$$\begin{array}{ccc} p_1^*\mathscr{F}' & \xrightarrow{\ \varphi\ } & p_2^*\mathscr{F}' \\ \downarrow{\scriptstyle p_1^* f} & & \downarrow{\scriptstyle p_2^* f} \\ p_1^*\mathscr{G}' & \xrightarrow{\ \psi\ } & p_2^*\mathscr{G}' \end{array}$$

is commutative.

Starting with a quasi-coherent S-module $\mathscr{F}$, we have a natural covering datum on $p^*\mathscr{F}$, which consists of the canonical isomorphism

$$p_1^*(p^*\mathscr{F}) \cong (p\circ p_1)^*\mathscr{F} = (p\circ p_2)^*\mathscr{F} \cong p_2^*(p^*\mathscr{F})\ .$$

So we can interpret the functor $\mathscr{F}\longmapsto p^*\mathscr{F}$ as a functor into the category of quasi-coherent S'-modules with covering data. It is this functor which will be of interest in the following. We will show that it is fully faithful if $p: S' \longrightarrow S$ is faithfully flat and quasi-compact, and that, furthermore, it is an equivalence of categories if, instead of covering data, we consider descent data; i.e., special covering data which satisfy a certain cocycle condition. The problem of descent can be viewed as a natural generalization of a patching problem; cf. Example 6.2/A.

As usual we will call a diagram

$$A \xrightarrow{\ \alpha\ } B \underset{\gamma}{\overset{\beta}{\rightrightarrows}} C$$

of maps between sets *exact* if α is injective and if $\operatorname{im}\alpha = \ker(\beta,\gamma)$, where $\ker(\beta,\gamma)$ consists of all elements $b\in B$ such that $\beta(b)=\gamma(b)$. Working in the category of abelian groups, the exactness of such a diagram is equivalent to the exactness of the sequence

$$O \longrightarrow A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta-\gamma\ } C\ .$$

Proposition 1. *Assume that $p: S'\longrightarrow S$ is faithfully flat and quasi-compact. Let $\mathscr{F}$ and $\mathscr{G}$ be quasi-coherent S-modules, and set $q := p\circ p_1 = p\circ p_2$. Then, identifying $q^*\mathscr{F}$ canonically with $p_i^*(p^*\mathscr{F})$ for $i=1,2$, likewise for $q^*\mathscr{G}$, the diagram*

$$\operatorname{Hom}_S(\mathscr{F},\mathscr{G}) \xrightarrow{\ p^*\ } \operatorname{Hom}_{S'}(p^*\mathscr{F},p^*\mathscr{G}) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \operatorname{Hom}_{S''}(q^*\mathscr{F},q^*\mathscr{G})$$

is exact. In other words, the functor $\mathscr{F}\longmapsto p^*\mathscr{F}$ from quasi-coherent S-modules to quasi-coherent S'-modules with covering data is fully faithful.

Proof. The assertion is local on S, so we can assume that S is affine. Then S' is quasi-compact, and it is covered by finitely many affine open subschemes $S_i'\subset S'$, $i\in I$. Consider the disjoint union $\overline{S}' := \coprod_{i\in I} S_i'$ of these schemes.

Let $u:\overline{S}'\longrightarrow S'$ be the canonical morphism, $\overline{p}:\overline{S}'\longrightarrow S$ its composition with $p: S'\longrightarrow S$, and let $\overline{p}_1, \overline{p}_2$ denote the projections of $\overline{S}'' := \overline{S}'\times_S \overline{S}'$ onto its factors. Then we obtain a diagram

$$\begin{array}{ccccc} \operatorname{Hom}_S(\mathscr{F},\mathscr{G}) & \xrightarrow{\ p^*\ } & \operatorname{Hom}_{S'}(p^*\mathscr{F},p^*\mathscr{G}) & \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} & \operatorname{Hom}_{S''}(q^*\mathscr{F},q^*\mathscr{G}) \\ \| & & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle (u\times u)^*} \\ \operatorname{Hom}_S(\mathscr{F},\mathscr{G}) & \xrightarrow{\ \overline{p}^*\ } & \operatorname{Hom}_{\overline{S}'}(\overline{p}^*\mathscr{F},\overline{p}^*\mathscr{G}) & \underset{\overline{p}_2^*}{\overset{\overline{p}_1^*}{\rightrightarrows}} & \operatorname{Hom}_{\overline{S}''}(\overline{q}^*\mathscr{F},\overline{q}^*\mathscr{G}) \end{array}$$

where $\overline{q} := \overline{p}\circ\overline{p}_1 = \overline{p}\circ\overline{p}_2$. The diagram is commutative if, in the right-hand square, we consider single horizontal arrows, either p_1^* and $\overline{p}_1^*$ or p_2^* and $\overline{p}_2^*$. Furthermore, u being faithfully flat, the vertical maps are injective. Using this fact, it is easily checked that the upper row is exact if the lower row has this property. In other words, we may replace $p: S'\longrightarrow S$ by $\overline{p}:\overline{S}'\longrightarrow S$ and thereby assume that S and S' are affine, say $S=\operatorname{Spec} R$ and $S'=\operatorname{Spec} R'$. Then the problem becomes a problem on R-modules.

Let

$$(*)\qquad R \longrightarrow R' \rightrightarrows R'\otimes_R R'$$

be the diagram which corresponds to the projections $S''\rightrightarrows S'\longrightarrow S$. We claim that the assertion of the proposition follows if we can show that the tensor product of $(*)$ with any R-module M yields an exact diagram. Namely, consider R-modules M and N such that $\mathscr{F}$ (resp. $\mathscr{G}$) is associated to M (resp. N), and assume that we have exact diagrams

$$M \longrightarrow M\otimes_R R' \rightrightarrows M\otimes_R R'\otimes_R R'\ ,$$

$$N \longrightarrow N\otimes_R R' \rightrightarrows N\otimes_R R'\otimes_R R'\ .$$

Then the injectivity of $N\longrightarrow N\otimes_R R'$ implies the injectivity of the map p^* in the assertion. Similarly, it is seen that any R'-homomorphism $M\otimes_R R'\longrightarrow N\otimes_R R'$, which corresponds to an element in $\ker(p_1^*,p_2^*)$, restricts to an R-homomorphism $M\longrightarrow N$. This yields $\operatorname{im} p^* \supset \ker(p_1^*,p_2^*)$. Since the opposite inclusion is trivial, our claim is justified. So, in order to finish the proof of the proposition, it remains to establish the following result:

Lemma 2. *Let $R\longrightarrow R'$ be a faithfully flat morphism of rings. Then, for any R-module M, the canonical diagram*

$$M \longrightarrow M\otimes_R R' \rightrightarrows M\otimes_R R'\otimes_R R'$$

is exact.

Proof. We may apply a faithfully flat base change over R, say with R'. Thereby we can assume that $R\longrightarrow R'$ admits a section $R'\longrightarrow R$. So all the maps in the above diagram have sections, and the exactness is obvious. □

Next we want to introduce descent data and the cocycle condition characterizing them. Set $S''' := S'\times_S S'\times_S S'$, and let $p_{ij}: S'''\longrightarrow S''$ be the projections onto the factors with indices i and j for $i<j$; $i,j=1,2,3$. In order that a quasi-coherent S'-module $\mathscr{F}'$ with covering datum $\varphi: p_1^*\mathscr{F}'\longrightarrow p_2^*\mathscr{F}'$ belongs to the essential image of the functor $\mathscr{F}\longmapsto p^*\mathscr{F}$, it is necessary that the diagram

$$\begin{array}{ccccc} p_{12}^*p_1^*\mathscr{F}' & \xrightarrow{\ p_{12}^*\varphi\ } & p_{12}^*p_2^*\mathscr{F}' = p_{23}^*p_1^*\mathscr{F}' & \xrightarrow{\ p_{23}^*\varphi\ } & p_{23}^*p_2^*\mathscr{F}' \\ \| & & & & \| \\ p_{13}^*p_1^*\mathscr{F}' & & \xrightarrow{\qquad p_{13}^*\varphi\qquad} & & p_{13}^*p_2^*\mathscr{F}' \end{array}$$

is commutative; the unspecified identities are the canonical ones. Namely, if $\mathscr{F}'$ is the pull-back under p of a quasi-coherent S-module and if φ is the natural covering datum on $\mathscr{F}'$, then the diagram is commutative, because all occurring isomorphisms are the canonical ones. The commutativity of the above diagram is referred to as the *cocycle condition* for φ; in short, we can write it as

$$p_{13}^*\varphi = p_{23}^*\varphi \circ p_{12}^*\varphi \ .$$

It corresponds to the usual cocycle condition on triple overlaps when a global object is to be constructed by gluing local parts. A covering datum φ on $\mathscr{F}'$ which satisfies the cocycle condition is called a *descent datum* on $\mathscr{F}'$. The descent datum is called *effective* if the pair $(\mathscr{F}', \varphi)$ is isomorphic to the pull-back $p^*\mathscr{F}$ of a quasi-coherent S-module $\mathscr{F}$ where, on $p^*\mathscr{F}$, we consider the canonical descent datum. Also we want to mention that the notions of covering and descent data are compatible with base change over S.

In the case where S and S' are affine, covering and descent data can be described in terms of modules over rings. Namely, let $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R'$, and consider a quasi-coherent S'-module $\mathscr{F}'$ with a covering datum $\varphi : p_1^*\mathscr{F}' \longrightarrow p_2^*\mathscr{F}'$, where $\mathscr{F}'$ is associated to the R'-module M'. Then $p_1^*\mathscr{F}'$ and $p_2^*\mathscr{F}'$ are associated to $M' \otimes_R R'$ and $R' \otimes_R M'$, both of which are viewed as $R' \otimes_R R'$-modules. Thus the covering datum φ on $\mathscr{F}'$ corresponds to an $R' \otimes_R R'$-isomorphism

$$M' \otimes_R R' \xrightarrow{\sim} R' \otimes_R M'$$

which, again, will be denoted by φ. Using the canonical map $M' \longrightarrow M' \otimes_R R'$ as well as the composition of the canonical map $M' \longrightarrow R' \otimes_R M'$ with φ^{-1}, we arrive at a *co-cartesian* diagram $M' \rightrightarrows M' \otimes_R R'$ over the canonical diagram $R' \rightrightarrows R' \otimes_R R'$. This means that, considering associated arrows in both diagrams, $M' \otimes_R R$ is obtained from M' by tensoring with $R' \otimes_R R'$ over R'. Conversely, any such co-cartesian diagram determines a covering datum on M' and, hence, on $\mathscr{F}'$.

If φ is a descent datum on $\mathscr{F}'$, we can pull it back with respect to the projections $p_{ij} : S''' \longrightarrow S''$. Due to the cocycle condition, the various pull-backs of $\mathscr{F}'$ to S''' can be identified via the pull-backs of φ. Thereby we obtain in a canonical way homomorphisms (depending on φ)

$$M' \otimes_R R' \overrightarrow{\rightrightarrows} M' \otimes_R R' \otimes_R R'$$

such that the diagram

$$(*) \qquad M' \rightrightarrows M' \otimes_R R' \overrightarrow{\rightrightarrows} M' \otimes_R R' \otimes_R R'$$

is co-cartesian over the canonical diagram

$$(**) \qquad R' \rightrightarrows R' \otimes_R R' \overrightarrow{\rightrightarrows} R' \otimes_R R' \otimes_R R' .$$

Furthermore, $(*)$ satisfies certain natural commutativity conditions just as we have them for $(**)$ or for the associated diagram

$$S''' \overrightarrow{\rightrightarrows} S'' \rightrightarrows S' ,$$

where $p_1 \circ p_{12} = p_1 \circ p_{13}$, etc. Conversely, one can show that each co-cartesian diagram $(*)$ over $(**)$, which satisfies the commutativity conditions, determines a

descent datum on M', and hence on $\mathscr{F}'$. It is clear that a descent datum φ on $\mathscr{F}'$ is effective if and only if the associated co-cartesian diagram $M' \rightrightarrows M' \otimes_R R'$ can be enlarged into a commutative co-cartesian diagram

$$M \longrightarrow M' \rightrightarrows M' \otimes_R R'$$

over the canonical diagram

$$R \longrightarrow R' \rightrightarrows R' \otimes_R R' .$$

Returning to the case where S and S' are arbitrary schemes, it is sometimes convenient to formulate the cocycle condition within the context of T-valued points of S', where T is an arbitrary S-scheme. So consider a quasi-coherent S'-module $\mathscr{F}'$ with a covering datum $\varphi : p_1^*\mathscr{F}' \longrightarrow p_2^*\mathscr{F}'$. For $t_1, t_2 \in S'(T)$, denote by

$$\varphi_{t_1,t_2} : t_1^*\mathscr{F}' \longrightarrow t_2^*\mathscr{F}'$$

the pull-back of φ under the morphism $(t_1, t_2) : T \longrightarrow S''$. Adding a third point $t_3 \in S'(T)$, we can consider the morphism

$$(t_1, t_2, t_3) : T \longrightarrow S'''$$

and compose it with each one of the projections $S''' \overrightarrow{\rightrightarrows} S''$. Then, pulling back φ to T, we see that φ satisfies the cocycle condition if and only if

$$\varphi_{t_1,t_3} = \varphi_{t_2,t_3} \circ \varphi_{t_1,t_2}$$

for all $t_1, t_2, t_3 \in S'(T)$ and all T. In particular, for $t = t_1 = t_2 = t_3$, the cocycle condition implies $\varphi_{t,t} = \varphi_{t,t}^2$ and, hence, $\varphi_{t,t} = \mathrm{id}$. For example, if $t : S' \longrightarrow S'$ is the universal point of S', we see that the pull-back of a descent datum $\varphi : p_1^*\mathscr{F}' \longrightarrow p_2^*\mathscr{F}'$ with respect to the diagonal morphism $\Delta : S' \longrightarrow S''$ yields the identity on $\mathscr{F}'$.

Lemma 3. *Assume that the morphism $p : S' \longrightarrow S$ admits a section. Then any descent datum φ on a quasi-coherent S'-module $\mathscr{F}'$ is effective. More precisely, the choice of a section $s : S \longrightarrow S'$ of p determines an S-module $\mathscr{F}$, namely $\mathscr{F} := s^*\mathscr{F}'$, such that $p^*\mathscr{F}$ is isomorphic to the pair $(\mathscr{F}', \varphi)$.*

Proof. Writing $T := S'$, let us consider the points $t := \mathrm{id}_{S'}$ and $\tilde{t} := s \circ p$ of $S'(T)$. Then $t^*\mathscr{F}' = \mathscr{F}'$ and $\tilde{t}^*\mathscr{F}' = p^*\mathscr{F}$, and we can consider the isomorphism

$$f = \varphi_{t,\tilde{t}} : \mathscr{F}' \xrightarrow{\sim} p^*\mathscr{F} .$$

It is enough to show that f is compatible with the descent datum on $p^*\mathscr{F}$; i.e., we have to show that the diagram

$$\begin{array}{ccc} p_1^*\mathscr{F}' & \xrightarrow{\ \varphi\ } & p_2^*\mathscr{F}' \\ \big\downarrow {\scriptstyle p_1^*f} & & \big\downarrow {\scriptstyle p_2^*f} \\ p_1^*p^*\mathscr{F} & = & p_2^*p^*\mathscr{F} \end{array}$$

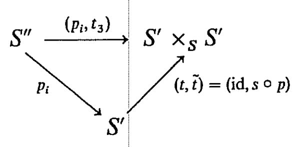
is commutative. In order to do this, consider the following S''-valued points of S':

$$p_1, \quad p_2, \quad \text{and} \quad t_3 := s \circ p \circ p_1 = s \circ p \circ p_2 .$$

Then $\varphi = \varphi_{p_1,p_2}$ since $(p_1,p_2) : S'' \longrightarrow S''$ is the identity, and we have

$$p_i^* f = p_i^* \varphi_{t,\tilde{t}} = \varphi_{p_i,t_3} \qquad \text{for } i = 1, 2 ,$$

since the diagram



is commutative. Now the cocycle condition for φ yields

$$\varphi_{p_1,t_3} = \varphi_{p_2,t_3} \circ \varphi_{p_1,p_2}$$

and thus

$$p_1^* f = p_2^* f \circ \varphi . \qquad \square$$

Now we are ready to prove the desired result on the descent of quasi-coherent S'-modules.

