi: A \rightarrow G$  is a homomorphism and  $\nabla$  is a connection on  $G \overset{A}{\parallel} B$  making

$$(e_{\phi}) \quad 0 \rightarrow G \rightarrow G \overset{A}{\parallel} B \rightarrow C \rightarrow 0$$

a  $\mathcal{G}$ -extension of  $C$  by  $G$ .

The set of  $(\epsilon)$ - $\mathcal{G}$  homomorphisms  $A \rightarrow G$  is made into a group by defining  $(\phi, \nabla) + (\phi', \nabla') = (\phi + \phi', \bar{\nabla})$  where  $\bar{\nabla}$  defines the structure of  $\mathcal{G}$ -extension on the "Baer sum" of  $(\epsilon_\phi)$  and  $(\epsilon_{\phi'})$ . We shall denote this group by  $(\epsilon)$ - $\mathcal{G} \text{Hom}(A, G)$ .

### (3.2) (The isomorphisms)

In this section we shall construct a homomorphism

$$(3.2.1) \quad \text{Ext}^{\mathcal{G}}(A, G) \rightarrow \text{Extrig}(A, G).$$

As a consequence, one then obtains a homomorphism

$$(3.2.2) \quad (\epsilon)\text{-}\mathcal{G} \text{Hom}(A, G) \rightarrow (\epsilon)\text{-Homrig}(A, G).$$

Later we shall prove that over an affine base  $S$  (3.2.2) is an isomorphism if  $G = G_m$ , and (3.2.1) is an isomorphism if  $G = G_m$  and  $A$  is an abelian scheme.

Let the  $\mathcal{G}$ -extension,

$$(\epsilon) \quad 0 \rightarrow G \rightarrow E \rightarrow A \rightarrow 0$$

be given.

Denote by  $i$ , the inclusion  $\text{Inf}^1(A) \hookrightarrow A$ ,  $\pi: \text{Inf}^1(A) \rightarrow S$  the structural morphism and by  $\tau: \text{Inf}^1(A) \rightarrow \Delta^1(A)$  the morphism determined by  $p_1 \circ \tau = e_A \circ \pi$ ,  $p_2 \circ \tau = i$ .

Since the  $\mathcal{G}$ -structure on  $E$  is given by an isomorphism  $\nabla: p_1^*(E) \xrightarrow{\sim} p_2^*(E)$ , we can "pull back"  $\nabla$  via  $\tau$  to obtain:

$$\tau^*(\nabla): \pi^* \circ e_A^*(E) \rightarrow i^*(E).$$

Since  $E$  is a group  $e_A^*(E)$  and hence  $\pi^* e_A^*(E)$  is equipped with an obvious choice of section, the unit section. Via  $\tau^*(\nabla)$  we transfer this section to obtain a section of  $i^*(E)$  and hence by composition with  $i^*(E) \rightarrow E$ , we obtain finally a morphism  $\sigma: \text{Inf}^1(A) \rightarrow E$ . It is this  $\sigma$ , which we shall choose to be the rigidification of the extension  $(\epsilon)$ . In order to show that this is legitimate let us verify that  $\sigma$  possesses the three properties required of a rigidification:

- 1)  $\sigma$  is a morphism of  $S$ -schemes
- 2) the following diagram commutes

$$\begin{array}{ccc} E & \longrightarrow & A \\ \sigma \nearrow & & \downarrow \\ & & \text{Inf}^1(A) \end{array}$$

- 3)  $\sigma$  is a morphism of  $S$ -pointed schemes.

$$\begin{aligned} \text{To check 1): } & \text{Inf}^1(A) \xrightarrow{\sigma} E \rightarrow S = \text{Inf}^1(A) \xrightarrow{\sigma} E \rightarrow A \rightarrow S \\ & = \text{Inf}^1(A) \rightarrow \text{Inf}^1(A) \times_A E \xrightarrow{\text{proj}} E \rightarrow A \rightarrow S = \\ & \text{Inf}^1(A) \rightarrow \text{Inf}^1(A) \times_A E \xrightarrow{\text{proj}} \text{Inf}^1(A) \hookrightarrow A \rightarrow S = \text{Inf}^1(A) \xrightarrow{\pi} S. \end{aligned}$$

To check 2) it suffices to observe that  $i^*(E) = \text{Inf}^1(A) \times_A E$  and that  $\sigma$  is the composition of a section in  $\tau(\text{Inf}^1(A), i^*(E))$  and the projection  $i^*(E) \rightarrow E$ .

Finally let us check that 3) holds. We are to show that  $S \rightarrow \text{Inf}^1(A) \xrightarrow{\sigma} E = S \xrightarrow{e} E$ . The left hand side can be computed as follows:

$$\begin{aligned} S & \hookrightarrow \text{Inf}^1(A) \xrightarrow{\sigma} E = S \hookrightarrow \text{Inf}^1(A) \rightarrow \text{Inf}^1(A) \times_A E \xrightarrow{\text{proj}} E \\ & = S \hookrightarrow \text{Inf}^1(A) \xrightarrow{(\text{id}, e \circ \pi)} \text{Inf}^1(A) \times_A E \xrightarrow{\tau^*(\nabla)} \text{Inf}^1(A) \times_A E \xrightarrow{\text{proj}} E \\ & = S \xrightarrow{u} \text{Inf}^1(A) \times_A E \xrightarrow{\tau^*(\nabla)} \text{Inf}^1(A) \times_A E \xrightarrow{\text{proj}} E \xrightarrow{i} A \end{aligned}$$

where the components of  $u: S \rightarrow \text{Inf}^1(A) \times E$  are  $e_{\text{Inf}^1(A)}$  and  $e_E$ .

Thus to conclude (3) it must be shown that  $\tau^*(\nabla)$  preserves the second component of this morphism. To do this let us return momentarily to the given  $\nabla: p_1^*(E) \rightarrow p_2^*(E)$ . By composing  $e_A: S \rightarrow A$  with  $\Delta: A \rightarrow \Delta^1(A)$ ,  $S$  may be viewed as a  $\Delta^1(A)$ -scheme. Via this both  $p_1^*(E)$  and  $p_2^*(E)$  have an obvious section  $\delta_1$  (resp  $\delta_2$ ) with values in the  $\Delta^1(A)$ -scheme  $S$ ; namely the section with components  $S \rightarrow \Delta^1(A)$  and  $S \xrightarrow{e_E} E$ . Under the identification of  $\Delta^1(A) \times p_1^*(E)$  with  $E$ , the unit section  $S \xrightarrow{e_E} E$  is identified with the section just described of  $p_1^*(E)$  with values in the  $\Delta^1(A)$ -scheme  $S$ . But by definition of a connection,  $\Delta^*(\nabla) = \text{id}_E$ , and hence  $\nabla$  must map  $\delta_1: S \rightarrow p_1^*(E)$  into the corresponding section  $\delta_2: S \rightarrow p_2^*(E)$ ; that is the second component remains  $S \xrightarrow{e_E} E$ .

Let us now consider the first factor  $S \xrightarrow{e_{\text{Inf}^1(A)}} \text{Inf}^1(A)$ . Because  $\tau^*e_{\text{Inf}^1(A)} = \Delta^*e_A$ , it follows immediately from the definitions that  $\tau^*(\delta_1) = u$ . This implies that  $\tau^*(\nabla) \circ u$  has as its second component the unit section  $e_E: S \rightarrow E$ , and completes the proof.

(3.2.3) Proposition: a) If  $A$  is an abelian scheme the homomorphism  $\text{Ext}^{\mathcal{L}}(A, G_m) \rightarrow \text{Extrig}(A, G_m)$  is an isomorphism.

b) The homomorphism

$(\epsilon)_{\mathcal{L}} \text{Hom}(A, G_m) \rightarrow (\epsilon)_{\mathcal{L}} \text{Homrig}(A, G_m)$  is an isomorphism if  $S$  is affine.

Proof: a) In order to prove  $\text{Ext}^{\mathcal{L}}(A, G_m) \rightarrow \text{Extrig}(A, G_m)$  is an isomorphism, let us construct an inverse. Assume given a

rigidified extension

$$(3.2.4) \quad \begin{array}{ccccccc} 0 & \rightarrow & G_m & \rightarrow & E & \xrightarrow{j} & A \rightarrow 0 \\ & & & & \sigma \swarrow & & \downarrow \\ & & & & & & \text{Inf}^1(A) \end{array}$$

$\sigma$  defines a section of  $i^*(E)$ , and hence a trivialization  $\rho: (e_A \circ \pi)^*(E) \xrightarrow{\sim} i^*(E)$ , via  $e \mapsto (\text{id}_{\text{Inf}^1(A)}, \sigma)$ .

By definition of  $\text{Inf}^1(A)$ , the map  $p_2 - p_1: \Delta^1(A) \rightarrow A$  factors through  $\text{Inf}^1(A)$ . Let us write it as  $\Delta^1(A) \xrightarrow{\eta} \text{Inf}^1(A) \xrightarrow{i} A$ . Thus  $\eta^*(\rho): (e_A \circ \pi_{\Delta^1(A)})^*(E) \xrightarrow{\sim} (p_2 - p_1)^*(E)$  is an isomorphism where  $\pi_{\Delta^1(A)}: \Delta^1(A) \rightarrow S$  is the structural morphism.

Multiplying both source and target of this map by  $p_1^*(E)$  and using the fact that  $E$  is a group we obtain a diagram where the lower horizontal arrow is defined so as to render it commutative

$$(3.2.5) \quad \begin{array}{ccc} (e_A \circ \pi_{\Delta^1(A)})^*(E) \wedge p_1^*(E) & \xrightarrow{\eta^*(\rho) \wedge p_1^*(E)} & (p_2 - p_1)^*(E) \wedge p_1^*(E) \\ \downarrow \} & & \downarrow \} \\ p_1^*(E) & \xrightarrow{\nabla'} & p_2^*(E) \end{array}$$

Our inverse mapping is now defined by associating to the rigidified extension above the  $\mathcal{L}$ -extension with the same underlying extension and the  $\mathcal{L}$ -structure defined by  $\nabla'$ . To show that the definition makes sense and actually gives an inverse, five statements must be proved:

- 1)  $\Delta^*(\nabla') = \text{id}_E$
- 2) The map  $\text{Extrig}(A, G_m) \rightarrow \text{Ext}^{\mathcal{L}}(A, G_m) \rightarrow \text{Extrig}(A, G_m)$  is the identity.

- 3) The map  $\text{Ext}^1(A, G_m) \rightarrow \text{Extrig}(A, G_m)$  is injective
- 4)  $\nabla'$  is integrable
- 5) The isomorphism  $\pi_1^*(E) \wedge^{G_m} \pi_2^*(E) \xrightarrow{\sim} s^*(E)$  is horizontal.

The proofs we give of the first two statements are entirely formal, while those of the remaining three actually use the assumptions that  $A$  is an abelian scheme and  $G = G_m$ .

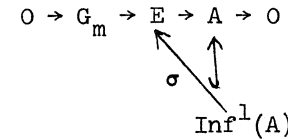
1) Since  $\Delta^*(\nabla')$  is a morphism over  $A$ , it suffices to show that it is the identity when  $E$  is viewed as a sheaf on  $\text{Sch}/S$ . Since our situation commutes with base change it suffices to show the mapping it induces,  $E(S) \rightarrow E(S)$ , is the identity. Let  $\zeta: S \rightarrow E$  be given so that  $\zeta$  defines morphisms  $\zeta_1: S \rightarrow P_1^*(E)$  and  $\zeta_2: S \rightarrow p_2^*(E)$ . Since  $\Delta: A \rightarrow \Delta^1(A)$  is a monomorphism it suffices to show that  $\nabla' \circ \zeta_1 = \zeta_2$ . To check that it is true let us recall the definition of the vertical isomorphisms in the diagram (3.2.5) above.

Let  $\alpha, \beta: T \rightarrow A$  be given and consider the torseurs  $E_\alpha, E_\beta, E_{\alpha+\beta}$  deduced from  $E$  by the corresponding base changes.  $E_\alpha \wedge E_\beta$  is a sheaf associated to the quotient of  $E_\alpha \times E_\beta$  by the action of  $G$ . Thus if  $T'$  is any  $S$ -scheme elements of  $\Gamma(T', E_\alpha \wedge E_\beta)$  are given locally by triples (of  $S$ -morphisms)  $x: T' \rightarrow E, y: T' \rightarrow E, t': T' \rightarrow T$  where  $T' \xrightarrow{x} E \rightarrow A = T' \rightarrow T \xrightarrow{\alpha} A, T' \xrightarrow{y} E \rightarrow A = T' \rightarrow T \xrightarrow{\beta} A$ . Thus the isomorphism in question is determined by associating to  $(t', x, y)$  the pair  $(t', x+y) \in \Gamma(T', E_{\alpha+\beta})$ .

Return now to diagram (3.2.4). Then  $\zeta_1 = (\Delta^0 j^0 \zeta, \zeta)$ ,  $\zeta_2 = (\Delta^0 j^0 \zeta, \zeta)$  and after the above explication of the vertical

isomorphism it is obvious that  $\zeta_1$  corresponds to the class of  $(\Delta^0 j^0 \zeta, e_E, \zeta)$ . On the other hand projection of  $(e_A \circ \pi_{\Delta^1(A)})^*(E) = \eta^*((e_A \circ \pi_{\text{Inf}^1(A)})^*(E))$  to  $(e_A \circ \pi_{\text{Inf}^1(A)})^*(E)$  assigns to  $(\Delta^0 j^0 \zeta, e_E)$  the pair  $(\eta^0 \Delta^0 j^0 \zeta, e_E)$  which it transforms via  $\rho$  into  $(\eta^0 \Delta^0 j^0 \zeta, \sigma \circ \eta^0 \Delta^0 j^0 \zeta)$ . Thus  $\eta^*(\rho)$  transforms  $(\Delta^0 j^0 \zeta, e_E)$  into  $(\Delta^0 j^0 \zeta, \sigma \circ \eta^0 \Delta^0 j^0 \zeta)$ . Therefore  $\eta^*(\rho) \wedge p_1^*(E)$  will transform the class of  $(\Delta^0 j^0 \zeta, e_E, \zeta)$  to the class of  $(\Delta^0 j^0 \zeta, \sigma \circ \eta^0 \Delta^0 j^0 \zeta, \zeta)$ . As  $e_A \circ \pi_A = (p_2 - p_1) \circ \Delta = i \circ \eta \circ \Delta$  and also  $e_A \circ \pi_A = i \circ e_{\text{Inf}^1(A)} \circ \pi_A$ , it follows that  $\eta \circ \Delta = e_{\text{Inf}^1(A)} \circ \pi_A$ . Hence  $\sigma \circ \eta \circ \Delta \circ j^0 \zeta = \sigma \circ e_{\text{Inf}^1(A)} \circ \pi_A \circ j^0 \zeta = e_E \circ \pi_A \circ j^0 \zeta = e_E$ . Thus under the isomorphism  $(p_2 - p_1)^*(E) \wedge p_1^*(E) \xrightarrow{\sim} p_2^*(E)$   $(\Delta^0 j^0 \zeta, \sigma \circ \eta \circ \Delta \circ j^0 \zeta, \zeta)$  corresponds to  $(\Delta^0 j^0 \zeta, \zeta)$  which shows (finally) that  $\Delta^*(\nabla') = \text{id}_E$ .