Theorem 4 (Grothendieck). *Let $p : S' \longrightarrow S$ be faithfully flat and quasi-compact. Then the functor $\mathscr{F} \longmapsto p^*\mathscr{F}$, which goes from quasi-coherent S-modules to quasi-coherent S'-modules with descent data, is an equivalence of categories.*

Proof. We know already from Proposition 1 that the functor in question is fully faithful. So it is enough to show that each descent datum on a quasi-coherent S'-module is effective. The latter is clear by Lemma 3 if $p : S' \longrightarrow S$ admits a section. We will reduce to this case.

First observe that we may replace the morphism $p : S' \longrightarrow S$ by a composition $\bar{p} : \bar{S}' \xrightarrow{u} S' \xrightarrow{p} S$, where $u : \bar{S}' \longrightarrow S'$ is faithfully flat and quasi-compact. This is true since the functor $\mathscr{F}' \longmapsto u^*\mathscr{F}'$ is fully faithful (see Proposition 1) and since descent data on $\mathscr{F}'$ (with respect to p) can easily be pulled back to descent data on $u^*\mathscr{F}'$ (with respect to $\bar{p}$). So, proceeding as in the proof of Proposition 1, we may assume that S and S' are affine, say $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$.

Let M' be an R'-module with descent datum $\varphi : M' \otimes_R R' \xrightarrow{\sim} R' \otimes_R M'$. Then φ determines a co-cartesian diagram $M' \rightrightarrows M' \otimes_R R'$ over $R' \rightrightarrows R' \otimes_R R'$. If M' descends to an R-module, we know from Lemma 2 that it must descend to the R-module

$$K := \ker(M' \rightrightarrows M' \otimes_R R') .$$

So let us work with this module. We claim that the diagram

$$K \longrightarrow M' \rightrightarrows M' \otimes_R R'$$

is commutative and co-cartesian over

$$R \longrightarrow R' \rightrightarrows R' \otimes_R R'$$

and, hence, that φ is effective. In order to verify this, we may apply a faithfully flat base change and thereby assume that $R \longrightarrow R'$ admits a section. Then it

follows from Lemma 3 that (M', φ) descends to an R-module M. More precisely, $M' \rightrightarrows M' \otimes_R R'$ extends to a commutative co-cartesian diagram

$$M \longrightarrow M' \rightrightarrows M' \otimes_R R'$$

over

$$R \longrightarrow R' \rightrightarrows R' \otimes_R R' .$$

Since M is mapped bijectively onto K by Lemma 2, our claim is justified. $\square$

Keeping the morphism $p : S' \longrightarrow S$, we want to study the problem of when an S'-scheme X' descends to an S-scheme X. The general setting will be the same as in the case of quasi-coherent modules, and the definitions we have given can easily be adapted to the new situation. For example, a *descent datum on an S'-scheme X'* is an S''-isomorphism

$$\varphi : p_1^* X' \longrightarrow p_2^* X'$$

which satisfies the cocycle condition; $p_i^* X'$ is the scheme obtained from X' by applying the base change $p_i : S'' \longrightarrow S'$. Again there is a canonical functor $X \longmapsto p^*X$ from S-schemes to S'-schemes with descent data. If $p : S' \longrightarrow S$ is faithfully flat and quasi-compact, we see from Theorem 4 that this functor gives an equivalence between affine S-schemes and affine S'-schemes with descent data. More generally, the same assertion is true with affine replaced by quasi-affine (use Theorem 6(b) below). Thus, in this case, descent data on affine or quasi-affine S'-schemes are always effective. Recall that an S'-scheme X' is called affine (resp. quasi-affine) over S' if, for each affine open subscheme $S'_0 \subset S'$, the open subscheme $S'_0 \times_{S'} X'$ of X' is affine (resp. quasi-affine). To be precise, one has, of course, to mention the fact that one can easily generalize Theorem 4 from quasi-coherent modules to quasi-coherent algebras, so that it can be applied to structure sheaves of schemes over S or S'. Working with an additional structure such as a multiplication on a quasi-coherent S'-module, this structure descends if it is compatible with the descent datum.

It is not true that descent data on schemes are always effective, even if $p : S' \longrightarrow S$ is faithfully flat and quasi-compact; see Section 6.7. So one needs criteria for effectiveness. First we mention that Lemma 3 carries over to the scheme situation. Since the proof was given by formal arguments, no changes are necessary.

Lemma 5. *Assume that $p : S' \longrightarrow S$ has a section. Then all descent data on S'-schemes are effective.*

In order to formulate another criterion, consider an S'-scheme X' with a descent datum $\varphi : p_1^* X' \longrightarrow p_2^* X'$, and let U' be an open subscheme of X'. Then U' is called *stable under* φ if φ induces a descent datum on U'; i.e., if φ restricts to an isomorphism $p_1^* U' \xrightarrow{\sim} p_2^* U'$.

Theorem 6. *Let $p : S' \longrightarrow S$ be faithfully flat and quasi-compact.*

(a) *The functor $X \longmapsto p^*X$ from S-schemes to S'-schemes with descent data is fully faithful.*

(b) *To simplify, assume S and S' affine. Then a descent datum φ on an S'-scheme X' is effective if and only if X' can be covered by quasi-affine* (or, *alternatively, by affine*) *open subschemes which are stable under φ.*

Proof. Assertion (a) is an immediate consequence of Proposition 1. Namely, consider S-schemes X and Y, and write X', Y' for the schemes obtained by the base change $p: S' \longrightarrow S$. Then it is to show that the sequence

$$\operatorname{Hom}_S(X,Y) \xrightarrow{p^*} \operatorname{Hom}_{S'}(X',Y') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \operatorname{Hom}_{S''}(X'',Y'')$$

is exact. The problem is local on S and Y. So we may assume that S and Y are affine. Furthermore, replacing S' by a finite disjoint sum of affine open parts of S', we may assume that S' is affine. Then, up to a local consideration on X, we can pose the problem in terms of quasi-coherent algebras so that Proposition 1 can be applied.

In order to verify the if-part of assertion (b), we may use (a) and assume that X' is quasi-affine. This means that X' is quasi-compact and can be realized as an open subscheme of an affine scheme or, equivalently, that the canonical map

$$X' \longrightarrow \operatorname{Spec} \Gamma(X', \mathcal{O}_{X'}) =: Z'$$

is a quasi-compact open immersion; cf. [EGA II], 5.1.2. Let $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$. Then, using the fact that, for quasi-compact R'-schemes, the functor of global sections commutes with flat extensions of R', the descent datum on X' gives a descent datum on the R'-module $\Gamma(X', \mathcal{O}_{X'})$ and hence on the affine S'-scheme Z'. Thus it follows from Theorem 4 that Z' descends to an affine S-scheme Z. Considering the canonical projections

$$Z'' \underset{q_2}{\overset{q_1}{\rightrightarrows}} Z' \xrightarrow{q} Z ,$$

where Z'' is obtained from Z by the base change $S'' \longrightarrow S$, we see $q_1^{-1}(X') = q_2^{-1}(X')$ since the descent datum of Z' is stable on X'. However, this implies $q^{-1}(q(X')) = X'$; in particular, the inverse image of $q(X')$ with respect to q is open. Using the fact that $q: Z' \longrightarrow Z$ is faithfully flat and quasi-compact and that therefore the Zariski topology on Z is the quotient of the Zariski topology on Z' (cf. [EGA IV_2], 2.3.12), we see that $q(X')$ is open. So X' descends to the quasi-affine piece $q(X')$ of Z. The only-if-part of assertion (b) is trivial. □

We want to add a criterion for the effectiveness of descent data on schemes which uses ample line bundles. Let us recall the definition of ampleness, cf. [EGA II], 4.5 and 4.6. An invertible sheaf $\mathscr{L}$ on a scheme X is called *ample* on X if X is quasi-compact and quasi-separated, and if for some $n > 0$ there are global sections $l_1, \ldots, l_r$ generating $\mathscr{L}^{\otimes n}$ such that X_{l_i}, the domain where the section l_i generates $\mathscr{L}^{\otimes n}$, is quasi-affine for each i. In fact, if $\mathscr{L}$ is ample on X, then, for any $n > 0$ and any global section l of $\mathscr{L}^{\otimes n}$, the open subscheme $X_l \subset X$ is quasi-affine as will follow from arguments given below. An invertible sheaf $\mathscr{L}$ on an S-scheme X is called *S-ample on X* (or *relatively ample over S*) if there exists an affine open covering $\{S_j\}$ of S such that the restriction of $\mathscr{L}$ onto $X \times_S S_j$ is ample for all j. The definition of S-ampleness is independent of the choice of the particular covering $\{S_j\}$, see [EGA

II], 4.6.4 and 4.6.6. If X admits an S-ample sheaf, then, by definition, it is automatically quasi-separated over S.

Consider now a quasi-compact and quasi-separated morphism $f: X \longrightarrow S$ and an invertible sheaf $\mathscr{L}$ on X. For each $n \in \mathbb{N}$, the direct image $f_*\mathscr{L}^{\otimes n}$ is a quasi-coherent sheaf on S, see [EGA I], 9.2.1. Let U_n be the open set of all points $x \in X$ such that the canonical morphism

$$(f^*f_*(\mathscr{L}^{\otimes n}))_x \longrightarrow \mathscr{L}_x^{\otimes n}$$

is surjective. Then U_n consists of all points $x \in X$ such that there is a section of $\mathscr{L}^{\otimes n}$ which is defined over the f-inverse of a neighborhood of $f(x)$ in S and which generates $\mathscr{L}^{\otimes n}$ at x. Denote by U the union of all U_n for $n \geq 1$. Let

$$\mathscr{M} = \bigoplus_{n \geq 0} f_*(\mathscr{L}^{\otimes n})$$

be the quasi-coherent graded S-algebra associated to $\mathscr{L}$, and set $P = \operatorname{Proj} \mathscr{M}$; see [EGA II], § 2. There is always a canonical S-morphism $r: U \longrightarrow P$. Namely, assuming S affine, one shows for each global section l of $\mathscr{L}^{\otimes n}$ with $n > 0$ that there is a canonical isomorphism

$$\Gamma(P_l, \mathcal{O}_P) \xrightarrow{\sim} \Gamma(X_l, \mathcal{O}_X) ,$$

use [EGA I], 9.3.1, and hence a morphism

$$X_l \longrightarrow P_l \hookrightarrow P .$$

The morphism is an open immersion if and only if X_l is quasi-affine over S. Thereby it is seen that *the sheaf $\mathscr{L}$ is S-ample on X if and only if $U = X$ and the canonical morphism $r: U \longrightarrow P$ is an open immersion.*

Returning to the problem of descent relative to a morphism $p: S' \longrightarrow S$, the notion of descent data generalizes naturally to pairs $(X', \mathscr{L}')$ where X' is an S'-scheme and $\mathscr{L}'$ is an invertible sheaf on X'. Namely, a descent datum on such a pair consists of a descent datum

$$\varphi: p_1^*X' \longrightarrow p_2^*X'$$

on X' and of an isomorphism

$$\lambda: \mathscr{L}_1' \longrightarrow \varphi^*\mathscr{L}_2'$$

where $\mathscr{L}_i'$ is the pull-back of $\mathscr{L}'$ with respect to the projection $p_i^*X' \longrightarrow X'$. Of course, λ must satisfy the cocycle condition, which is a cocycle condition over the cocycle condition for φ. More precisely, introducing the total space L' associated to $\mathscr{L}'$, we can say that a descent datum on $(X', \mathscr{L}')$ is a commutative diagram

$$\begin{array}{ccc} p_1^*L' & \xrightarrow{\lambda} & p_2^*L' \\ \downarrow & & \downarrow \\ p_1^*X' & \xrightarrow{\varphi} & p_2^*X' , \end{array}$$

where the vertical maps are the projections of the linear fibre spaces p_i^*L' onto their bases $p_i^* X'$, where φ and λ are descent data for schemes, and where λ is an isomorphism of linear fibre spaces over φ. Another possibility is to view the descent datum φ as a cartesian diagram

$$\begin{array}{ccccc} X' \times_S S' \times_S S' & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & X' \times_S S' & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & X' \\ \downarrow & & \downarrow & & \downarrow \\ S''' & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & S'' & \rightrightarrows & S' \end{array}$$

with natural commutativity conditions (similar to what we have explained for S'-modules), and to view λ as an isomorphism

$$\lambda : q_1^* \mathscr{L}' \longrightarrow q_2^* \mathscr{L}' .$$

The cocycle condition for λ can then be formulated as usual by using pull-backs with respect to the projections $X' \times_S S' \times_S S' \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} X' \times_S S'$.

Theorem 7 (Grothendieck). *Let $p : S' \longrightarrow S$ be faithfully flat and quasi-compact. Let X' be a quasi-compact S'-scheme, and consider an invertible sheaf $\mathscr{L}'$ which is S'-ample on X'. Then, if there is a descent datum on $(X', \mathscr{L}')$, the descent is effective on X', and the pair $(X', \mathscr{L}')$ descends to a pair $(X, \mathscr{L})$ with an S-ample invertible sheaf $\mathscr{L}$ on X.*

We give only a sketch of proof for the case where S and S' are affine. First, using Theorem 4, the graded S'-algebra $\mathscr{M}' = \bigoplus_{n \ge 0} f'_*(\mathscr{L}'^{\otimes n})$, where $f' : X' \longrightarrow S'$ is the structural morphism, descends to a graded S-algebra $\mathscr{M} = \bigoplus_{n \ge 0} \mathscr{M}_n$. Next, let l' be a global section in some $\mathscr{L}'^{\otimes n}$. Then we can write

$$l' = \sum_{i=1}^{m} a_i \otimes l_i$$

with global sections a_i of $\mathscr{O}_{S'}$ and global sections l_i of $\mathscr{M}_n$. If l' generates $\mathscr{L}'^{\otimes n}$ at a certain point $x \in X'$, at least one of the global sections $1 \otimes l_i$ must generate $\mathscr{L}'^{\otimes n}$ at this point. Thereby it is seen that X' can be covered by quasi-affine open pieces X'_l where l is a global section in some $\mathscr{L}'^{\otimes n}$ which descends to a global section in $\mathscr{M}$. Then the descent datum is stable on the X'_l, and X' descends to X by Theorem 6. Finally, $\mathscr{L}'$ descends to $\mathscr{L}$ with respect to the canonical projection $X' \longrightarrow X$ since one can use Theorem 4 again.