2) Let us begin with the rigidified extension



We associate a connection  $\nabla'$  on  $E$  to  $\sigma$  and then a rigidification  $\sigma'$  is associated to  $\Delta'$ . It is to be shown that  $\sigma' = \sigma$ . Using the definition of  $\sigma'$  it is the projection onto  $E$  of  $\tau^*(\nabla')(\text{id}_{\text{Inf}^1(A)}, e_E \circ \pi_{\text{Inf}^1(A)})$ . Hence it is the projection onto  $E$  of  $\nabla'(\tau, e_E \circ \pi_{\text{Inf}^1(A)})$ . But as it follows from the definition of  $\nabla'$  in terms of the diagram (3.2.5) above this projection is simply the sum:

$$\begin{aligned}
& \text{proj. onto } E(\eta^*(\rho)(\tau, e_E \circ \pi_{\text{Infl}}^1(A))) + e_E \circ \pi_{\text{Infl}}^1(A) \\
&= \text{proj. onto } E(\eta^*(\rho)(\tau, e_E \circ \pi_{\text{Infl}}^1(A))) \\
&= \text{proj. onto } E(\rho(\eta \circ \tau, e_E \circ \pi_{\text{Infl}}^1(A)))
\end{aligned}$$

But since  $i \circ \eta \circ \tau = (p_2 - p_1) \circ \tau = p_2 \circ \tau - p_1 \circ \tau = i - e_A \circ \pi_{\text{Infl}}^1(A) = i$ , and since  $i$  is a monomorphism, it follows that  $\eta \circ \tau = \text{id}_{\text{Infl}}^1(A)$ . This implies, by the very definition of  $\rho$ , that  $\sigma' = \sigma$ .

3) To show the map  $\text{Ext}_{\mathcal{L}}^1(A, G_m) \rightarrow \text{Extrig}(A, G_m)$  is injective, we must show that if  $\nabla$  defines a  $\mathcal{L}$ -structure on the trivial extension

$$0 \rightarrow G_m \rightarrow G_m \times A \rightarrow A \rightarrow 0$$

whose associated rigidification,  $\sigma$ , is trivial, then  $\nabla$  is trivial. But  $\nabla$  is determined by giving a section of  $\Gamma(\mathcal{O}_{\Delta^1(A)}^* \otimes \omega_A^1)$  of the form  $1 + \omega$ ,  $\omega \in \Gamma(A, \omega_A^1)$ . The corresponding  $\rho$  (associated to the rigidification  $\sigma$ ) is, because it is an automorphism of  $G_{m, \text{Infl}}^1(A)$ , determined by a unit in  $\Gamma(\mathcal{O}_{\text{Infl}}^* \otimes \omega_A^1)$  of the form  $1 + \omega'$ ,  $\omega' \in \Gamma(S, \omega_A)$ . One has:  $\omega' = \tau^*(\omega)$ . But, because  $A$  is an abelian scheme, this mapping  $\Gamma(A, \omega_A^1) \rightarrow \Gamma(S, \omega_A)$  is an isomorphism.

4) To show the connection  $\nabla'$  is integrable we shall use a trick which will be repeated below in showing that  $\nabla'$  is compatible with the group structure on  $E$ . The curvature tensor  $c(\nabla')$  is an element of  $\Gamma(S, \pi_{A*}(\omega_{A/S}^2))$ . As mentioned in (3.1),  $E$  corresponds to a line bundle  $\mathcal{L}_E$  and  $\nabla'$  to a connection on this line bundle. Thus because  $A$  is an abelian scheme, and hence all global 1-forms are closed, the curvature  $c(\nabla)$  is

actually independent of the connection on  $E$ . Notice this allows us to define a morphism  $\text{Ext}(A, G_m) \rightarrow \pi_{A*}(\omega_{A/S}^2)$ . Namely if  $S'$  is an (absolutely) affine  $S$ -scheme and  $0 \rightarrow G_{m, S'} \rightarrow E' \rightarrow A_{S'} \rightarrow 0$  is an extension, we can take any structure of rigidified extension on it, then by the above procedure put a connection on  $E'$  and hence finally obtain the curvature tensor which lies in  $\Gamma(S', \pi_{A_{S'}*}(\omega_{A_{S'}/S'}^2)) = \Gamma(S', \pi_{A*}(\omega_{A/S}^2)_{S'})$ . Passing to the associated sheaves gives the morphism  $\text{Ext}(A, G_m) \rightarrow \pi_{A*}(\omega_{A/S}^2)$ . Since  $\text{Ext}(A, G_m)$  is an abelian scheme and  $\pi_{A*}(\omega_{A/S}^2)$  is a vector group, this morphism is constant. Clearly the image of the trivial extension is zero and thus the map is identically zero implying that the connection  $\nabla'$  is integrable.

5) To show the connection  $\nabla'$  is compatible with the group structure let us replace  $E$  by the corresponding line bundle  $\mathcal{L}_E$ . Then we are to show the isomorphism  $s^*(\mathcal{L}_E) \xrightarrow{\sim} \pi_1^*(\mathcal{L}_E) \otimes \pi_2^*(\mathcal{L}_E)$  is horizontal. Using this isomorphism the problem can be interpreted as that of showing that two connections on  $s^*(\mathcal{L}_E)$  are the same. Taking their "difference" we obtain a section,  $\delta(\nabla')$  in  $\Gamma(S, \omega_{A \times A})$ . In order to imitate the trick used in 4) above, we will use the following lemma.

(3.2.6) Lemma: Let  $X/S$  be a scheme,  $\mathcal{L}_1, \mathcal{L}_2$  line bundles on  $X$ ,  $\nabla_1, \nabla_2, \nabla_1', \nabla_2'$  connections on  $\mathcal{L}_1$ ,  $\phi: \mathcal{L}_1 \rightarrow \mathcal{L}_2$  an isomorphism. Let  $\delta$  (resp.  $\delta'$ ) denote the "difference" between  $\phi^*(\nabla_2)$  and  $\nabla_1$  (resp.  $\phi^*(\nabla_2')$  and  $\nabla_1'$ ). Then we have the following formula  $\delta - \delta' =$  "difference" between  $\nabla_2$  and  $\nabla_2'$  - "difference" between

$\nabla_1$  and  $\nabla'_1$ .

Proof: The assertion is local hence we can assume  $X = \text{Spec}(B)$   $S$  affine,  $\mathcal{L}_1, \mathcal{L}_2$  trivial. Translating then  $\nabla_1, \nabla'_1, \nabla_2, \nabla'_2$  corresponding to differential forms  $\omega_1, \omega'_1, \omega_2, \omega'_2 \in \Omega_B^1$  and  $\phi$  corresponds to the mapping multiplication by a unit  $b \in B^*$ . Thus  $\phi^*(\nabla_2)$  corresponds to  $\frac{db}{b} + \omega_2$  so  $\phi^*(\nabla_2) - \nabla_1 = \frac{db}{b} + (\omega_2 - \omega_1)$  and analogously  $\phi^*(\nabla'_2) - \nabla'_1 = \frac{db}{b} + (\omega'_2 - \omega'_1)$ . Subtracting we find the result.