6.2 Some Standard Examples of Descent

We start with an example which shows that the problem of descent occurs as a natural generalization of a patching problem.

Example A (*Zariski coverings*). Consider a quasi-separated scheme S and a finite affine open covering $(S_i)_{i \in I}$ of S. Let $S' := \coprod_{i \in I} S_i$ be the disjoint union of the S_i, and let $p : S' \longrightarrow S$ be the canonical projection. Note that p is faithfully flat and quasi-compact. A quasi-coherent S'-module $\mathscr{F}'$ may be thought of as a family of S_i-modules $\mathscr{F}_i$. Under what conditions does $\mathscr{F}'$ descend to a quasi-coherent S-module $\mathscr{F}$; i.e., under what conditions can one glue the $\mathscr{F}_i$ in order to obtain a quasi-coherent S-module $\mathscr{F}$ from them? By Theorem 6.1/4 we need a descent datum for $\mathscr{F}'$ with respect to $p : S' \longrightarrow S$. Such a datum consists of an isomorphism $\varphi : p_1^* \mathscr{F}' \xrightarrow{\sim} p_2^* \mathscr{F}'$ satisfying the cocycle condition, where p_1 and p_2 are the projections from S'' onto S'. In our case, we have

$$S'' = S' \times_S S' = \coprod_{i,j \in I} S_i \times_S S_j = \coprod_{i,j \in I} S_i \cap S_j ,$$

and on $S_i \times_S S_j = S_i \cap S_j$, the first projection p_1 is the inclusion of $S_i \cap S_j$ into S_i whereas p_2 is the inclusion of $S_i \cap S_j$ into S_j. Thus the isomorphism φ consists of a family of isomorphisms

$$\varphi_{ij} : \mathscr{F}_i|_{S_i \cap S_j} \xrightarrow{\sim} \mathscr{F}_j|_{S_i \cap S_j}$$

satisfying the cocycle condition, namely, the condition that

$$\varphi_{ik}|_{S_i \cap S_j \cap S_k} = \varphi_{jk}|_{S_i \cap S_j \cap S_k} \circ \varphi_{ij}|_{S_i \cap S_j \cap S_k}$$

for all $i, j, k \in I$. So the descent datum φ is equivalent to patching data for the S_i-modules $\mathscr{F}_i$, and the cocycle condition assures that the patching data are compatible on triple overlaps.

Example B (*Galois descent*). Let $p : S' \longrightarrow S$ be a finite and faithfully flat morphism of schemes, and assume that p is a *Galois covering*; i.e., there is a finite group Γ of S-automorphisms of S' such that the morphism

$$\Gamma \times S' \longrightarrow S'' , \qquad (\sigma, x) \longmapsto (\sigma x, x) ,$$

is an isomorphism; $\Gamma \times S'$ is the disjoint union of copies of S', parametrized by Γ. For example, if K'/K is a finite Galois extension of fields, the morphism $p : \operatorname{Spec} K' \longrightarrow \operatorname{Spec} K$ is such a Galois covering. Similarly, for a pair of discrete valuation rings $R \subset R'$, the morphism $p : \operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$ is a Galois covering if R is henselian, R' is (finite) étale over R, and the residue extension of R'/R is Galois; use 2.3/7 and the fact that R' is henselian. We want to describe the descent of schemes with respect to $p : S' \longrightarrow S$.

Consider an S'-scheme X' with an action $\Gamma \times X' \longrightarrow X'$ which is compatible with the action of Γ on S'; i.e., we require that, for each $\sigma \in \Gamma$, the diagram

$$(*) \qquad \begin{array}{ccc} X' & \xrightarrow{\ \sigma\ } & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\ \sigma\ } & S' \end{array}$$

is commutative (for simplicity, automorphisms given by σ are again denoted by σ). Notice that the diagram is cartesian. We claim that an action on X' of the type just described is equivalent to a descent datum on X'.

Namely, from the isomorphism

$$\Gamma \times S' \xrightarrow{\sim} S'' , \qquad (\sigma, x) \longmapsto (\sigma x, x) ,$$

we obtain an isomorphism

$$\Gamma \times \Gamma \times S' \xrightarrow{\sim} S''' , \qquad (\sigma, \tau, x) \longmapsto ((\sigma \circ \tau)x, \tau x, x) .$$

Taking these isomorphisms as identifications, the projections $p_{ij} : S''' \longrightarrow S''$ and $p_i : S'' \longrightarrow S'$ define projections

$$\Gamma \times \Gamma \times S' \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \Gamma \times S' \rightrightarrows S'$$

which are described by

$$(0) \qquad \begin{array}{ll} (\sigma, \tau, x) \overset{p_{12}}{\longmapsto} (\sigma, \tau x) & \\ (\sigma, \tau, x) \overset{p_{13}}{\longmapsto} (\sigma \circ \tau, x) , & (\sigma, x) \overset{p_1}{\longmapsto} \sigma x \\ (\sigma, \tau, x) \overset{p_{23}}{\longmapsto} (\tau, x) & (\sigma, x) \overset{p_2}{\longmapsto} x \end{array}$$

Now assume that we have an action of Γ on X' which is compatible with the action of Γ on S'. Then we can use the same definitions (0) in order to define "projections" from $\Gamma \times \Gamma \times X'$ to $\Gamma \times X'$ and from the latter to X'. Thereby we obtain a diagram

$$(**) \qquad \begin{array}{ccccc} \Gamma \times \Gamma \times X' & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & \Gamma \times X' & \rightrightarrows & X' \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma \times \Gamma \times S' & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & \Gamma \times S' & \rightrightarrows & S' \end{array}$$

where the vertical maps are the canonical ones. Since the diagram (*) is cartesian, all squares above are cartesian if in the first and second rows maps are considered which correspond to each other. Furthermore, in the last row we have the usual commutativity relations

$$\text{(i)} \quad p_1 \circ p_{12} = p_1 \circ p_{13} ,$$

$$\text{(ii)} \quad p_1 \circ p_{23} = p_2 \circ p_{12} ,$$

$$\text{(iii)} \quad p_2 \circ p_{23} = p_2 \circ p_{13} .$$

The same relations hold for the first row. Indeed, (ii) and (iii) are trivial whereas (i) is equivalent to the associativity condition

$$\sigma(\tau x) = (\sigma \circ \tau)x ; \qquad \sigma, \tau \in \Gamma , \qquad x \in X' .$$

So it is clear that (**) yields a descent datum on X', the associativity of the action accounting for the cocycle condition.

Conversely, start with a descent datum φ on X'. Then, applying the base change $X' \longrightarrow S'$ to the morphism

$$\Gamma \times \Gamma \times S' \xrightarrow{p_{23}} \Gamma \times S' \xrightarrow{p_2} S' ,$$

we obtain the following canonical diagram

$$(***) \qquad \begin{array}{ccccc} \Gamma \times \Gamma \times X' & \longrightarrow & \Gamma \times X' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma \times \Gamma \times S' & \longrightarrow & \Gamma \times S' & \longrightarrow & S' \end{array}$$

which has cartesian squares. In particular, we have canonical isomorphisms

$$\Gamma \times X' \xrightarrow{\sim} S' \times_S X' ,$$

and

$$\Gamma \times \Gamma \times X' \xrightarrow{\sim} S' \times_S S' \times_S X' .$$

Therefore we can write the descent datum φ in the form of a diagram (**). Furthermore, we may assume that (***) forms a part of (**), the one, where in both rows of (**) only the lower morphisms are considered. We claim that the morphism $\Gamma \times X' \longrightarrow X'$ over $p_1 : \Gamma \times S' \longrightarrow S'$ defines the desired action on X'. To justify this, note first that each $\sigma \in \Gamma$ acts as an automorphism on X'. Next, the commutativity conditions (ii) and (iii) imply that the morphisms

$$\Gamma \times \Gamma \times X' \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \Gamma \times X'$$

are defined as in (0) and, finally, as before, condition (i) accounts for the associativity of the action of Γ on X'.

As for the effectiveness of the descent, one may look at the condition given in Theorem 6.1/6. Assuming S and, hence, S' affine, as well as X' quasi-separated, a necessary and sufficient condition is that the Γ-orbit of each point $x \in X'$ is contained in a quasi-affine open subscheme of X'. Namely, considering translates of such subschemes under elements $\sigma \in \Gamma$ and taking their intersections, we can cover X' by quasi-affine open pieces which are Γ-invariant and hence stable under the descent datum. For example, if $X' \longrightarrow S'$ is quasi-projective, the condition is fulfilled, and the descent is always effective.

Example C (*Descent from R' to R, where $R \subset R'$ is an étale extension of discrete valuation rings with same residue field*). Let K (resp. K') be the field of fractions of R (resp. R'). We want to show the following result on the descent from R' to R, which will be further generalized in Example D.

Proposition C.1. *The functor which associates to an R-scheme X the triple (X_K, X', τ), consisting of the K-scheme $X_K := X \otimes_R K$, the R'-scheme $X' := X \otimes_R R'$, and the canonical isomorphism $\tau : X_K \otimes_K K' \xrightarrow{\sim} X' \otimes_{R'} K'$, is fully faithful. Its essential image consists of all triples (X_K, X', τ) which admit a quasi-affine open covering.*

The notion of an open covering of a triple (X_K, X', τ) is meant in the obvious way. Such a covering consists of a family of triples $(U_{K,i}, U'_i, \tau_i)$, where the $U_{K,i}$ (resp. the U'_i) form an open covering of X_K (resp. X'), and where τ restricts to an isomorphism $\tau_i : U_{K,i} \otimes_K K' \longrightarrow U_i' \otimes_{R'} K'$. The covering is called quasi-affine if all $U_{K,i}$ and all U_i' are quasi-affine.

Starting with a triple (X_K, X', τ), we have the canonical descent datum on $X_K \otimes_K K'$. Transporting it with τ, we obtain a descent datum on the generic fibre $X' \otimes_{R'} K'$ of X', and by the lemma below, this descent datum extends canonically to a descent datum on X'. Then the assertion of Proposition C.1 is a consequence of 6.1/6. So it is enough to show:

Lemma C.2. *For each R'-scheme X', any descent datum with respect to $K \longrightarrow K'$ on the generic fibre of X' extends canonically to a descent datum with respect to $R \longrightarrow R'$ on X'.*

Proof. Let us use the notations R'' and R''' for $R' \otimes_R R'$ and $R' \otimes_R R' \otimes_R R'$. Since R' is étale over R, the diagonal embedding $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R''$ is open (cf. 2.2/2). Thus its image, the diagonal Δ'' of $\operatorname{Spec} R''$, is a connected component of $\operatorname{Spec} R''$. Furthermore, since the residue extension of R'/R is trivial, the special fibre of Δ'' coincides with the special fibre of $\operatorname{Spec} R''$; i.e., $\operatorname{Spec} R'' = \Delta'' \,\dot{\cup}\, T''$ where the special fibre of T'' is empty. A similar assertion is true for the diagonal Δ''' in $\operatorname{Spec} R'''$.

Write K'' and K''' for the two- and threefold tensor products of K' over K. Furthermore, consider an R'-scheme X' and a descent datum with respect to $K \longrightarrow K'$ on its generic fibre. Indicating generic fibres by an index K, the descent datum on X_K' corresponds to a diagram

$$(*)\qquad \begin{array}{ccccc} X_K''' & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & X_K'' & \rightrightarrows & X_K' \\ \downarrow{\scriptstyle p'''} & & \downarrow{\scriptstyle p''} & & \downarrow{\scriptstyle p'} \\ \operatorname{Spec} K''' & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & \operatorname{Spec} K'' & \rightrightarrows & \operatorname{Spec} K' \end{array}$$

with cartesian squares such that the rows satisfy the usual commutativity conditions. In order to extend the descent datum to a descent datum on X', it is enough to extend the diagram $(*)$ to a diagram

$$(**)\qquad \begin{array}{ccccc} X''' & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & X'' & \rightrightarrows & X' \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} R''' & \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} & \operatorname{Spec} R'' & \rightrightarrows & \operatorname{Spec} R' \end{array}$$

of the same type. In order to do this, we have to realize that, by restriction, the lower row in $(*)$ gives rise to unique isomorphisms

$$(***)\qquad \Delta_K''' \xrightarrow{\sim} \Delta_K'' \xrightarrow{\sim} \operatorname{Spec} K' ,$$

and that the upper row in $(*)$ gives rise to unique isomorphisms

$$(p''')^{-1}(\Delta_K''') \xrightarrow{\sim} (p'')^{-1}(\Delta_K'') \xrightarrow{\sim} X_K' .$$

That the maps $X_K'' \rightrightarrows X_K'$ coincide on the p''-inverse of Δ_K'' follows from the fact that the pull-back of descent data with respect to diagonal maps always yields the identity map (cf. 6.1). A similar reasoning applies to the maps $X_K''' \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_K''$.

Now it is easy to extend $(*)$ into $(**)$. Since the special fibre of $\operatorname{Spec} R''$ is concentrated at the open and closed subscheme Δ'', similarly for $\operatorname{Spec} R'''$ and its diagonal Δ''', we have just to extend the part of $(*)$ which lies over $(***)$. However this is trivial by the above isomorphisms.

Example D (*Descent from R' to R where $R \subset R'$ is a pair of discrete valuation rings with same uniformizing element π and with same residue field*). The situation is more general than the one considered in Example C. For example, R' can be the maximal-adic completion of the discrete valuation ring R. But we will see that, nevertheless, the results C.1 and C.2 remain valid in this case.

Consider a pair of discrete valuation rings $R \subset R'$ as required, and denote their fields of fractions by K and K'. By an index K we will indicate tensor products with K over R. Let $\delta : \operatorname{Spec} R' \longrightarrow \operatorname{Spec} R''$ be the diagonal embedding where, as usual, $R'' = R' \otimes_R R'$.

Lemma D.1. *Let M'' be an R''-module and denote by M' its pull-back with respect to δ. Assume that the quotient M''/T'' is flat over R'' where T'' is the kernel of the canonical map $M'' \longrightarrow M_K''$. Then the canonical diagram*

$$\begin{array}{ccc} M'' & \longrightarrow & M' \\ \downarrow & & \downarrow \\ M_K'' & \longrightarrow & M_K' \end{array}$$

is cartesian; i.e., M'' is a fibred product of M_K'' and M' over M_K' (in the category of sets, resp. R-modules, resp. R''-modules).

For example, the flatness condition on M''/T'' is satisfied if we start with an R'-module M' and take for M'' the pull-back of M' with respect to a projection $p_i : \operatorname{Spec} R'' \longrightarrow \operatorname{Spec} R'$.

Proof. Since the horizontal maps are surjective, we may extend the diagram to a commutative diagram of exact sequences

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M'' & \longrightarrow & M' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_K & \longrightarrow & M_K'' & \longrightarrow & M_K' & \longrightarrow & 0 \end{array}$$

The second row can be thought of as being obtained from the first one by taking tensor products over R with K. We claim that the map $L \longrightarrow L_K$ is an isomorphism; i.e., that L is already a K-vector space. Then it is immediately clear that M'' is the fibred product of M''_K and M' over M'_K; the universal property is checked by means of diagram chasing in $(*)$.