In applying the lemma take  $\mathcal{L}_1 = s^*(\mathcal{L})$ ,  $\mathcal{L}_2 = \pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L})$  and for any two connections  $\bar{\nabla}, \bar{\bar{\nabla}}$  on  $\mathcal{L}$  let  $\nabla_1 = s^*(\bar{\nabla})$ ,  $\nabla'_1 = s^*(\bar{\bar{\nabla}})$ ,  $\nabla_2 = \pi_1^*(\bar{\nabla}) \otimes \pi_2^*(\bar{\nabla})$ ,  $\nabla'_2 = \pi_1^*(\bar{\bar{\nabla}}) \otimes \pi_2^*(\bar{\nabla})$ . Then if  $\bar{\nabla} - \bar{\bar{\nabla}} = \psi \in \Gamma(A, \Omega_A^1)$  the lemma says that  $\delta(\bar{\nabla}) - \delta(\bar{\bar{\nabla}}) = \pi_1^*(\psi) + \pi_2^*(\psi) - s^*(\psi)$ . But because  $A$  is an abelian scheme  $\psi$  is primitive and hence  $\delta(\bar{\nabla}) = \delta(\bar{\bar{\nabla}})$ .

Because  $\delta(\bar{\nabla})$  is independent of the connection put on the line bundle  $\mathcal{L}$ , we can just as in 4) above define a morphism  $\text{Ext}(A, G_m) \rightarrow \underline{u}_{A \times A}$ . As the trivial connection on the trivial extension is compatible with the group structure, any connection placed on any extension is similarly compatible since the morphism is constantly zero.

b) Assume  $S$  is affine and consider the extension of finite locally free-groups:

$$(\mathcal{E}) \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

From (2.2.1) there is an exact sequence

$$(3.2.7) \quad 0 \rightarrow \Gamma(S, \underline{u}_C) \rightarrow (\mathcal{E})\text{-Homrig}(A, G_m) \rightarrow \text{Hom}_{S\text{-gr}}(A, G_m) \rightarrow 0$$

The indeterminacy in putting a structure of  $\mathcal{H}$ -extension on the trivial extension  $0 \rightarrow G_m \rightarrow G_m \times C \rightarrow C \rightarrow 0$  is given by  $\Gamma(S, \underline{u}_C)$  since the differential form defining the connection must be primitive (i.e. translation invariant). Thus there is also an exact sequence

$$(3.2.8) \quad 0 \rightarrow \Gamma(S, \underline{u}_C) \rightarrow (\mathcal{E})\text{-}\mathcal{H}\text{Hom}(A, G_m) \rightarrow \text{Hom}_{S\text{-gr}}(A, G_m)$$

Obviously (3.2.7) receives a map from (3.2.8) which is the identity on  $\Gamma(S, \underline{u}_C)$  and on  $\text{Hom}_{S\text{-gr}}(A, G_m)$ , and which is the map (3.2.2) on the middle terms. Hence to conclude that (3.2.2) is an isomorphism it suffices to prove that the map  $(\mathcal{E})\text{-}\mathcal{H}\text{Hom}(A, G_m) \rightarrow \text{Hom}_{S\text{-gr}}(A, G_m)$  is surjective.

Let  $\phi: A \rightarrow G_m$  be a homomorphism and consider the corresponding extension

$$(\mathcal{E}_\phi) \quad 0 \rightarrow G_m \rightarrow E \rightarrow C \rightarrow 0, \quad E = G_m \overset{A}{\text{fl}} B.$$

If the set of structures of  $\mathcal{H}$ -extension on  $(\mathcal{E}_\phi)$  is not empty it is principal homogeneous under  $\Gamma(S, \underline{u}_C)$ . Replacing  $S$  by an arbitrary  $S$ -scheme  $S'$  we see that for variable  $S'$  the functor  $S' \mapsto \text{set of structures of } \mathcal{H}\text{-extension on } (\mathcal{E}_\phi)_{S'}$  is formally principal homogeneous under  $\underline{u}_C$ . Since  $C$  is finite and locally free  $\text{Ext}^1(C, G_m) = (0)$  and hence locally for the f.p.p.f. topology, the extension  $(\mathcal{E}_\phi)$  is trivial. This implies that we actually have a torsour. By descent it is locally trivial for the Zariski topology and thus because  $S$  is affine it is trivial. Hence  $(\mathcal{E}_\phi)$  actually admits a structure of  $\mathcal{H}$ -extension; which proves  $(\mathcal{E})\text{-}\mathcal{H}\text{Hom}(A, G_m) \rightarrow \text{Hom}_{S\text{-gr}}(A, G_m)$  is surjective.

§4. THE RELATION BETWEEN ONE-DIMENSIONAL DE RHAM COHOMOLOGY  
AND THE LIE ALGEBRA OF THE UNIVERSAL EXTENSION

Let  $A$  be an abelian scheme over  $S$ . We shall establish an isomorphism between  $H_{DR}^1(A)$  and the lie algebra of  $\text{Extrig}(A, G_m)$ . The most convenient way to do this is to find yet another interpretation of  $\text{Extrig}(A, G_m)$ , this time in terms of differential forms (see the construction of  $E^4$  below).

(4.1). The Definition of  $E^4$ .

(4.1.1). Let  $A/S$  be an abelian scheme. Its De Rham cohomology is quite simple:

- a) all the  $H_{DR}^i(A)$ ,  $H^q(\Omega_A^p)$  are locally free (and hence their formation commutes with base change).
- b) The Hodge-DR spectral sequence degenerates at  $E_1$ .
- c)  $H_{DR}^*(A) = \wedge^* H_{DR}^1(A)$

The first thing we do is give a geometric interpretation to a portion of the long exact sequence of (hyper) cohomology associated to the short exact sequence of complexes

$$\begin{array}{ccccccc}
 & & 0 & & \mathcal{O}^* & & \mathcal{O}^* \\
 & & \downarrow & & \downarrow & \text{dlog} & \downarrow \\
 0 & \longrightarrow & \Omega^1 & \longrightarrow & \Omega^1 & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \Omega^2 & & \Omega^2 & & 0 \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

which we abbreviate to:  $0 \rightarrow \tau_1(\Omega) \rightarrow \Omega^* \rightarrow \mathcal{O}^* \rightarrow 0$

Define a functor on  $S$ -schemes by:

$S' \mapsto$  the group of isomorphism classes of line bundles on  $A_{S'}$  endowed with an integrable connection.

Write  $P^4$  for this functor and  $\underline{P}^4$  for the associated Zariski sheaf. For any  $S'$ , there is the forgetting map:

$$P^4(S') \rightarrow H^1(\mathcal{O}_{A_{S'}}^*)$$

which by passage to the associated sheaves yields (since  $A$  is an abelian scheme) a homomorphism

$$\underline{P}^4 \xrightarrow{\pi} \underline{\text{Pic}}(A)$$

Because global 1-forms on an abelian scheme are closed, and because the map  $H^0(\mathcal{O}_A^*) \xrightarrow{d \log} H^0(\Omega_A^1)$  is the zero map, the indeterminacy in putting an integrable connection on the trivial bundle  $\mathcal{O}$  is precisely  $\Gamma(A, \Omega_A^1) = \Gamma(S, \underline{\omega}_A)$ . Passing to the associated sheaves we find the kernel of the map  $\pi$  to be  $\underline{\omega}_A$ . What is the obstruction to putting an integrable connection on a line bundle  $\mathcal{L}$  (over  $A$ )? The obstruction to putting any connection on  $\mathcal{L}$  is furnished by the cocycle arising as the logarithmic derivative of the transition function defining  $\mathcal{L}$ :

$$H^1(\mathcal{O}_A^*) \rightarrow H^1(\Omega_A^1), \quad (f_{ij}) \mapsto \frac{df_{ij}}{f_{ij}}$$

There is an obvious map

$$H^1(\mathcal{O}_A^*) \rightarrow H^2(\tau_1(\Omega_A^*))$$

given in terms of Čech cocycles (for some affine open cover  $\mathcal{U}$  of  $A$ ) by

$$(f_{ij}) \mapsto \left( \left( \frac{df_{ij}}{f_{ij}} \right), 0 \right) \in C^1(\mathcal{U}, \Omega_A^1) \oplus C^0(\mathcal{U}, \Omega_A^2)$$

If this cocycle is a coboundary there are closed 1-forms  $\omega_i$

such that  $\frac{df_{ij}}{f_{ij}} = \omega_i - \omega_j$  and hence  $\mathcal{L}$  will admit an integrable connection. The converse is equally trivial.

(4.1.2) Proposition:

$$P^g(S) \rightarrow H^1(\Omega_A^*) .$$

Proof: To any line bundle with integrable connection  $(\mathcal{L}, \nabla)$  we associate the cohomology class of the Čech cocycle

$((f_{ij}), (\omega_i)) \in C^1(\mathcal{O}_A^*) \oplus C^0(\Omega_A^1)$  where  $f_{ij}$  are the transition functions and  $\omega_i$  is the "connection form" for the induced connection on  $\mathcal{L}|_{U_i}$ .

Q.E.D.

Thus we have arrived at the geometrical description of a portion of the above mentioned cohomology sequence:

$$(4.1.3) \quad \begin{array}{ccccccc} 0 & \rightarrow & H^0(\Omega_A^1) & \longrightarrow & P^g(S) & \longrightarrow & \text{Pic}(A) \longrightarrow H^2(\tau_1(\Omega_A^*)) \\ & & \parallel & & \downarrow S & & \parallel \\ 0 & \rightarrow & H^1(\tau_1(\Omega_A^*)) & \rightarrow & H^1(\Omega_A^*) & \rightarrow & H^1(\mathcal{O}_A^*) \rightarrow H^2(\tau_1(\Omega_A^*)) \end{array}$$

Now we shall consider Lie algebras. For any group functor  $G$  on  $\text{Sch}/S$ , the formation of  $\text{Lie}(G)$  commutes with taking of the associated Zariski sheaf. Thus to calculate the Lie algebra of  $P^g$  it suffices to calculate that of  $P^g$ .

(4.1.4) Proposition:  $H_{\text{DR}}^1(A/S)$  is canonically isomorphic to  $\text{Lie}(P^g)$ .

Proof: We must examine  $\text{Ker}(P^g(S[\epsilon]) \rightarrow P^g(S))$  which by (4.1.2) can be regarded as the kernel of

$$H^1(\Omega_{A/S[\epsilon]}^*) \rightarrow H^1(\Omega_A^*) .$$

But we have a split exact sequence of complexes of sheaves of abelian groups on  $A$ :

$$0 \rightarrow \Omega_A^* \rightarrow \Omega_{A/S[\epsilon]}^* \rightarrow \Omega_A^* \rightarrow 0$$

and hence (at least as abelian groups)  $H^1(\Omega_A^*) \xrightarrow{\sim} \text{Lie}(P^g)(S)$

The fact that the module structures coincide is a straightforward verification. Passing to the associated sheaves we find  $H_{\text{DR}}^1(A/S) \xrightarrow{\sim} \text{Lie}(P^g)$  as desired.

(4.1.5) Lemma.  $H^*(\tau_1(\Omega_A^*))$  is locally free (and hence commutes with arbitrary base change).

Proof: From the exact sequence  $0 \rightarrow \tau_1(\Omega_A^*) \rightarrow \Omega_A^* \rightarrow \mathcal{O}_A \rightarrow 0$ ,

using the local freeness of  $H_{\text{DR}}^*(A)$ ,  $H^*(\mathcal{O}_A)$  and the degeneration of Hodge  $\Rightarrow$  De Rham, we read the result from the short exact sequences:  $0 \rightarrow H^1(\tau_1(\Omega_A^*)) \rightarrow H^1(\Omega_A^*) \rightarrow H^1(\mathcal{O}_A) \rightarrow 0$

Knowing  $H^2(\tau_1(\Omega_A^*))$  is a locally free module commuting with base change we obtain the exact sequence of Zariski sheaves on  $\text{Sch}/S$ .

$$0 \rightarrow \underline{\omega}_A \rightarrow \underline{P}^g \rightarrow \underline{\text{Pic}}(A) \rightarrow H^2(\tau_1(\Omega_A^*))$$

Let us consider the dual abelian scheme  $A^* = \underline{\text{Pic}}^0(A)$  and the composite of its inclusions into  $\underline{\text{Pic}}(A)$  with the map  $\underline{\text{Pic}}(A) \rightarrow H^2(\tau_1(\Omega_A^*))$ . This composite is zero because there are no non-trivial homomorphisms from a abelian scheme to a (locally-free) quasi-coherent module. Hence the image of  $\underline{P}^g$  in  $\underline{\text{Pic}}(A)$  contains  $A^*$  and there is an exact sequence

$$0 \rightarrow \underline{\omega}_A \rightarrow \underline{P}^g \times_{\underline{\text{Pic}}(A)} A^* \rightarrow A^* \rightarrow 0$$

(4.1.6) Definition  $E^{\zeta} = \underline{P}^{\zeta} \times_{\underline{\text{Pic}}(A)} A^*$

Thus  $E^{\zeta}$  is actually a smooth group scheme which is obtained by considering the Zariski sheaf associated to the presheaf assigning to  $S'/S$  the set of isomorphism classes of  $(\mathcal{L}, \nabla)$  where the cohomology class of  $\mathcal{L}$  is primitive or equivalently the  $G_{m_{A/S}}$  torsor corresponding to  $\mathcal{L}$  is an extension of  $A_{S'}$  by  $G_{m_{S'}}$ .

(4.1.7) Proposition:  $H_{\text{DR}}^1(A/S)$  is canonically isomorphic to  $\underline{\text{Lie}}(E^{\zeta})$ .

Proof:  $\underline{\text{Lie}}(\underline{P}^{\zeta} \times_{\underline{\text{Pic}}(A)} A^*) \simeq \underline{\text{Lie}}(\underline{P}^{\zeta}) \times_{\underline{\text{Lie}}(\underline{\text{Pic}}(A))} \underline{\text{Lie}}(A^*)$

and as is well known  $\underline{\text{Lie}}(A^*) \rightarrow \underline{\text{Lie}}(\underline{\text{Pic}}(A))$  is an isomorphism.

(4.2) The isomorphism between  $\text{Ext}_{\zeta}^{\zeta}(A, G_m)$  and  $E^{\zeta}$ .