So it remains to show that L is already a K-vector space. Let us consider the first row of $(*)$ for the special case where $M'' = R''$. Thereby we obtain the exact sequence

$$(**) \qquad 0 \longrightarrow \mathfrak{J}'' \longrightarrow R'' \longrightarrow R' \longrightarrow 0$$

of R-modules (or, alternatively, R''-modules). In terms of R-modules, the sequence is split exact, since $R'' \longrightarrow R'$ admits a section. In particular, taking the tensor product of $(**)$ over R with $R/\pi^n R$ for any $n > 0$ gives a split exact sequence

$$0 \longrightarrow \mathfrak{J}''/\pi^n\mathfrak{J}'' \longrightarrow R''/\pi^n R'' \longrightarrow R'/\pi^n R' \longrightarrow 0 .$$

By the assumptions on R and R', we see that the map

$$R''/\pi^n R'' \longrightarrow R'/\pi^n R'$$

is bijective. Thus, for $n = 1$, we have $\mathfrak{J}''/\pi\mathfrak{J}'' = 0$ and, therefore, $\mathfrak{J}'' = \pi\mathfrak{J}''$. So $\mathfrak{J}''$ is a K-vector space since R'' and, hence, $\mathfrak{J}''$ have no π-torsion. Now, tensoring $(**)$ over R'' with M'' and using the fact that M' is the pull-back of M'' with respect to the diagonal morphism $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R''$, we get the exact sequence $\mathfrak{J}'' \otimes_{R''} M'' \longrightarrow M'' \longrightarrow M' \longrightarrow 0$. Comparing it with the first row in $(*)$, we have a surjective R-homomorphism $\mathfrak{J}'' \otimes_{R''} M'' \longrightarrow L$. Therefore, since $\mathfrak{J}''$ is a K-vector space, the same must be true for L, provided L has no π-torsion.

Thus it remains to show that the π-torsion of L is trivial. To do this we consider first the case where $M'' = T''$. Using a limit argument, we may assume $\pi^n M'' = 0$ for some integer n. But then the isomorphism $R''/\pi^n R'' \xrightarrow{\sim} R'/\pi^n R'$ yields an isomorphism

$$M'' = M''/\pi^n M'' \xrightarrow{\sim} M'/\pi^n M' = M'$$

so that L is trivial in this case. In the general case we tensor the exact sequence

$$0 \longrightarrow T'' \longrightarrow M'' \longrightarrow M''/T'' \longrightarrow 0$$

over R'' with R', thereby obtaining the sequence

$$0 \longrightarrow T'' \otimes_{R''} R' \longrightarrow M' \longrightarrow (M''/T'') \otimes_{R''} R' \longrightarrow 0 .$$

The latter is exact because M''/T'' is flat over R''. By the same reason, $(M''/T'') \otimes_{R''} R'$ is flat over R' and, thus, $T' := T'' \otimes_{R''} R'$ is the torsion-submodule of M'. Since the canonical homomorphism $M'' \longrightarrow M'$ maps T'' surjectively onto T', the first row of the diagram $(*)$ yields an exact sequence

$$0 \longrightarrow L \cap T'' \longrightarrow T'' \longrightarrow T' \longrightarrow 0$$

and it follows from the special case considered above that $L \cap T''$ must be trivial. So the π-torsion of L is trivial and we see that L is a K-vector space. □

Reversing arrows in the definition of cartesian diagrams and fibred products, one arrives at the notions of *co-cartesian diagrams* and *amalgamated sums*. We want to translate the assertion of the above lemma into a statement on amalgamated sums of schemes. First note that Lemma D.1 remains true if we work in the category

of R-algebras or R''-algebras. So it yields a statement on amalgamated sums in the category of affine R-schemes or R''-schemes. We want to generalize it to the case of not necessarily affine schemes. Set $S = \operatorname{Spec} R$, $S' = \operatorname{Spec} R'$, $S'' = \operatorname{Spec} R''$, and let $\delta : S' \longrightarrow S''$ be the diagonal embedding. For any R-scheme X, let $X_K = X \otimes_R K$ be its generic fibre.

Proposition D.2. *Let X' be an S'-scheme and let X'' be its pull-back with respect to one of the projections $p_i : S'' \longrightarrow S'$. Then the canonical diagram*

$$\begin{array}{ccc} X'_K = \delta^* X''_K & \longrightarrow & X''_K \\ \downarrow & & \downarrow \\ X' = \delta^* X'' & \longrightarrow & X'' \end{array}$$

is co-cartesian in the category of R-schemes (resp. R''-schemes); i.e., in this category, X'' is the amalgamated sum of X' and X''_K under X'_K.

Proof. In order to reduce the assertion of the proposition to Lemma D.1, we need to know that a subset $F \subset X''$ is closed if and only if $F \cap X'$ is closed in X' and $F \cap X''_K$ is closed in X''_K; note that, in terms of sets, the above diagram consists of injections and that $X'' = X' \cup X''_K$, due to the assumption on R and R'. The necessity of the condition is clear. In order to show that it is sufficient, we may assume that X' is affine, say $X' = \operatorname{Spec} A'$. Then the above diagram of schemes corresponds to a diagram of R''-algebras

$$\begin{array}{ccc} A'' & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A''_K & \longrightarrow & A'_K , \end{array}$$

which is cartesian in the category of sets. Now assume that $F \cap X'$ is closed in X' and that $F \cap X''_K$ is closed in X''_K. Let $\mathfrak{J}' \subset A'$ and $\mathfrak{J}''_K \subset A''_K$ be the corresponding reduced ideals. Since $F \cap X'$ coincides with $F \cap X''_K$ on X'_K, we have

$$\operatorname{rad}(A'_K \mathfrak{J}') = \operatorname{rad}(A'_K \mathfrak{J}''_K) .$$

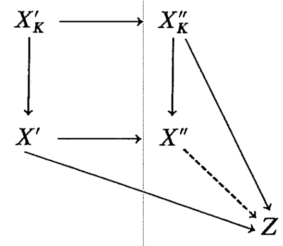
The fibred product of $\mathfrak{J}'$ and $\mathfrak{J}''_K$ over A'_K exists in the category of sets. Denoting it by $\mathfrak{J}''$, we see that we have a canonical inclusion $\mathfrak{J}'' \hookrightarrow A''$; furthermore, it is easily verified that $\mathfrak{J}''$ is an ideal in A''. We claim

$$\operatorname{rad}(\mathfrak{J}'' A') = \mathfrak{J}' \qquad \text{and} \qquad \operatorname{rad}(\mathfrak{J}'' A''_K) = \mathfrak{J}''_K .$$

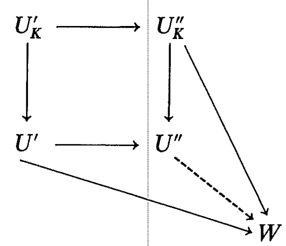
The inclusion "$\subset$" is trivial in both cases. To justify the opposite inclusions, consider an element $f \in \mathfrak{J}'$. Using the equation between radicals above, it is seen that a power of f has an inverse image in $\mathfrak{J}''$; so $f \in \operatorname{rad}(\mathfrak{J}'' A')$. Similarly, if $f \in \mathfrak{J}''_K$, a power of π times a power of f has an inverse image in $\mathfrak{J}''$ and, hence, $f \in \operatorname{rad} \mathfrak{J}'' A''_K$. This justifies the above description of $\mathfrak{J}'$ and $\mathfrak{J}''_K$, and it follows that the closed subset of X'' given by $\mathfrak{J}''$ coincides with $F \cap X'$ on X' and with $F \cap X''_K$ on X''_K. Hence F is closed in

X'', since $X'' = X' \cup X''_K$. Thereby we have proved the desired topological characterization of closed sets in X''. Looking at complements of closed sets, we see that a subset of X'' is open if and only if its intersection with X' is open in X' and its intersection with X''_K is open in X''_K.

Now it is easy to verify the assertion of the proposition. Consider a scheme Z and a commutative diagram



where the solid arrows are given and where the square is the canonical diagram. It has to be shown that the diagram can be supplemented by a unique morphism $X'' \dashrightarrow Z$. Let W be an open affine subscheme of Z, let U' be its inverse image in X' and U''_K its inverse image in X''_K. Then by the above topological characterization, $U'' := U' \cup U''_K$ is an open subscheme of X'' which extends U''_K and whose pull-back with respect to the diagonal embedding $\delta : X' \longrightarrow X''$ yields U'. So we can look at the problem



Working locally on U'' and applying Lemma D.1, we want to show that it has a unique solution. To do this, it is enough to verify the flatness condition of Lemma D.1 or, equivalently, the fact that the schematic closure $\bar{X}''$ of X''_K in X'' is flat over R''. Since the projection p_i we are considering is flat, we see that $\bar{X}''$ can be interpreted as the pull-back under p_i of the schematic closure $\bar{X}'$ of X'_K in X'; cf. 2.5/2. However, $\bar{X}'$ is flat over R' by its definition. So $\bar{X}''$ is flat over R'' and Lemma D.1 is applicable. It follows that the above local problem has a unique solution $U'' \longrightarrow W$ and, by working with respect to an affine open covering of Z, that the above global problem has a unique solution $X'' \longrightarrow Z$. □

Now we want to explain how the results D.1 and D.2 imply that descent data with respect to $\operatorname{Spec} K' \longrightarrow \operatorname{Spec} K$ extend to descent data with respect to $S' \longrightarrow S$.

Lemma D.3. *Consider an R'-module M' (resp. an R'-scheme X') and a descent datum φ_K with respect to $K \longrightarrow K'$ on M'_K (resp. on X'_K). Then φ_K extends uniquely to a descent datum with respect to $R \longrightarrow R'$ on M' (resp. on X').*

Proof. A descent datum with respect to $R \longrightarrow R'$ on M' may be viewed as a commutative diagram

$$\begin{array}{ccc} M' \otimes_R R' & \xrightarrow{\varphi} & R' \otimes_R M' \\ \downarrow & & \downarrow \\ M' & = & M' \end{array},$$

where φ is an isomorphism satisfying the cocycle condition and where the vertical maps are the canonical ones obtained from the diagonal map $\delta : S' \longrightarrow S''$. Similarly, for the descent datum φ_K on the generic fibre of M', we get the upper square of the following commutative diagram

$$\begin{array}{ccc} (M' \otimes_R R')_K & \xrightarrow{\varphi_K} & (R' \otimes_R M')_K \\ \downarrow & & \downarrow \\ M'_K & = & M'_K \\ \uparrow & & \uparrow \\ M' & = & M' \end{array}.$$

Then, taking the fibred product of the first and third rows over the second row, Lemma D.1 shows that φ_K extends uniquely to an R''-isomorphism

$$\varphi : M' \otimes_R R' \longrightarrow R' \otimes_R M' ,$$

whose pull-back with respect to the diagonal map $\delta : S'' \longrightarrow S'$ yields the identity on M'. That φ satisfies the cocycle condition follows in a similar way from Lemma D.1. Thus φ is a descent datum on M' which extends φ_K; it is unique. For the case of schemes, the assertion is deduced in formally the same way from Proposition D.2. □

Now, applying Theorems 6.1/4 and 6.1/6, we can derive from the above lemma the desired generalization of Proposition C.1.

Proposition D.4. (a) *The functor which associates to each R-module M the triple (M_K, M', τ), where $M_K := M \otimes_R K$, $M' := M \otimes_R R'$, and $\tau : M_K \otimes_K K' \xrightarrow{\sim} M' \otimes_R K'$ is the canonical isomorphism, is an equivalence of categories.*

(b) *The functor which associates to each R-scheme X the triple (X_K, X', τ) consisting of the K-scheme $X_K := X \otimes_R K$, the R'-scheme $X' = X \otimes_R R'$, and the canonical isomorphism $\tau : X_K \otimes_K K' \xrightarrow{\sim} X' \otimes_{R'} K'$, is fully faithful. Its essential image consists of all triples (X_K, X', τ) which admit a quasi-affine open covering.*

Finally, we want to mention that it is an easy exercise to verify assertion (a) of the proposition by a direct argument. Applying a limit argument, one reduces to the case of finitely generated R- or R'-modules, where it is possible to treat the case of torsion and of free modules separately. However, for the purpose of assertion (b), it was necessary to prove more precise results also in the module case.

6.3 The Theorem of the Square

Let S be a scheme, let X be an S-scheme, and consider an S-group scheme G which acts on X. Using the notion of T-valued points for arbitrary S-schemes T, such an action corresponds to an S-morphism

$$G \times_S X \longrightarrow X\,, \qquad (g,x) \longmapsto gx\,,$$

where

$$g(g'x) = (gg')x \qquad \text{and} \qquad 1_T x = x$$

for arbitrary points $g, g' \in G(T)$, $x \in X(T)$, and for the unit element $1_T \in G(T)$. Alternatively, interpreting G (resp. X) as a functor from the category of S-schemes to the category of groups (resp. sets), we can say that the group functor G acts on X; i.e., that, for each S-scheme T, we have an action of $G(T)$ on $X(T)$ which is compatible with S-morphisms $T' \longrightarrow T$ in the usual way. Similarly as in the case of group schemes, one defines for any $g \in G(T)$ the translation

$$\tau_g : X \longrightarrow X\,, \qquad x \longmapsto gx\,,$$

where, more precisely, τ_g has to be interpreted as a T-morphism from X_T to X_T.

Now let us fix an invertible sheaf $\mathscr{L}$ on X. Its pull-back to X_T will again be denoted by $\mathscr{L}$. So we can talk about the pull-back of $\mathscr{L}$ with respect to a translation τ_g, $g \in G(T)$, thus obtaining the invertible sheaf

$$\mathscr{L}_g := \tau_g^* \mathscr{L}$$

on X_T. Let $P_{X/S}$ be the functor which associates to any S-scheme T the group

$$\operatorname{Pic}(T \times_S X)/p^*\operatorname{Pic}(T)\,;$$

i.e., the group of invertible sheaves on $X \times_S T$ modulo the pull-back under the projection $p : T \times_S X \longrightarrow T$ of invertible sheaves on T. Then $P_{X/S}$ is a commutative group functor, and we can consider the morphism

$$\varphi_{\mathscr{L}} : G \longrightarrow P_{X/S}\,, \qquad g \longmapsto \text{class of} \quad \mathscr{L}_g \otimes \mathscr{L}^{-1}\,,$$

which is a functorial morphism between functors from the category of S-schemes to the category of sets. We will say that $\mathscr{L}$ *satisfies the theorem of the square* if $\varphi_{\mathscr{L}}$ respects group structures and, thus, is a functorial morphism between group functors. We do this in analogy to the classical case, where X is an abelian variety over a field K, and where the action of G on X is given by translation. In this case, the functor $P_{X/S}$ coincides with the relative Picard functor $\operatorname{Pic}_{X/S}$ (see 8.1/4), and the classical theorem of the square asserts that, for each invertible sheaf $\mathscr{L}$ on X, the morphism $\varphi_{\mathscr{L}}$ is a morphism of group functors. For proofs see Weil [2], § VIII, n°57, Thm. 30, Cor. 2, as well as Lang [1], Chap. III, § 3, Cor. 4, and Mumford [3], Chap. II, § 6, Cor. 4.

The purpose of the present section is to extend the classical theorem of the square to a more general situation. For the applications we have in mind, it is enough to

know that, for each invertible sheaf $\mathscr{L}$ on X, a power $\mathscr{L}^{\otimes n}$ satisfies the theorem of the square.