For any abelian scheme  $A/S$  define a homomorphism,

$$\text{Ext}_{\zeta}^{\zeta}(A, G_m) \rightarrow E^{\zeta} = \underline{P}^{\zeta} \times_{\underline{\text{Pic}}(A)} \underline{\text{Pic}}^0(A)$$

as follows: Any element  $e$  in  $\text{Ext}_{\zeta}^{\zeta}(A, G_m)$  may be regarded as an isomorphism class of invertible sheaves on  $A$  endowed with an integrable connection and with a horizontal isomorphism

$$s^*(G) \xrightarrow{\epsilon} p_1^*(L) \otimes p_2^*(L)$$

where  $p_1, p_2: A \times A \rightarrow A$  are the projections and  $s = p_1 + p_2$  is the sum morphism. By forgetting  $\epsilon$ , (resp. the connection) we obtain an element of  $\underline{P}^{\zeta}$  (resp.  $\underline{\text{Pic}}^0(A)$ ).

(4.2.1) Proposition

The above morphism is an isomorphism,

$$\text{Ext}_{\zeta}^{\zeta} \xrightarrow{\sim} E^{\zeta}.$$

Proof: It is injective. Any two horizontal isomorphisms between line bundles differ by multiplication by a unit in  $\Gamma(S, \mathcal{O}_S)$ . Thus if there is a horizontal isomorphism, an isomorphism compatible with the  $\epsilon$ 's is also horizontal.

To show that it is surjective, we shall define a morphism of  $S$ -schemes  $\eta: A^* \rightarrow \underline{U}_{A \times A}$  which expresses the obstruction to surjectivity of  $\zeta$ : Let  $L$  be in  $\text{Ext}(A, G_m)$ . Choose any integrable connection  $\nabla$  on  $L$ . This induces connections on  $s^*(L), p_1^*(L), p_2^*(L), p_1^*(L) \otimes p_2^*(L)$ .

The extension-structure of  $L$  gives us an explicit isomorphism,

$$s^*(L) \xrightarrow{\epsilon} p_1^*(L) \otimes p_2^*(L).$$

Consider the difference between the connection on  $s^*(L)$  and the pullback of the connection on  $p_1^*(L) \otimes p_2^*(L)$  via the above morphism. This difference  $i(\nabla)$  is a section of  $\underline{U}_{A \times A}$ . By (3.2.6)  $i(\nabla)$  depends only on  $L$  and not on the integrable connection  $\nabla$  chosen.

We define  $\eta(L) = i(\nabla)$ . Since  $A^*$  is an abelian scheme and  $\underline{U}_{A \times A}$  is a locally free module,  $\eta$  is a constant map. Since  $\eta(0) = 0$ ,  $\eta$  is identically zero. It follows that  $\epsilon$  is horizontal and  $\zeta$  is surjective.

(4.6.3) The sheaf  $\underline{P}^{\zeta}$  in concrete terms.

Consider the morphism of complexes  $\Omega_{A/S}^* \rightarrow \mathcal{O}_A^*$  and the

corresponding mapping induced on the exact sequence of terms of low degree, from the Leray spectral reference:

$$\begin{array}{ccccccc} 0 \rightarrow & H^1(\mathcal{O}_S^*) & \rightarrow & H^1(\mathcal{O}_A^*) & \rightarrow & \Gamma(S, R^1 f_* (\mathcal{O}_A^*)) & \rightarrow 0 \\ & \downarrow \cong & & \downarrow & & \downarrow & \\ 0 \rightarrow & H^1(\mathcal{O}_S^*) & \rightarrow & H^1(A, \mathcal{O}_A^*) & \rightarrow & \Gamma(S, \text{Pic}(A)) & \rightarrow 0 \end{array}$$

Consider on the other hand the group  $\text{Pic}_e^h(A) =_{\text{dfn}} \{\text{isomorphism class of triples } (\mathcal{L}, \alpha, \nabla) \text{ where } (\mathcal{L}, \alpha) \text{ is an e-rigidified line bundle on } A \text{ and } \nabla \text{ an integrable connection on } \mathcal{L}\}$ . Here isomorphisms are to be horizontal and respect the e-rigidification.

There is an obvious map  $\text{Pic}_e^h(A) \rightarrow P^h(S)$   
 $(\mathcal{L}, \alpha, \nabla) \mapsto (\mathcal{L}, \nabla)$

If  $(\mathcal{L}, \nabla), (\mathcal{L}', \nabla')$  are isomorphic, an isomorphism compatible with the rigidifications can be chosen since to modify an isomorphism we use a global section of  $\Gamma(A, \mathcal{O}_A^*) = \Gamma(S, \mathcal{O}_S^*)$  and clearly this will not alter the horizontality. Hence the map is injective. We obviously have a commutative diagram:

$$\begin{array}{ccc} (\mathcal{L}, \alpha, \nabla) & \xrightarrow{\quad} & \text{Pic}_e^h(A) \longrightarrow P^h(S) \\ \downarrow & \searrow & \downarrow \\ (\mathcal{L}, \alpha) & \xrightarrow{\quad} & \text{Pic}_e(A) \longrightarrow H^1(A, \mathcal{O}_A^*) \end{array}$$

Given  $(\mathcal{L}, \nabla)$  in  $P^h(S)$ ,  $\mathcal{L} \otimes f^*e^*(\mathcal{L}^{-1})$  is rigidified and  $f^*e^*(\mathcal{L}^{-1})$  can be given the "stupid" connection so that it is in the image of  $H^1(S, \mathcal{O}_S^*) \rightarrow H^1(\mathcal{O}_A^*)$ . Thus the map  $\text{Pic}_e^h(A) \rightarrow P^h(S) = H^0(\mathbb{R}^1 f_* (\mathcal{O}_A^*))$  is surjective. If  $(\mathcal{L}, \alpha, \nabla) \mapsto 0$ , then  $\mathcal{L} = f^*(\mathcal{L}')$ ,  $\nabla =$  trivial connection, and  $\mathcal{O}_S \simeq e^*(\mathcal{L}) = e^*f^*(\mathcal{L}') = \mathcal{L}' \Rightarrow \mathcal{L} \simeq \mathcal{O}_A$ , and  $\nabla =$  trivial connection, which obviously implies  $(\mathcal{L}, \alpha, \nabla) \simeq (\mathcal{O}_A, \text{obv}, 0)$ . Thus the map  $\text{Pic}_e^h(A) \rightarrow P^h(S)$

is an isomorphism and we have the desired description of  $P^h(S)$  as  $\{\text{e-rig line bundles} + \nabla\}$ , a description which is obviously compatible with the description of  $\Gamma(S, \text{Pic}(A))$  as  $\{\text{e-rigidified line bundles.}\}$

Since  $E^h =_{\text{dfn}} \frac{P^h \times \text{Ext}(A, G_m)}{\text{Pic}(A)}$ , it is clear that

$E^h$  admits the following description, its points with values in  $S$  (or for that matter any  $S$ -scheme  $S'$ ) consist of isomorphism classes of extensions  $0 \rightarrow G_m \rightarrow E \rightarrow A \rightarrow 0$  such that  $E$  is as  $G_{m,A}$ -torseur endowed with an integrable connection.

(4.4) The Universal Extension of an Abelian Scheme in the Analytic Category over  $\mathbb{C}$ .