Theorem 1. *Let S be a Dedekind scheme and let G be a smooth S-group scheme with connected fibres which acts on an S-scheme X, where $X \longrightarrow S$ is smooth, of finite type, and has geometrically connected generic fibres. Then, for any invertible sheaf $\mathscr{L}$ on X, there is an integer $n > 0$ such that $\mathscr{L}^{\otimes n}$ satisfies the theorem of the square.*

If the generic fibres of X are proper or if the local rings $\mathcal{O}_{S,\xi}$ at generic points $\xi \in S$ are perfect fields, the assertion holds for $n = 1$.

We will reduce the theorem to the classical situation where S consists of a field. In fact, we will show that $\mathscr{L}$ satisfies the theorem of the square if and only if this is the case over each generic point of S; see Lemma 2. In order to carry out this reduction step, it is necessary to write down somewhat more explicitly the condition of $\varphi_{\mathscr{L}} : G \longrightarrow P_{X/S}$ being a morphism of group functors. Let m be the group law on G. Set $T := G \times_S G$, and consider the projections $p_1, p_2 : G \times_S G \rightrightarrows G$ as T-valued points of G. Furthermore, let

$$f : G \times_S G \times_S X \longrightarrow G \times_S G$$

be the projection onto the first two factors. Then we claim that $\varphi_{\mathscr{L}}$ is a morphism of group functors if and only if

$$\mathscr{M} := \mathscr{L}_{m(p_1,p_2)} \otimes \mathscr{L}_{p_1}^{-1} \otimes \mathscr{L}_{p_2}^{-1} \otimes \mathscr{L}\,,$$

as an invertible sheaf on $G \times_S G \times_S X$, is isomorphic to the pull-back $f^*\mathscr{N}$ of an invertible sheaf $\mathscr{N}$ on $G \times_S G$.

In fact, the class of $\mathscr{M}$ in $P_{X/S}(G \times_S G)$ is given by

$$\varphi_{\mathscr{L}}(m(p_1,p_2)) - \varphi_{\mathscr{L}}(p_1) - \varphi_{\mathscr{L}}(p_2)\,.$$

Thus it is trivial if $\varphi_{\mathscr{L}}$ is a morphism of group functors. In order to show the converse, we mention the following fact:

For an arbitrary S-scheme T and two points $g, g' \in G(T)$, the invertible sheaf $\mathscr{L}_{m(g,g')} \otimes \mathscr{L}_g^{-1} \otimes \mathscr{L}_{g'}^{-1} \otimes \mathscr{L}$ is the pull-back of $\mathscr{M}$ with respect to the morphism

$$(g,g') \times_S \mathrm{id}_X : T \times_S X \longrightarrow G \times_S G \times_S X\,.$$

So if $\mathscr{M} \cong f^*\mathscr{N}$ for some invertible sheaf $\mathscr{N}$ on $G \times_S G$, the commutative diagram

$$\begin{array}{ccc} T \times_S X & \xrightarrow{(g,g') \times_S \mathrm{id}_X} & G \times_S G \times_S X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{(g,g')} & G \times_S G \end{array}\quad,$$

where p is the projection onto the first factor, yields

$$\mathscr{L}_{m(g,g')} \otimes \mathscr{L}_g^{-1} \otimes \mathscr{L}_{g'}^{-1} \otimes \mathscr{L} \cong p^*((g,g')^*(\mathscr{N}))$$

and, hence,

$$\varphi_{\mathscr{L}}(m(g,g')) = \varphi_{\mathscr{L}}(g) + \varphi_{\mathscr{L}}(g') .$$

This justifies our claim. We will now reduce the theorem of the square to generic fibres.

Lemma 2. *Let S, G, X and $\mathscr{L}$ be as in Theorem 1, and let $\mathscr{M}$ be the invertible sheaf on $G \times_S G \times_S X$ which has been defined above. Then the following conditions are equivalent:*

(a) *There exists an invertible sheaf $\mathscr{N}$ on $G \times_S G$ such that $\mathscr{M}$ is isomorphic to the pull-back $f^*\mathscr{N}$ of $\mathscr{N}$ with respect to the projection $f: G \times_S G \times_S X \longrightarrow G \times_S G$; i.e., $\mathscr{L}$ satisfies the theorem of the square.*

(b) *For each generic point ξ of S, the invertible sheaf $\mathscr{L}$ satisfies the theorem of the square after performing the base change* $\operatorname{Spec} k(\xi) \longrightarrow S$.

Proof. The fact that an invertible sheaf on X satisfies the theorem of the square is preserved by any base change. Thus the implication (a) $\Longrightarrow$ (b) is obvious.

In order to show the converse, we may assume that S is irreducible with generic point ξ. If condition (b) is given, there is an invertible sheaf $\mathscr{N}_\xi$ on $(G \times_S G)_\xi$ satisfying

$$\mathscr{M}_\xi \cong f_\xi^*(\mathscr{N}_\xi) ,$$

where the index ξ indicates restrictions to generic fibres. We can extend $\mathscr{N}_\xi$ to an invertible sheaf $\mathscr{N}$ on $G \times_S G$ because $G \times_S G$ is regular. For example, this can be done by considering a divisor on $(G \times_S G)_\xi$ which corresponds to $\mathscr{N}_\xi$. Taking its schematic closure in $G \times_S G$, the associated invertible sheaf on $G \times_S G$ may be viewed as an extension of $\mathscr{N}_\xi$.

Now consider the invertible sheaf $\mathscr{M}' := \mathscr{M} \otimes (f^*(\mathscr{N}))^{-1}$ on $G \times_S G \times_S X$. Using the projection $p: G \times_S G \times_S X \longrightarrow S$, we claim there is a divisor Δ on S such that

$$\mathscr{M}' \cong p^*(\mathscr{O}_S(\Delta)) .$$

Namely, $\mathscr{M}'_\xi$ is trivial. So we can choose a global generator and view it as a meromorphic section of $\mathscr{M}'$. Then it generates $\mathscr{M}'$ over an open subset of $G \times_S G \times_S X$ whose complement consists of at most finitely many fibres over closed points in S. Thus there is a divisor D on $G \times_S G \times_S X$ whose support meets only finitely many fibres of p over closed points of S such that

$$\mathscr{M}' \cong \mathscr{O}_{G \times_S G \times_S X}(D) .$$

Now look at the projection

$$p_3 : G \times_S G \times_S X \longrightarrow X .$$

Since the structural morphism $G \times_S G \longrightarrow S$ is smooth and has geometrically irreducible fibres, the same is true for p_3 and it is easily seen that the pull-back of a prime divisor on X yields a prime divisor on $G \times_S G \times_S X$. Hence, the Weil divisor

D, whose support is not dominant over X, is of the tye $p_3^*(\Delta')$ with a Weil divisor Δ' of X. So we have

$$(*) \qquad \mathscr{M}' = \mathscr{M} \otimes (f^*(\mathscr{N}))^{-1} \cong p_3^*(\mathscr{O}_X(\Delta')) ,$$

and it remains to show that $\mathscr{O}_X(\Delta')$ is the pull-back of an invertible sheaf on S. If X has irreducible fibres over S, a similar argument as above shows that Δ' is pull-back of a divisor on S. In the general case, consider the morphism

$$q = (\varepsilon, \varepsilon, \mathrm{id}_X) : X \longrightarrow G \times_S G \times_S X ,$$

where ε is the composition of the structural morphism $X \longrightarrow S$ with the unit section $S \longrightarrow G$. Pulling back $(*)$ with respect to q, we get on the right-hand side $\mathscr{O}_X(\Delta')$. On the left-hand side, the pull-back of $\mathscr{M}$ is trivial; it is the evaluation of $\mathscr{M}$ at the unit section of $G \times_S G$. Furthermore, since $f \circ q : X \longrightarrow G \times_S G$ factors through S, we see that $q^*(f^*(\mathscr{N}))$ is the pull-back of an invertible sheaf on S. So $\mathscr{O}_X(\Delta')$ is the pull-back of an invertible sheaf on S as claimed; we can write it in the form $\mathscr{O}_S(\Delta)$ with a divisor Δ on S.

Now, looking at the isomorphism

$$\mathscr{M} \cong f^*(\mathscr{N}) \otimes p^*(\mathscr{O}_S(\Delta))$$

obtained from $(*)$, we can replace $\mathscr{N}$ by its tensor product with the pull-back of $\mathscr{O}_S(\Delta)$ to $G \times_S G$. Then the resulting invertible sheaf, again denoted by $\mathscr{N}$, satisfies $\mathscr{M} \cong f^*(\mathscr{N})$. Thus $\mathscr{M}$ is as required in condition (a). □

The essence of the lemma consists in the fact that an invertible sheaf $\mathscr{L}$ on X satisfies the theorem of the square as soon as the pull-back of $\mathscr{L}$ to each generic fibre of X satisfies this theorem. So, in order to establish Theorem 1, it can be assumed that S is the spectrum of a field.

In the main case where $G = X$ is an abelian variety we are done by the classical theorem of the square. For the general case, we refer to Raynaud [4], Thm. IV. 3.3, in order to see that a power of $\mathscr{L}$ satisfies the theorem of the square. In fact, one shows that $\mathscr{L}$ itself satisfies the theorem of the square if the field K is replaced by a finite radicial extension; cf. Raynaud [4], Thm. IV. 2.6.

We want to add two possibilities of obtaining the theorem of the square in special situations, always assuming that the base is a field. First, let us consider the case where X is proper. In order to show that

$$\varphi_{\mathscr{L}} : G \longrightarrow P_{X/K}$$

is a morphism of group functors, look at the relative Picard functor $\mathrm{Pic}_{X/K}$ (cf. Section 8.1). Since X is proper, smooth, and geometrically connected over K, the canonical morphism

$$P_{X/K} \longrightarrow \mathrm{Pic}_{X/K}$$

is injective (cf. 8.1/4). So it is enough to show that $\varphi_{\mathscr{L}}$ defines a morphism of group functors

$$\varphi'_{\mathscr{L}} : G \longrightarrow \mathrm{Pic}_{X/K} .$$

Now we use the fact that $\mathrm{Pic}_{X/K}$ is representable by a group scheme over K (cf. 8.2/3) and that $(\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$ is an abelian variety A over K; cf. [FGA], n°236, Cor. 3.2. Since $\varphi'_{\mathscr{L}}$ maps unit sections onto each other, it must factor through A. Then the rigitiy lemma (cf. Lang [1], Chap. II, § 1, Thm. 4) shows that the resulting morphism

$$G \longrightarrow A$$

is a morphism of group functors. Hence, it follows that $\mathscr{L}$ satisfies the theorem of the square.

The second method we want to mention applies to the case where X is a torsor under G. The applications of Theorem 1 we have in mind refer to this situation. Still considering the case where S consists of a field K and replacing K by its algebraic closure, we may assume that X coincides with G and, thus, is an algebraic group over an algebraically closed field. Then, by the theorem of Chevalley 9.2/1, there is an exact sequence of algebraic groups over K

$$1 \longrightarrow G_{\text{aff}} \longrightarrow G \longrightarrow A \longrightarrow 1$$

where G_{aff} is smooth, connected, and affine, and where A is an abelian variety. Since the Picard group of the affine group G_{aff} consists only of torsion, one can show that a power of $\mathscr{L}$ is the pull-back of an invertible sheaf on A. So one is essentially reduced to the case where G is an abelian variety.

6.4 The Quasi-Projectivity of Torsors

We want to introduce the notion of torsors, a notion which is closely related to the concept of group schemes. Consider a base scheme S, an S-scheme X, and an S-group scheme G which acts on X by means of a morphism

$$G \times_S X \longrightarrow X\,, \qquad (g,x) \longmapsto gx\,.$$

Assume that G is (faithfully) flat and locally of finite presentation over S. Then X is called a *torsor* (with respect to the fppf-topology), more precisely, an *S-torsor under G* if

(i) the structural morphism $X \longrightarrow S$ is faithfully flat and locally of finite presentation, and

(ii) the morphism $G \times_S X \longrightarrow X \times_S X$, $(g,x) \longmapsto (gx, x)$, is an isomorphism.

Viewing $G \times_S X$ and $X \times_S X$ as X-schemes with respect to the second projections, we see that the isomorphism in (ii) is, in fact, an X-isomorphism. In other words, applying the base change $X \longrightarrow S$ to X and G, both schemes become isomorphic. The same is, of course, true for any base change $Y \longrightarrow S$ which factors through X. In particular, if $X(S) \neq \emptyset$, the choice of an S-valued point of X gives rise to an S-isomorphism from G to X, and there is no essential difference between G and the torsor X. We say that the torsor X is *trivial* in this case. Furthermore, $X \longrightarrow S$ satisfies any of the conditions listed in [EGA IV_2], 2.7.1 and [EGA IV_4], 17.7.4, for example, being smooth, separated, or of finite type, provided these conditions are satisfied by $G \longrightarrow S$. Namely, in order to apply the cited results, it is enough to consider the case where S is affine. Then, since $X \longrightarrow S$ is open, there exists a quasi-compact open subscheme Y of X such that $Y \longrightarrow S$ is surjective and, hence, faithfully flat as well as locally of finite presentation. So, what we have claimed follows from the isomorphism $G \times_S Y \xrightarrow{\sim} X \times_S Y$ by faithfully flat and quasi-compact descent. In particular, if G is smooth, X is smooth and it can be trivialized after a surjective étale base change $S' \longrightarrow S$ because, after performing a suitable base change of this type, X will have sections by 2.2/14.

Examples of torsors are easy to describe. Consider a finite Galois extension L/K of fields. Then $\operatorname{Spec} L$ is a $(\operatorname{Spec} K)$-torsor under the constant group $\operatorname{Gal}(L/K)$. Or, consider an invertible sheaf $\mathscr{L}$ on a scheme X and remove the zero section from its associated total space. The resulting scheme is an X-torsor under the multiplicative group $(\mathbb{G}_m)_X$. It is trivial if and only if $\mathscr{L}$ is trivial. We want to formulate now the main result to be proved in this section.

Theorem 1. *Let S be a Dedekind scheme, and let X be a torsor under an S-group scheme G. Assume that G is smooth, separated, and of finite type over S. Then X is quasi-projective over S. In particular, G itself is quasi-projective over S.*

For the proof we have to construct an S-ample invertible sheaf $\mathscr{L}$ on X. In order to do so, we use the theorem of the square.

First we show that, for any effective divisor D on X, the associated invertible sheaf $\mathscr{L} := \mathcal{O}_X(D)$ is S-ample if $X - \operatorname{supp}(D)$ satisfies certain properties.

Proposition 2. *Let S be a Dedekind scheme and let G be a smooth S-group scheme with connected fibres which acts on an S-scheme X, where X is smooth and of finite type over S. Assume there exists an open subscheme $U \subset X$ such that U is affine over S and such that U meets all G-orbits of points in X; i.e., such that the action of G induces a surjective morphism $G \times_S U \longrightarrow X$. Then, for any effective divisor D on X with support $X - U$, the invertible sheaf $\mathscr{L} = \mathcal{O}_X(D)$ is S-ample.*

For example, $X - U$ provided with its reduced structure gives rise to such a divisor D; cf. [EGA IV_4], 21.12.7.

Proof. In a first step we want to reduce to the case where S is local. So assume $\mathscr{L}$ is an invertible sheaf on X such that, for each $s \in S$, the pull-back $\mathscr{L}(s)$ of $\mathscr{L}$ to $X(s) := X \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is ample. Then there exist global sections $l_1, \ldots, l_r$ generating a certain power $\mathscr{L}(s)^{\otimes n}$ such that the open subscheme $X(s)_{l_i} \subset X(s)$ where l_i generates $\mathscr{L}(s)^{\otimes n}$ is *affine*; use [EGA II], 4.5.2, or the characterization of ample invertible sheaves given in Section 6.1. By a limit argument, the l_i extend to sections l_i' of $\mathscr{L}^{\otimes n}$ over some neighborhood S' of $s \in S$ and, by [EGA IV_3], 8.10.5, we may assume that the l_i' generate $\mathscr{L}^{\otimes n}$ over S', that the projection $X \times_S S' \longrightarrow S'$ is separated, and that the open subscheme $X'_{l_i'} \subset X \times_S S'$ where l_i' generates $\mathscr{L}^{\otimes n}$ is affine. Thereby we see that $\mathscr{L}$ is ample over a neighborhood of each point $s \in S$ and, thus, that $\mathscr{L}$ is S-ample on X.

Let us assume now that S is the spectrum of a local ring R. Since ampleness can be checked after faithfully flat and quasi-compact base change, as follows from [EGA IV_2], 2.7.2, it is enough to treat the case where R is strictly henselian. Using the fact that G has geometrically connected fibres, we see that G operates on the connected components of X. So we can assume that X is connected. We claim that it is enough to consider the case where the structural morphism $X \longrightarrow S$ is surjective. In fact, $X \longrightarrow S$ is open and, if $X \longrightarrow S$ is not surjective, we replace S by the image of X. However, doing so, we may loose the property of S being local and strictly henselian. In this case we have to go back to the beginning and to start the proof anew. Therefore, by induction on the dimension of S, we are reduced to the case where S is local and strictly henselian, where $X \longrightarrow S$ is surjective, and where X is connected. Then X has sections by 2.3/5 and, thus, its generic fibre is *geometrically connected* by [EGA IV_2], 4.5.13.1.

In this situation, we want to establish the assertion of the proposition. Replacing the divisor D by a multiple of itself, we can assume that the invertible sheaf $\mathscr{L} = \mathcal{O}_X(D)$ satisfies the theorem of the square; see 6.3/1. Then the divisor $D_g + D_{g^{-1}}$ is linearly equivalent to $2D$, where we have written D_g for the translate of D under g. Hence there is a section $l \in \Gamma(X, \mathscr{L}^{\otimes 2})$ such that

$$X_l = X - \operatorname{supp}(D_g + D_{g^{-1}}) = gU \cap g^{-1}U .$$

As the intersection of two affine open subschemes of a noetherian scheme, X_l is quasi-affine. Furthermore, it follows that $\mathscr{L}$ is ample, provided we can show that the open subschemes $gU \cap g^{-1}U$ cover X if g varies over $G(S)$.

So it remains to verify the latter fact. Fix a point $x \in X$. Write s for its image in S and set $k = k(s)$. We claim that there is a dense open subscheme $Z_s \subset G_s$ such that

$$x \in gU_s \cap g^{-1}U_s$$

for each $g \in Z_s(k)$. To see this, we may assume that x is a closed point of X_s. Then we apply the base change $k \longrightarrow k'$ to fibres over s, where $k' = k(x)$ is finite over k. Let W be the inverse of $U_s \otimes_k k'$ under the morphism

$$G_s \otimes_k k' \longrightarrow X_s \otimes_k k' , \qquad a \longmapsto ax ,$$

and write W^{-1} for its inverse under the group law on $G_s \otimes_k k'$. Then, since U meets all G-orbits of points in X and since G has geometrically connected fibres, $W \cap W^{-1}$ is a dense open subscheme of $G_s \otimes_k k'$. Furthermore, the relation $x \in g(U_s \otimes_k k') \cap g^{-1}(U_s \otimes_k k')$ is equivalent to $g^{-1}x \in U_s \otimes_k k'$ and $gx \in U_s \otimes_k k'$. Thus $x \in g(U_s \otimes_k k') \cap g^{-1}(U_s \otimes_k k')$ for all $g \in (W \cap W^{-1})(k')$. Then, using methods of descent, we find a dense open subscheme of $W \cap W^{-1}$ descending to a dense open subscheme Z_s of G_s such that $x \in gU_s \cap g^{-1}U_s$ for all $g \in Z_s(k)$.

Now it is easy to see that the open subschemes $gU \cap g^{-1}U$ cover X if g varies over $G(S)$. Namely, we have only to realize that, for each dense open subscheme $Z_s \subset G_s$ of a fibre over a point $s \in S$, there exists a section in $G(S)$ which, by restriction to G_s, yields a section of Z_s. If s is the closed point of S, this follows from 2.3/5. If s belongs to the generic fibre of S, we can consider the schematic closure of $G_s - Z_s$ in G. Its special fibre is nowhere dense in the special fibre of G so that an argument as the one given before will finish the proof of Proposition 2.

Later, in 6.6/1, we will use the same idea of proof again without the restriction that the base S is of dimension ≤ 1. In this case, one can apply the assertion of 5.3/7 in order to end the proof. □

In order to derive the assertion of Theorem 1 from Proposition 2, we need some further preparations. Let G^0 be the identity component of G; i.e., G^0 is the open subscheme of G which is the union of all identity components of the fibres G_s over points $s \in S$ (cf. [EGA IV_3], 15.6.5). Then G^0 has geometrically connected fibres, and it acts on X. Therefore we can apply Proposition 2 if we can find an open subscheme $U \subset X$ such that U is affine over S and such that U meets all G^0-orbits

of points in X. As is easily checked on geometric fibres, the latter condition is equivalent to the fact that U is S-dense in X:

Lemma 3. *Let X be a torsor under a smooth S-group scheme G which is of finite type over S, and consider an open subscheme $U \subset X$. Then U meets all G^0-orbits of points in X if and only if U is S-dense in X.*

In order to really construct an open subscheme $U \subset X$ as required in Proposition 2, we have to derive some information on the existence of affine open subschemes of X.

Lemma 4. *Let $S = \operatorname{Spec} R$ be an affine scheme which is noetherian, and consider an S-scheme X of finite type which is normal and separated. Let $(x_i)_{i \in I}$ be a finite family of points of codimension $\leqq 1$ in X. Then there exists an affine open subscheme $U \subset X$ containing all points x_i.*

Proof. We may assume that X is connected with field of rational functions L and, furthermore, that all x_i are of codimension 1. Then the local rings $\mathcal{O}_{X,x_i}$ are discrete valuation rings contained in L, and they are pairwise different since X is separated. So we can use the approximation theorem for inequivalent valuations (cf. Bourbaki [2], Chap. VI, § 7, n°1, Prop. 1) and see that

$$A := \bigcap_{i \in I} \mathcal{O}_{X,x_i}$$

is a semi-local ring with local components $\mathcal{O}_{X,x_i}$. We can write A as a direct limit of R-algebras A_j of finite type. Interpreting the elements of each A_j as rational functions on X, we obtain for each j a rational map

$$u_j : X \dashrightarrow \operatorname{Spec} A_j$$

which is an S-morphism in a neighborhood of each x_i. Since X and A_j are of finite type over R, our construction shows that u_j is an open immersion at each x_i if j is big enough; cf. [EGA IV_2], 8.10.5. Thus we have reduced the assertion of the lemma to the case where X is quasi-affine and where it is easily verified. □

Now we are able to *prove the assertion of Theorem* 1. As explained before, we have only to construct an S-dense open subscheme $U \subset X$ which is affine over S. In order to do this, fix a closed point $s \in S$. Working over an affine neighborhood S' of s in S and applying Lemma 4, there is an affine open subscheme $U' \subset X \times_S S'$ which contains all generic points of $X \times_S S'$ and all generic points of the fibre X_s. The complement of U' in $X \times_S S'$ equals the support of finitely many prime divisors $D_1, \ldots, D_r$ of $X \times_S S'$. Removing from S' all closed points s' such that the support of some D_j is contained in $X_{s'}$, we may assume that U' is S'-dense in $X \times_S S'$. Proceeding this way with all closed points in S, and using a quasi-compactness argument, we obtain affine open subschemes $U^1, \ldots, U^n$ of X such that U^i is S^i-dense over an affine open part S^i of S and such that the S^i cover S. For simplicity, assume that S is irreducible with generic point ξ. Let D_ξ be an effective divisor on X_ξ with support

$$X_\xi - \bigcap_{i=1}^{n} U_\xi^i ,$$

let D be its schematic closure in X, and set $U := X - \operatorname{supp} D$. Then U is S-dense in X since all U_ξ^i are dense in X_ξ and since $\operatorname{supp} D$ cannot contain components of closed fibres of X. Furthermore, U is affine over S. Namely, $U \times_S S^i$ is contained in U^i; it differs from the affine scheme U^i by the support of a divisor. Therefore the inclusion $U \times_S S^i \hookrightarrow U^i$ is affine, as can be checked locally, and it follows that $U \times_S S^i$ must be affine itself; cf. [EGA II], 1.3.4. So we have constructed U as required in Proposition 2, thereby finishing the proof of Theorem 1. □

6.5 The Descent of Torsors

In this section we want to apply the descent techniques of 6.1 to torsors under group schemes. So far we have dealt only with the descent of schemes without considering a group structure or a structure of torsor on them; however, we will see that the methods of 6.1 apply immediately to the new situation. Namely, consider a faithfully flat and quasi-compact morphism of schemes $p : S' \longrightarrow S$ as well as an S'-group scheme G'. As in 6.1, set $S'' := S' \times_S S'$, and let $p_1, p_2 : S'' \longrightarrow S'$ be the projections. Recall that, in terms of schemes, a descent datum on G' with respect to p consists of an S''-isomorphism

$$\varphi : p_1^* G' \longrightarrow p_2^* G'$$

satisfying the cocycle condition. Using the canonical isomorphisms

$$p_i^*(G' \times_{S'} G') \cong p_i^* G' \times_{S''} p_i^* G' , \qquad i = 1, 2,$$

one obtains from φ a descent datum

$$\varphi \times \varphi : p_1^*(G' \times_{S'} G') \longrightarrow p_2^*(G' \times_{S'} G')$$

on $G' \times_{S'} G'$. Talking about descent data on group schemes, it is required that the descent datum φ on G' is *compatible with the group multiplication* $m : G' \times_{S'} G' \longrightarrow G'$; i.e., that the diagram

$$\begin{array}{ccc} p_1^*(G' \times_{S'} G') & \xrightarrow{\ \varphi \times \varphi\ } & p_2^*(G' \times_{S'} G') \\ \downarrow{\scriptstyle p_1^*(m)} & & \downarrow{\scriptstyle p_2^*(m)} \\ p_1^* G' & \xrightarrow{\ \varphi\ } & p_2^* G' \end{array}$$

is commutative. Viewing $p_i^* G'$ as the S''-group scheme obtained from G' by means of the base change $p_i : S'' \longrightarrow S'$, the condition simply says that the descent datum

$$\varphi : p_1^* G' \longrightarrow p_2^* G'$$

is an isomorphism of S''-*group* schemes. Then, if the descent is effective, i.e., if G' descends to an S-scheme G, Theorem 6.1/6 implies readily that the group structure descends from G' to G and, hence, that G is an S-group scheme.

The procedure is similar for torsors. Consider an S'-scheme X' which is a torsor under an S'-group scheme G'. Let φ be a descent datum on G' which is compatible with the group multiplication on G'. Then a descent datum ψ on X' is said to be compatible with the structure of X' as a torsor under G' if the action

$$G' \times_{S'} X' \longrightarrow X'$$

is compatible with the descent data φ and ψ. If φ and ψ are effective, G' descends to an S-group scheme G and X' to an S-scheme X which is a torsor under G.

In the following, we want to exploit the existence of ample invertible sheaves in order to treat the descent of torsors over discrete valuation rings. Since it is necessary to study the problems on generic fibres first, our considerations will include the more or less trivial case where the base consists of a field.

Theorem 1. *Let $R \longrightarrow R'$ be a faithfully flat extension of discrete valuation rings (resp. of fields). Let G' be an R'-group scheme which is smooth, separated, and of finite type over R', and let X' be an R'-torsor under G'. Furthermore, assume that there are descent data with respect to $R \longrightarrow R'$ on G' and X' such that these data are compatible with the group structure on G' and with the action of G' on X'. Then G' descends to an R-group scheme G, and X' descends to an R-torsor X under G. Furthermore, by the properties of descent, G and X are smooth, separated, and of finite type over R.*

Before we give the proof, let us discuss some applications of the theorem. First we go back to Section 5, where we have studied the problem of associating group schemes to birational group laws; cf. 5.1/5. In 5.2/3, which applies to strict birational group laws, we had worked out a solution for the case where the base consists of a strictly henselian local ring R which is noetherian and normal. Now, using descent, we can show that 5.2/3 remains true if we work over a discrete valuation ring or over a field, without assuming that the latter is strictly henselian. Thereby we will fill the gap which was left in the proof of 5.1/5; we refer to Section 6.6 for a more rigorous approach to the problem.

Corollary 2. *Let R be a discrete valuation ring or a field, and let m be a strict birational group law on an R-scheme U which is separated, smooth, faithfully flat, and of finite type over R. Then there exists an open immersion $U \hookrightarrow G$ with R-dense image into a smooth and separated S-group scheme G such that the group law on G restricts to m on U. The group scheme G is unique up to canonical isomorphism.*

Proof. Write R' for a strict henselization of R. Then, applying the base change $R \longrightarrow R'$ to our situation, we obtain a strict birational group law m' on the R'-scheme $U' = U \otimes_R R'$. It has a unique solution by 5.2/3; i.e., there is an open immersion $U' \hookrightarrow G'$ into an R'-group scheme G', just as we have claimed for U and m.

In order to prove the corollary, it is enough to extend the canonical descent datum on U' to a descent datum on G' which is compatible with the group structure on G'. Then Theorem 1 can be applied. As usual, set $R'' = R' \otimes_R R'$ and write p_1,

p_2 for the projections from Spec R'' to Spec R'. The canonical descent datum on U' consists of the canonical isomorphism

$$p_1^* U' \xrightarrow{\sim} p_2^* U' .$$

Working over the base R'', we see immediately from the uniqueness assertion in 5.1/3 that this isomorphism extends to an isomorphism of R''-group schemes

$$p_1^* G' \xrightarrow{\sim} p_2^* G' .$$

A similar argument shows that the isomorphism satisfies the cocycle condition; so we have a descent datum on G' as required. □

As a second application, we want to discuss the existence of Néron models for torsors in the local case. Since, over strictly henselian valuation rings, torsors under smooth group schemes are trivial, the problem is a question of descent.