Let  $A/S$  be an abelian scheme over  $S$ , where  $S$  is a scheme locally of finite type over  $\mathbb{C}$ . We may view  $A/S$  as a family of complex analytic spaces. The theory of Extrig carries over, with no significant change, in the analytic category. One thus obtains the analytic versions and natural maps below:

$$\begin{array}{l} [\text{Extrig}(A, G_m)]^{\text{an}} \rightarrow \text{Extrig}(A^{\text{an}}, G_m^{\text{an}}) \\ [\text{Extrig}(A, G_a)]^{\text{an}} \rightarrow \text{Extrig}(A^{\text{an}}, G_a^{\text{an}}) \end{array}$$

(4.4.1) Proposition: The morphisms above are isomorphisms.

Proof: This follows for each fibre (over  $S$ ) by GAGA. Consequently our morphisms are analytic morphisms bijective on underlying pointsets. By consideration of vertical and horizontal tangent vectors one checks that the jacobian criterion is satisfied.

Q.E.D.

As a consequence, the exponential sequence of analytic groups over  $\mathbb{C}$

$$0 \rightarrow 2\pi i\mathbb{Z} \rightarrow G_a \xrightarrow{\exp} G_m \rightarrow 0$$

gives rise to the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow \psi_A & & \downarrow \psi_A & & \\
 & & \text{Extrig}(A^{\text{an}}, G_a) & \xrightarrow{\cong} & \text{Extrig}(A^{\text{an}}, G_m) & & \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow & R^1 f_* \mathbb{Z} & \rightarrow & R^1 f_* \mathcal{O}_{A^{\text{an}}} & \rightarrow & \text{Pic}^0(A) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & 
 \end{array}$$

which gives us (using the snake lemma) the following exact sequence:

$$0 \rightarrow H^1(A^{\text{an}}, \mathbb{Z}) \rightarrow \text{Extrig}(A^{\text{an}}, G_a) \rightarrow \text{Extrig}(A^{\text{an}}, G_m) \rightarrow 0$$

over any affine base  $S$ .

(4.4.2) Corollary: One has an exact sequence of analytic groups over  $S$ :

$$0 \rightarrow R^1 f_* \mathbb{Z} \rightarrow H_{\text{DR}}^1(A^{\text{an}}/S) \rightarrow E(A^*)^{\text{an}} \rightarrow 0$$

Proof: Note that  $H_{\text{DR}}^1$  refers to relative de Rham cohomology over the base  $S$ .  $R^1 f_* \mathbb{Z}$  refers to the locally constant sheaf of abelian groups.

The corollary follows from our identifications

$$\text{Extrig}(A, G_m) = E(A^*)$$

$$\text{Extrig}(A, G_a) = H_{\text{DR}}^1(A/S)$$

## §5. FRAGMENTARY COMMENTS CONCERNING NÉRON MODELS AND UNIVERSAL EXTENSIONS

Let  $S$  be a connected Dedekind scheme. ( $S = \text{Spec } D$  where  $D$  is a Dedekind domain). Let  $N$  be a Néron model over  $S$ . This means that there is a nonempty open  $U \subset S$  such that  $N/U$  is an abelian scheme, and  $N/S$  is the Néron model of  $N/U$ . Let  $N'/U$  denote the dual abelian scheme and let  $N'/S$  be its Néron model over  $S$ . Define  $N^0 \subset N$  to be the open subgroup scheme all of whose fibres are connected.

The easy part of an unpublished duality theorem of Artin and Mazur asserts

(5.1) Lemma The duality of Abelian schemes

$$\underline{\text{Ext}}_U^1(N_U, G_m) \xrightarrow{\cong} N'_U$$

extends to an isomorphism of functors evaluated on smooth  $S$ -schemes:

$$\underline{\text{Ext}}_S^1(N^0, G_m) \xrightarrow{\cong} N'$$

We sketch a proof of this lemma by showing that  $\underline{\text{Ext}}_S^1(N^0, G_m)$  enjoys the Néron property [11, SGA<sub>7</sub> IX, 1]. To do this one must take  $T/S$  a smooth "test" scheme and consider the diagram with exact rows, [11, SGA<sub>7</sub> VII 1.3.5, 1.3.8]:

$$\begin{array}{ccccc}
 0 \rightarrow & \underline{\text{Ext}}_S^1(N^0, G_m)(T) & \rightarrow & \text{Pic}_S(N^0)(T) & \xrightarrow{\sigma} & \text{Pic}_S(N^0 \times_S N^0)(T) \\
 & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
 0 \rightarrow & \underline{\text{Ext}}_U^1(N^0, G_m)(T_U) & \rightarrow & \text{Pic}_U(N^0)(T_U) & \xrightarrow{\sigma} & \text{Pic}_U(N^0 \times_U N^0)(T_U)
 \end{array}$$

where  $\sigma = \text{proj}_1^* + \text{proj}_2^* - \text{sum}^*$

Since  $N^{\circ} \times_S T$  and  $N^{\circ} \times_S N^{\circ} \times_S T$  are regular schemes and since  $N^{\circ} \times_S T/T$  and  $N^{\circ} \times_S N^{\circ} \times_S T/T$  have connected geometric fibres,  $\beta$  and  $\gamma$  are isomorphisms. Thus  $\alpha$  is an isomorphism as well, and the sketch of the proof of (5.1) is concluded.

(5.2) Corollary There is an exact sequence of smooth groups/S:

$$0 \rightarrow \underline{w}_N \rightarrow \underline{\text{Extrig}}_S(N^{\circ}, G_m) \rightarrow N \rightarrow 0$$

Proof:  $f_*(\mathcal{O}_{N^{\circ}}) = \mathcal{O}_S$ . Thus there is an exact sequence of Zariski sheaves on the category of smooth  $S$ -schemes.

$$0 \rightarrow \underline{w}_N \rightarrow \underline{\text{Extrig}}(N^{\circ}, G_m) \rightarrow \underline{\text{Ext}}(N^{\circ}, G_m) \rightarrow 0$$

(c.f. the discussion preceeding (2.6.1))

From the lemma,  $N \cong \underline{\text{Ext}}(N^{\circ}, G_m)$ , and hence  $\underline{\text{Extrig}}(N^{\circ}, G_m)$  is a smooth group.

Write  $\underline{E}(N) = \underline{\text{Extrig}}_S(N^{\circ}, G_m)$ .

A surprise is that the exact sequence

$$(5.2.1) \quad 0 \rightarrow \underline{w}_N \rightarrow \underline{E}(N) \rightarrow N \rightarrow 0$$

is not necessarily the universal extension of  $N$ . In fact, as L. Breen and M. Raynaud have shown: there are Néron models  $N$  which possess no universal extension. A sketch of their elegant argument is included below. Therefore we refer to (5.2.1) as the canonical extension of a Néron model  $N$  by a vector group.

It appears to us that this canonical extension deserves systematic study, and indeed the first question one may ask about it is the following, which we pose in purposely vague language:

Find a functorial characterization of the canonical extension (5.2.1) of a Néron model.

It is especially interesting to consider the canonical extension over the base  $S = \text{Spec}(\mathbb{Z})$ .

Let  $M = N(\mathbb{Z}) \cong N(\mathbb{Q})$  denote the Mordell-Weil group of the abelian variety  $N_{\mathbb{Q}}$ . This is a finitely generated group. Let  $M^* = \underline{E}(N)(\mathbb{Z})$ . Since  $S$  is affine, (5.2.1) gives the exact sequence

$$(5.2.2) \quad 0 \rightarrow \underline{w}_N(\mathbb{Z}) \rightarrow M^* \rightarrow M \rightarrow 0$$

Since  $\underline{w}_N(\mathbb{Z})$  is a free abelian group whose rank is  $\dim N = d$ , we see that  $M^*$  is a finitely generated abelian group of rank  $d + \text{rank}(M)$ . What is curious is that  $M^*$  has a strong tendency to be free. Explicitly:

(5.3) Theorem: If  $p$  divides the order of the torsion subgroup of  $M^*$  then either  $p = 2$  or  $p$  is a prime of bad reduction for  $N$ .