Corollary 3. *Let $R \subset R' \subset R^{sh}$ be discrete valuation rings, where R^{sh} is a strict henselization of R, and let K, K' and K^{sh} denote the fields of fractions of R, R' and R^{sh}. Furthermore, let X_K be a K-torsor under a smooth K-group scheme G_K of finite type, and assume that, after the base change $K \longrightarrow K'$, there are Néron models G' of $G_{K'}$ and X' of $X_{K'}$ over R'. Then G' (resp. X') descends to a Néron model G of G_K (resp. X of X_K) over R. Furthermore, if the torsor X_K is unramified, i.e., if $X_K(K^{sh}) \neq \varnothing$, the structure of X_K as a torsor under G_K extends uniquely to a structure of X as a torsor under G.*

Postponing the proof for a moment, let us first explain why X might not be a torsor under G. The universal mapping property of Néron models implies that the action of G_K on X_K extends uniquely to an action of G on X giving rise to an isomorphism

$$G \times_R X \longrightarrow X \times_R X , \qquad (g, x) \longmapsto (gx, x) .$$

However, in general, X will not be a torsor under G, since the structural morphism $X \longrightarrow \operatorname{Spec} R$ might not be surjective; i.e., it can happen that the special fibre of X is empty. Due to 2.3/5, the latter is the case if and only if $X(R^{sh})$ is empty or, by the Néron mapping property, if and only if $X_K(K^{sh})$ is empty. The torsor X_K is called *ramified* if $X_K(K^{sh}) = \varnothing$, and *unramified* if $X_K(K^{sh}) \neq \varnothing$. Combining the assertion of 1.3/1 with the preceding corollary, we can say:

Corollary 4. *Let R, K, K^{sh} be as before, and let X_K be a K-torsor under a smooth K-group scheme G_K of finite type. Then the following conditions are equivalent:*

(a) *X_K admits a Néron model over R.*
(b) *$X_K(K^{sh})$ is bounded in X_K.*
(c) *X_K is ramified or $G_K(K^{sh})$ is bounded in G_K.*

Proof of Corollary 3. As far as the Néron model of X_K is concerned, the assertion is trivial if X' has empty special fibre and thus coincides with $X_{K'}$. So assume that the latter is not the case and, hence, that X' is a torsor under G'. We claim it is enough to verify that the canonical descent data on $G_{K'}$ and $X_{K'}$ extend to descent data on G' and X'. Namely, the extensions are unique since both G' and X' are flat and separated over R'. By the same reason, we obtain the compatibility of the descent data with the group structure of G' and the structure of X' as a torsor under G'. Then Theorem 1 is applicable, and it follows that the pair (G', X') descends to a pair (G, X) over R. That G and X satisfy the universal mapping property of Néron models is a consequence of 6.1/6 (a) and, again, of the fact that G' and X' are flat and separated over R'. So, as claimed, it is enough to construct extensions of the canonical descent data on $G_{K'}$ and $X_{K'}$. Next, observe that G' and X' are of finite type over R'. Since $R' \subset R^{sh}$, we see by a limit argument that G' and X' (as well as the group structure of G' and the structure of X' as a torsor under G') are already defined over an étale extension of R. So it is enough to consider the case where R' is étale over R.

Now write $R'' := R' \otimes_R R'$ and let $p_i : \operatorname{Spec} R'' \longrightarrow \operatorname{Spec} R'$, $i = 1, 2$, be the projections. Then, since the formation of Néron models is compatible with étale base change (cf. 1.2/2), we see that $p_i^*(X')$ is a Néron model of $p_i^*(X'_{K'})$ over Spec R''. Thus, by the Néron mapping property, the canonical descent datum

$$\varphi_{K'} : p_1^*(X'_{K'}) \longrightarrow p_2^*(X'_{K'})$$

extends to an isomorphism

$$\varphi : p_1^*(X') \longrightarrow p_2^*(X')$$

which, in fact, constitutes a descent datum on X'. In the same way, the canonical descent datum on $G'_{K'}$ is extended to a descent datum on G. □

Remark 5. The assertion of Corollary 3 remains valid if, instead of a pair $R \subset R'$ where R' is contained in a strict henselization of R, one considers a pair of discrete valuation rings $R \subset R'$ such that a uniformizing element of R gives rise to a uniformizing element of R' and such that the residue extension of R'/R is trivial. For example, R' can be the maximal-adic completion of R (actually, it is only necessary to require that R' is of ramification index 1 over R; see 7.2/1). Namely, reviewing the proof of Corollary 3, the first part, which reduces the assertion to the problem of extending descent data from $G_{K'}$ to G' (resp. $X'_{K'}$ to X'), remains valid. That the required extensions of descent data exist is a consequence of Lemma 6.2/D.3.

It remains to give the *proof of Theorem* 1. For the applications in Corollaries 2 to 4 which have just been discussed, the theorem is not needed in its full generality. Namely, in the first case (Corollary 2), we know that

(a) *there exists an R'-dense open subscheme $U' \subset X'$, stable under the descent datum of X', such that the descent is effective on U',*

whereas in the second case (Corollaries 3 and 4) we know that

(b) *K', the field of fractions of R', is algebraic over K, the field of fractions of R.*

Both properties can simplify the proof substantially. In order to demonstrate this, we will first establish the theorem under the additional assumption (a), and then under (b). Finally, we will indicate how to reduce the general case to the situation (a). Also we want to mention that we have only to work out the descent for the torsor X', because G' can be handled in the same way by viewing it as a trivial torsor under itself.

As a first step we show that, independently of conditions (a) or (b), the descent we have to perform is always effective on generic fibres. So consider the extension $K \longrightarrow K'$ of the fields of fractions of $R \longrightarrow R'$. Since $X'_{K'}$ is of finite type over K', we may use a limit argument and thereby replace K' by a K-subalgebra C of finite type. Then the quotient $C/\mathfrak{m}$ by some maximal ideal $\mathfrak{m} \subset C$ is a finite extension of K. If $[C/\mathfrak{m} : K] = 1$, the morphism $\operatorname{Spec} C \longrightarrow \operatorname{Spec} K$ has a section, and the descent with respect to it is effective by 6.1/5. If $[C/\mathfrak{m} : K] > 1$, the same argument applies to $\operatorname{Spec}(C \otimes_K C/\mathfrak{m}) \longrightarrow \operatorname{Spec} C/\mathfrak{m}$ so that we may replace K' by $C/\mathfrak{m}$. Thereby we are reduced to the case where $[K' : K] < \infty$, and we may assume that K' is quasi-Galois, or since the descent is trivial for radicial extensions, that K' is Galois over K. Then the descent on $X'_{K'}$ is a Galois descent (see Example 6.2/B) and, in order to show it is effective, it is enough to know that finitely many given points of $X'_{K'}$ are always contained in an affine open subscheme of $X'_{K'}$. That the latter condition is fulfilled can be seen either from the quasi-projectivity of $X'_{K'}$ (use 6.4/1) or, in a more elementary way, by using standard translation arguments. So the descent is effective, and $X'_{K'}$ descends to a K-scheme X_K. This settles the assertion of Theorem 1 for the case where R and R' are fields.

Next, let us assume that condition (a) is satisfied. Then U' descends to an R-scheme U, where U_K is open in X_K. Applying Lemma 6.4/4 to U, we can find an R-dense affine open subscheme of U, and hence, by pulling it back to U', an R'-dense affine open subscheme of U' which is stable under the descent datum on X'. In other words, we can assume that U' is affine. We claim one can find an effective divisor D' on X' with support $X' - U'$ such that D' is stable under the descent datum on X'. Denoting the descent datum on X' by $\varphi : p_1^* X' \longrightarrow p_2^* X'$, the latter means that $p_1^* D'$ corresponds to $p_2^* D'$ under the isomorphism φ. In order to obtain such a divisor D', choose an effective divisor D_K on X_K with support $X_K - U_K$ (cf. [EGA IV$_4$], 21.12.7), and define D' as the schematic closure of the pull-back of D_K to $X'_{K'}$. By the properties of the schematic closure, the descent datum on X' extends to a descent datum on the pair $(X', \mathscr{L}')$ where $\mathscr{L}' := \mathscr{O}_X(D')$. Considering the action of the identity component of G' on X', we conclude from 6.4/2 and 6.4/3 that $\mathscr{L}'$ is ample. Hence, 6.1/7 shows that the descent is effective on X'. This settles the assertion of Theorem 1 if condition (a) is given.

Now let us assume that condition (b) is satisfied. We want to reduce to condition (a). Applying Lemma 6.4/4, there is an R'-dense affine open subscheme $\Omega' \subset X'$. In particular, $F'_{K'} := X'_{K'} - \Omega'_{K'}$ is nowhere dense in $X'_{K'}$ and, since K' is algebraic over K, its image F_K in X_K is nowhere dense. Set $U_K := X_K - F_K$. Then $U'_{K'} := U_K \otimes_K K'$ is a dense open subscheme of $\Omega'_{K'}$. Subtracting from X' the schematic closure of $X'_{K'} - U'_{K'}$ we arrive at an R'-dense open subscheme U' of X' whose generic fibre is $U'_{K'}$. Furthermore, by construction, U' is stable under the descent datum on X', and it is quasi-affine since $U' \subset \Omega'$. The latter inclusion is verified by using the fact that $X' - \Omega'$ is the support of a divisor and that, since Ω' is R'-dense in X', the schematic closure of $X'_{K'} - \Omega'_{K'}$ in X' coincides with $X' - \Omega'$. In particular, the descent is effective on U' by 6.1/6, and we have thus reduced assumption (b) to assumption (a).

In order to prove Theorem 1 in its general version, some preparations are necessary. Consider a smooth and separated scheme X of finite type over a discrete valuation ring R. Let K be the field of fractions of R, and let k be the residue field of R. Writing $A := \Gamma(X, \mathscr{O}_X)$, we have a canonical morphism

$$u : X \longrightarrow \operatorname{Spec} A$$

whose formation is compatible with flat base change. For each $f \in A$, we denote by A_f the localization of A by f and by

$$u_f : X_f \longrightarrow \operatorname{Spec} A_f$$

the morphism obtained from u by the base change $\operatorname{Spec} A_f \longrightarrow \operatorname{Spec} A$.

In this situation, u is of finite type since X is of finite type over R. Furthermore, $\operatorname{Spec} A$ is flat over R and normal since the same is true for X. Since the formation of global sections on X commutes with flat base change, there are canonical isomorphisms

$$A_K := A \otimes_R K \cong \Gamma(X_K, \mathscr{O}_X)$$

and, for $f \in A$,

$$A_f \cong \Gamma(X_f, \mathscr{O}_X) .$$

Moreover, we have a canonical injection

$$A_k := A \otimes_R k \hookrightarrow \Gamma(X_k, \mathscr{O}_{X_k}) .$$

So a global section $h \in A$ vanishes on the special fibre X_k if and only if $h \in \pi A$, where π is a uniformizing element of R.

Lemma 6. *Let $u : X \longrightarrow \operatorname{Spec} A$ be as above and assume that the generic fibre X_K is affine. Then $u_K : X_K \longrightarrow \operatorname{Spec} A_K$ is an isomorphism and, if $X_k \neq \varnothing$, there exists an element $f \in A$ such that $X_f \cap X_k \neq \varnothing$ and such that $u_f : X_f \longrightarrow \operatorname{Spec} A_f$ is an isomorphism.*

Proof. The first assertion is clear. Next, assume $X_k \neq \varnothing$. Using the separatedness of X, we can apply Lemma 6.4/4 and find an R-dense affine open subscheme $U \subset X$. Since $u : X \longrightarrow \operatorname{Spec} A$ is an isomorphism on generic fibres, there is an $f \in A_K$, we may assume $f \in A$, such that $(X_f)_K \subset U_K$. Furthermore, X_k is not empty, so we may assume $f \in A - \pi A$. Then consider the schematic closure of $X_K - (X_f)_K$ in X; it is contained in $X - X_f$. Similarly, since U is R-dense and affine in X, its complement $X - U$ is of pure codimension 1 by [EGA IV$_4$], 21.12.7, and we see that it equals the schematic closure of $X_K - U_K$ in X. So we obtain from $(X_f)_K \subset U_K$ the inclusions

$$X_K - (X_f)_K \supset X_K - U_K$$

and, hence,

$$X - X_f \supset X - U .$$

Therefore $X_f \subset U$ and, thus, $X_f = U_f$ is affine. Interpreting A_f as the ring of global sections on X_f, the morphism $u_f : X_f \longrightarrow \operatorname{Spec} A_f$ is an isomorphism. Consequently, since f does not vanish identically on X_k, the assertion of the lemma follows. □

It should be realized that, in the situation of Lemma 6, we cannot expect to find a global section $f \in A$ such that $u_f : X_f \longrightarrow \operatorname{Spec} A_f$ is an isomorphism and X_f is *R-dense* in X. For example, consider an irreducible conic $C \subset \mathbb{P}^2_R$ whose special fibre consists of two projective lines P_1 and P_2. Assume that C admits an R-valued point meeting P_2, but not P_1. Removing this point from C, we obtain an R-scheme X whose generic fibre is affine and whose special fibre consists of two components, one of them P_1. Since each global section of $\mathscr{O}_X$ must be constant on P_1, we see that any subscheme $X_f \subset X$, as in Lemma 6, must be disjoint from P_1. So X_f cannot be R-dense in this case.

Returning to the proof of Theorem 1, it is enough to construct an open subscheme $U' \subset X'$ as required in condition (a). In order to do this, we will forget about the special situation given in Theorem 1 and assume only that X' is a smooth and separated R'-scheme of finite type with a descent datum on it, which is effective on the generic fibre $X'_{K'}$. In particular, we may apply the above considerations to X' as a scheme over R' (and to suitable open subschemes of it). First we reduce to the case where the generic fibre of X' is affine; then Lemma 6 is applicable. Let $K \longrightarrow K'$ be the extension of fields of fractions corresponding to $R \longrightarrow R'$. We know already that the generic fibre $X'_{K'}$ descends to a K-scheme X_K. Choose an affine dense open subscheme $U_K \subset X_K$ and consider its pull-back $U'_{K'}$ to $X'_{K'}$. Then $X'_{K'} - U'_{K'}$ is thin in $X'_{K'}$, and its schematic closure is R'-thin in X'. If we remove it from X', we obtain an R'-dense open subscheme whose generic fibre is affine and which is stable under the descent datum on X'. We can replace X' by this subscheme and thereby assume that the generic fibre of X' is affine.

Now set $A' = \Gamma(X', \mathcal{O}_{X'})$ and consider the canonical morphism $u' : X' \longrightarrow \operatorname{Spec} A'$. Then the descent datum on X' yields a descent datum on $\operatorname{Spec} A'$ such that the morphism u' is compatible with these descent data. Let U' be the open subscheme of X' consisting of all points of X' where u is quasi-finite. We claim that

(i) the generic fibre of U' coincides with X'_K, and the special fibre of U' is non-empty,

(ii) U' is stable under the descent datum of X', and

(iii) U' is quasi-affine; in particular, the descent datum is effective on U'.

Namely, property (i) is a consequence of Lemma 6, whereas property (ii) follows from the fact that, for a morphism of finite type, quasi-finiteness at a certain point can be tested after surjective base change such as provided by the projections $\operatorname{Spec} R' \times_R \operatorname{Spec} R' \rightrightarrows \operatorname{Spec} R'$. In order to justify the latter claim, observe that quasi-finiteness can be tested on fibres. So it is enough to consider a field as base and a field extension as base change. In this situation, a dimension argument gives the desired assertion. Finally, property (iii) follows from Zariski's Main Theorem (in the version 2.3/2'); it implies that $u' : X' \longrightarrow \operatorname{Spec} A'$ restricts to an open immersion on U'. So U' is quasi-affine, and the descent is effective on U' by 6.1/4.