(5.4) Corollary: If the order of the torsion subgroup of  $M$  is relatively prime to

$2 \times$  product of primes of bad reduction of  $N$

then  $M^*$  is a free abelian group.

Proof of Theorem: Let  $x^* \in M^*$  be a nontrivial element of order  $p$ . Since  $E(N)$  is separated it suffices to show  $x^*$  is zero, after having base changed to  $S = \text{Spec}(\mathbb{Z}_p)$ .

By our assumption,  $N$  is an abelian scheme over  $S$ , and  $E(N)$  is the universal extension of  $N$ . Let  $N(p)_{/S}$  be the Barsotti-Tate group associated to the abelian scheme  $N/S$ . Then over  $S_\nu = \text{Spec}(\mathbb{Z}/p^\nu)$  for any  $\nu$ ,  $E(N)(p)$  is the universal extension of the Barsotti-Tate group  $N(p)$ . The element  $x^* \in M^*$  may be viewed as a section of  $E(N)(p)$  and its image,  $y$ , in  $N(p)$  generates a finite flat group  $G$  over  $S$  of order  $p$ . Since  $p \neq 2$ , and since  $G$  has a nontrivial rational section, by the classification theory of finite flat groups of order  $p$  over  $\mathbb{Z}_p$  [20, Theorem 2],  $G = \mathbb{Z}/p$ .

Let the subscript  $\nu \geq 1$  denote restriction to the base  $S_\nu = \text{Spec} \mathbb{Z}/p^\nu$ .

Let  $N(p)^{\text{et}}$  denote the étale quotient of  $N(p)$ , and let  $E(N(p)^{\text{et}})_\nu$  denote the universal extension of  $N(p)_\nu^{\text{et}}$ . We have the diagram

$$\begin{array}{ccc} E(N)(p)_\nu & \longrightarrow & N(p)_\nu \\ \downarrow & & \downarrow \\ E(N(p)^{\text{et}})_\nu & \longrightarrow & N(p)_\nu^{\text{et}} \end{array}$$

Since  $G = \mathbb{Z}/p$  the image of  $G$  in  $N(p)^{\text{et}}$  is nonzero. Consequently the image of the section  $x^*$  in  $N(p)^{\text{et}}$  is nonzero. It follows that the image of  $x^*$  in  $E(N(p)^{\text{et}})_\nu$  is nonzero. But this is a contradiction because the universal extension of an étale  $p$ -divisible group over  $\hat{S} = \text{Spf}(\mathbb{Z}_p)$  has no nontrivial section of order  $p$ .

(5.5) As a special case of the above theorem, take an elliptic curve  $C$  over  $\mathbb{Q}$  whose Mordell-Weil group is a finite group  $F$  of odd order relatively prime to the conductor of  $C$ .

Since any odd finite group of real points of  $C$  is cyclic,  $F$  is a cyclic group.

Making a choice of sign of the Néron differential of  $C$  enables us to identify  $\omega_N(\mathbb{Z}) \cong \mathbb{Z}$  (where  $N$  is the Néron model of  $C$ ) and consequently the exact sequence (5.2.2) becomes

$$(5.5.1) \quad 0 \rightarrow \mathbb{Z} \rightarrow M^* \rightarrow F \rightarrow 0$$

But the theorem implies, under our hypotheses that  $M^*$  is free, and consequently the exact sequence (5.5.1) becomes:

$$(5.5.2) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{\pi} F \rightarrow 0$$

where  $\varphi$  consists in multiplication by the order of  $F$ .

Consequently the canonical extension of the Néron model of  $C$  determines in this case a canonical free resolution of the Mordell-Weil group of  $C$ . In particular, choosing a Néron differential of  $C$  (there are two possible choices, and to choose one of these two amounts to the same as orienting the real locus of  $C$ ) gives (in the case considered above) a canonical generator of the Mordell-Weil group, defined to be the image of  $1 \in \mathbb{Z}$  under  $\pi$  in (5.5.2). (call this the generator defined by the canonical extension) It may occur to the reader that the topology of the real locus of  $C$  enables one to obtain yet another canonical generator of  $F$ : Since  $F$  is a finite subgroup of the connected component of the real locus of  $C$ , which is a circle (oriented, after a choice of Néron differential), it

makes sense to consider that element of  $F$ , closest to the origin in the circle, where "closest" means in the direction of orientation of the circle. Call this the topologically-defined generator.

Tate has made some computations which abundantly support the opinion that there is no relation at all between the generator defined by the canonical extension and the topologically defined generator.

(5.6) Example of Breen and Raynaud.

The following is taken from a letter of L. Breen.

Let  $R$  be a discrete valuation ring with uniformizer  $\pi$  and residue field  $k$ . Let  $N/R$  be the Néron model of an elliptic curve. Let  $\bar{N}$  denote its fibre at  $k$ . Suppose one of two special cases

$$\text{I) } \bar{N} = G_a$$

$$\text{II) } \bar{N} = G_m$$

Consider the short exact sequence of Zariski sheaves on the smooth site over  $S = \text{Spec } R$ ,

$$0 \rightarrow G_a \xrightarrow{\text{mult. by } \pi} G_a \rightarrow i_* G_a \rightarrow 0$$

(Here  $i : \text{Spec } k \rightarrow S$  is the canonical injection).

This induces the exact sequence

$$0 \rightarrow \text{Hom}_S(N, i_* G_a) \rightarrow \text{Ext}^1(N, G_a) \xrightarrow{\pi} \text{Ext}^1(N, G_a) \rightarrow \text{Ext}_S^1(N, i_* G_a)$$

But

$$\begin{aligned} \text{Hom}_S(N, i_* G_a) &= \text{Hom}_k(\bar{N}, G_a) \\ \text{Ext}_S^1(N, i_* G_a) &= \text{Ext}_k^1(\bar{N}, G_a) \end{aligned}$$

and consequently

(\*)  $\text{Ext}^1(N, G_a) \xrightarrow{\pi} \text{Ext}^1(N, G_a)$  enjoys the following properties in each of our two special cases:

Case I: (\*) is not injective

Case II: (\*) is surjective

(5.6.1) Corollary: In either case,  $\text{Ext}^1(N, G_a)$  is not a locally free sheaf of  $\mathcal{O}_S$ -modules, and there is no universal extension of  $N$  by a vector group (over  $R$ ).

Proof: If  $\text{Ext}^1(N, G_a)$  were locally free then  $\text{Ext}^1(N, G_a) = H^0(S, \text{Ext}^1(N, G_a))$  would be a free  $R$ -module and consequently multiplication by  $\pi$  would be injective and not surjective on it. (N.B.  $\text{Ext}^1(N, G_a)$  is not zero since the canonical extension is non-trivial). Moreover if there were an extension of  $N$  by a vector group  $V$ , which was universal we would have

$$\text{Hom}(V, G_a) \cong \text{Ext}^1(N, G_a)$$

consider

$$\text{Hom}_S(V, G_a) \xrightarrow{\pi} \text{Hom}_S(V, G_a).$$

Since  $V$  is a vector group,  $\pi$  is not surjective and is injective, contradicting the situation that obtained in either case I or case II.