If U' is R'-dense in X', we have obtained an open subscheme of X' as required in condition (a). If U' is not R'-dense in X', remove from X' all components of the special fibre which meet U'. The resulting open subscheme of X', call it X'_1, is again stable under the descent datum. So, concluding as before, X'_1 contains an open subscheme U'_1 satisfying conditions (i) to (iii). Continuing this way, we can work up the finitely many components of X_k and thereby obtain finitely many open subschemes $U', U'_1, \ldots, U'_n \subset X'$ satisfying conditions (i) to (iii). Then the union of these subschemes is R'-dense in X' and, hence, gives rise to an open subscheme of X' as required in condition (a), thereby finishing the proof of Theorem 1. □

6.6 Applications to Birational Group Laws

In this section, we want to sharpen M. Artin's result on the construction of group laws from birational group laws, which is explained in [SGA 3_{II}], Exp. XVIII. Let S be a scheme, and consider an S-birational group law m on a smooth S-scheme X. It is shown in [SGA 3_{II}], Exp. XVIII, that, if m is strict in the sense of 5.2/1, there exists a solution $\bar{X}$ in the category of algebraic spaces such that $\bar{X}$ contains X as an S-dense open subspace; for the notion of algebraic spaces see Section 8.3. We will admit this result. However, if the base S is normal, it could also have been obtained by the construction technique of Section 5.3. The latter method yields even more, namely that $\bar{X}$ is a scheme for the étale topology of S. Using the descent techniques of Section 6.5, we want to show that $\bar{X}$ is already a scheme. So, we will mainly be concerned with the representability of a smooth group object in the category of algebraic spaces.

Theorem 1. *Let S be a scheme, and let m be an S-birational group law on a smooth and separated S-scheme X which is faithfully flat and of finite presentation over S. Then there exists a smooth and separated S-group scheme $\bar{X}$ of finite presentation with a group law $\bar{m}$, together with an S-dense open subscheme $X' \subset X$ and an open immersion $X' \hookrightarrow \bar{X}$ having S-dense image such that $\bar{m}$ restricts to m on X'.*

The group scheme $\bar{X}$ is unique up to canonical isomorphism. If the S-birational group law m is strict, the assertion is true with X' replaced by X.

Proof. Due to the uniqueness assertion 5.1/3, we may assume that S is affine and, using limit arguments, that S is noetherian. If the S-birational law is strict, it follows from the result of M. Artin that there exists a solution $\bar{X}$ of the strict law in the category of algebraic spaces containing X as an S-dense open subspace of $\bar{X}$. As we will see by the theorem below, the solution is represented by a scheme. Thereby, Theorem 1 will be proved for the case where the S-birational group law is strict. Now we want to treat the general case accepting the assertion of Theorem 1 for strict S-birational laws.

Let U be the largest open subscheme of S such that the S-birational group law has a solution over U; here and in the following, solutions are meant in the category of schemes. If $U \neq S$ choose the generic point s of an irreducible component of $S - U$. Since we consider only S-schemes of finite presentation, it suffices to verify that there exists a solution after the base change $\operatorname{Spec}(\mathcal{O}_{S,s}) \longrightarrow S$. So we may assume that S is a local scheme, and that s is the closed point of S; then $U = S - \{s\}$.

Assume first that, for each component X^i_s of X_s, there exists a section σ_i of X over S crossing the given component. Let $X(\sigma_i)$ be the union of all components of the fibres of X meeting the section σ_i; due to [EGA IV_3], 15.6.5, $X(\sigma_i)$ is an open subscheme of X. Denote by X_0 the union of the $X(\sigma_i)$; note that X_0 might not be S-dense in X. Then m induces an S-birational group law m_0 on X_0. Moreover, due to the construction, the components of the fibres of X are geometrically irreducible. Now one can proceed as in the proof of 5.2/2. The set Z (in the proof of 5.2/2) will provide an S-dense open subscheme X'_0 of X_0 such that m_0 induces a strict law m'_0 on X'_0. Namely, set

$$\Omega_1 = \bigcup_i \left(\bigcap_j p_1(Z \cap (X(\sigma_i) \times_S X(\sigma_j))) \right)$$

where $p_1 : X \times_S X \longrightarrow X$ is the first projection. Then Ω_1 is S-dense open in X_0, and $Z \cap (\Omega_1 \times_S X_0)$ is Ω_1-dense in $\Omega_1 \times_S X_0$. Defining Ω_2 in a similar way by using the second projection, the intersection $\Omega_1 \cap \Omega_2$ defines an S-dense open subscheme X'_0 of X_0. As in 5.2/2, one shows that the restriction m'_0 of m to X'_0 is strict. As we have said above, there is a solution $\bar{X}'_0$ of the strict law m'_0 which contains X'_0 as an S-dense open subscheme. Since $\bar{X}'_0 \times_S U$ is an open subscheme of the solution $\bar{X}_U$ of the restriction of m to U, one can glue $\bar{X}'_0$ and $\bar{X}_U$ along $\bar{X}'_0 \times_S U$ in order to get a solution of m.

In the general case, one performs first an étale surjective extension $S^* \longrightarrow S$ of the base in order to get enough sections of X. So one obtains a solution $\bar{X}^*$ of the S^*-birational group law $m \times_S S^*$. Now consider the S^*-birational map

$$\iota : X \times_S S^* \dashrightarrow \bar{X}^* .$$

The canonical descent datum extends to a descent datum on $\bar{X}^*$ by the uniqueness of solutions; cf. 5.1/3. Furthermore, there exists a largest open subscheme X^* of

$X \times_S S^*$, where the map ι is defined and where ι is an open immersion; use the separatedness of $X \times_S S^*$ and of $\bar{X}^*$ as well as the birationality of ι. Since the domain of definition is compatible with flat base change (cf. 2.5/6), the formation of the largest open subscheme where ι is defined and where ι is an open immersion is compatible with flat base change. So X^* is stable under the descent datum and, hence, there exists an open subscheme X' of X which is S-dense in X such that $X' \times_S S^* = X^*$. Then it is easy to see that the S-birational law m on X restricts to a strict law on X'. □

In order to complete the proof of the preceding theorem, it remains to show the following result on the representability of algebraic spaces with group action.

Theorem 2. *Let S be a locally noetherian scheme and let G be a group object in the category of algebraic spaces over S. Assume that G is smooth over S and that G has connected fibres over S. Let X be a smooth algebraic space over S and let*

$$\sigma : G \times_S X \longrightarrow X$$

be a group action on X. Let Y be an open subspace of X. Then the image GY of $G \times_S Y$ in X is an open subspace of X. If GY equals X, the following assertions hold:

(a) *If Y is separated (resp. of finite type) over S, the same is true for X.*

(b) *If Y is a scheme, then X is a scheme.*

(c) *If S is affine and if Y is quasi-affine, any finite set of points of X is contained in an affine open subset of X.*

(d) *If S is normal and if Y is affine over S, any effective Weil divisor of X with support $X - Y$ is a Cartier divisor, and is S-ample. In particular, X is quasi-projective over S.*

Corollary 3. *Let S be a Dedekind scheme, and let G be a group object in the category of algebraic spaces over S. Assume that G is separated, smooth, and of finite type over S. Then G is a scheme.*

Proof of Corollary 3. Let Y be the open subspace of G consisting of all points which admit a scheme-like neighborhood. Due to Raynaud [6], Lemme 3.3.2, Y contains all the generic points of the fibres of G over S. Hence, Y is S-dense in G. So Theorem 2 yields that G is a scheme. □

Proof of Theorem 2. The group action σ is the composition of the maps

$$G \times_S X \xrightarrow{(p_1,\sigma)} G \times_S X \xrightarrow{p_2} X$$

where p_i is the projection onto the i-th factor, $i = 1, 2$. The first map is an isomorphism, and the second one is smooth, since G is smooth over S. Hence, the map σ is open, and the image GY is an open subspace of X.

(a) In order to prove the separatedness of X, we can use the valuative criterion. So, we may assume that S consists of a discrete valuation ring R with field of fractions K and residue field k. Then we have to show that any two R-valued points $x_1, x_2 \in X(R)$ which coincide on the generic fibre are equal. Let $\bar{x}_1, \bar{x}_2$ be the induced

closed points. Since the sets

$$\bar{U}_i = \{\bar{g} \in G \times_S k, \bar{g}^{-1}\bar{x}_i \in Y \times_S k\}, \qquad i = 1, 2,$$

are open and non-empty, they are dense in $G \times_S k$. Due to the smoothness of G over S, there exist an étale surjective base extension $R \longrightarrow R'$ and a section $g \in G(R')$ inducing a point of $\bar{U}_1 \cap \bar{U}_2$. Thus $\bar{x}_i \in gY$ and, hence, $x_i \in gY$ for $i = 1, 2$. Since Y is separated over S, we see that $x_1 = x_2$.

In order to show that X is of finite type over S if Y is, it suffices to verify that X is quasi-compact if S is affine. Since the map

$$\sigma : G \times_S Y \longrightarrow X$$

is surjective, the assertion follows from the fact that G is quasi-compact, as can easily be deduced from Lemma 5.1/4.

(d) We may assume that S is affine. Due to assertion (a), X is of finite presentation and separated over S. Let D be an effective Weil divisor with support $X - Y$. Due to the theorem of Ramanujam-Samuel [EGA IV$_4$], 21.14.3, D is a relative Cartier divisor. Namely, as can be seen by an étale localization on X, this theorem carries over to the case of algebraic spaces. Next we want to show that $\mathscr{L} = \mathcal{O}_X(D)$ is S-ample. To do this, we need the fact that $\mathscr{L}^{\otimes n}$ satisfies the theorem of the square for large integers n if the generic fibres of X over S are geometrically irreducible, cf. Section 6.3. Namely, after étale localization of the base, X can be covered by open subspaces of type X_l where l varies over the global sections of $\mathscr{L}^{\otimes n}$. The X_l are affine as intersections of translates of Y; cf. the proof of 6.4/2 or Raynaud [4], Thm. V.3.10, p. 88. In order to verify that $\mathscr{L}^{\otimes n}$ satisfies the theorem of the square for large integers n, one proceeds as follows:

Similarly as in the proof of 6.3/2, one reduces to the case where S consists of a field. Then G is a scheme; cf. Section 8.3. We claim that X is a scheme, too. Let U be the set consisting of all points of X admitting a scheme-like neighborhood. Using finite Galois descent, one easily shows that U is invariant under G, since any finite set of points of U is contained in an affine open subscheme of U. In our case, due to the assumption $X = GY$, one has $U = X$. So, X is a scheme, and the assertion follows from Raynaud [4], Thm. IV 3.3 (d), p. 72.

Finally, since $Y \longrightarrow S$ is affine, the reduced subscheme with support $X - Y$ is a Weil divisor by [EGA IV$_4$], 21.12.7, and thus an S-ample Cartier divisor. Therefore $X \longrightarrow S$ is quasi-projective.

(c) First, let us show assertion (c) under the additional assumption that S is normal. Let $x_1, \ldots, x_n$ be finitely many points of X, and let $s_1, \ldots, s_n$ be their images in S. Since Y is quasi-affine, there exists an affine open subscheme Y^* of Y which gives rise to a dense open subscheme of the fibres $Y_{s_1}, \ldots, Y_{s_n}$. Then the points $x_1, \ldots, x_n$ are contained in the image X^* of $G \times_S Y^*$ under σ. We may replace X by X^*, and so we may assume that Y is affine. In this case, the assertion follows from assertion (d). Namely, X admits a relatively ample line bundle, since $X - Y$ with its reduced structure gives rise to a Weil divisor; cf. [EGA IV$_4$], 21.12.7. So, X is quasi-projective over S, and hence X satisfies assertion (c).

Now let us consider the general case. Using limit arguments, we may assume that S is of finite type over the ring of integers $\mathbb{Z}$. Let $\tilde{S}$ be the normalization of S, and set $\tilde{X} = X \times_S \tilde{S}$ and $\tilde{G} = G \times_S \tilde{S}$. Then $\tilde{X}$ is a scheme by what we have just proved, and any finite set of points of $\tilde{X}$ is contained in an affine open subscheme of $\tilde{X}$. Furthermore, $X' = X \times_S S'$ is a scheme after étale surjective base extension $S' \longrightarrow S$, since there are finitely many sections of G such that X can be covered by the translates of Y under these sections, as follows from 5.3/7; see also 6.4/2. In order to show the effectivity of the canonical descent datum on X' we make use of the following result which is contained in Raynaud [3], Cor. 3.8 and Thm. 4.2:

Let S be a locally noetherian scheme, let $S' \longrightarrow S$ be a faithfully flat quasi-compact morphism of schemes, and let $\tilde{S} \longrightarrow S$ be a finite surjective morphism of schemes. Let X be a sheaf for the fppf-topology of S (cf. Section 8.1). Assume that $X' = X \times_S S'$ is represented by an S'-scheme which is locally of finite presentation, and that $\tilde{X} = X \times_S \tilde{S}$ is represented by an $\tilde{S}$-scheme. Then

(i) *X is represented by an S-scheme of finite presentation if and only if, for each point $\tilde{x}$ of $\tilde{X}$, there exists an affine open subscheme of $\tilde{X}$ which contains all points of $\tilde{X}$ giving rise to the same point of X as $\tilde{x}$.*

(ii) *If $\tilde{X}$ satisfies the property that any finite set of points of $\tilde{X}$ is contained in an open affine subscheme, so does X.*

Thus we see that X is a scheme, and any finite set of points of X is contained in an affine open subscheme of X, since $\tilde{X}$ has this property.

Assertion (b) follows from (c). □

6.7 An Example of Non-Effective Descent

Let R be a discrete valuation ring with field of fractions K and residue field k. In the present section we will consider relative curves over R; i.e., flat R-schemes X whose fibres are of pure dimension 1. We assume that, in addition, X is normal and proper over R and that the generic fibre X_K is connected. Then X_K is regular (in fact, smooth over K if char $K = 0$), and the set of singular points x of X (i.e., of those points where the local ring $\mathcal{O}_{X,x}$ is not regular) is a finite subset of the special fibre X_k; see [EGA IV_2], 5.8.6, and [EGA IV_2], 6.12.6. The example we want to present is based on the fact that, after replacing the base R by a henselization R^h, irreducible components of X_k can be contracted in X whereas, over a non-henselian ring R, such a procedure is not always possible.

To construct an R-curve with a non-effective descent datum on it, set $A = \mathbb{C}[\tau, \tau^{-1}]$, where τ is an indeterminate, and start out from a smooth and proper elliptic curve E over $S = \operatorname{Spec} A$ which has non-constant j-invariant. Alternatively, we can consider the ring $A = \mathbb{Q}[\tau, \tau^{-1}]$ and the elliptic curve with constant j-invariant $E \subset \mathbb{P}^2_S$ which is given by the equation

$$y^2 z = x^3 + \tau x z^2 .$$

Replacing A by the local ring $R = \mathcal{O}_{S,t}$ at a closed point $t \in S$ if $A = \mathbb{C}[\tau, \tau^{-1}]$ (resp. at a suitable closed point $t \in S$ corresponding to a maximal ideal $(\tau - t) \subset A$ with $t \in \mathbb{Q}^*$ if $A = \mathbb{Q}[\tau, \tau^{-1}]$), we will show in Proposition 5 that there exists a rational

point $a_k \in E_k$ such that none of the multiples ra_k with $r > 0$ admits a lifting to an R-valued point of E. Blowing up a_k in E yields a proper curve X over R which is regular. Its special fibre X_k consists of two components, the strict transform $\tilde{E}_k$ of E_k and the inverse image of a_k which is a projective line P_k; both intersect transversally at a single point.

In this situation we will see in Lemma 6 that one cannot contract the component $\tilde{E}_k$ in X; i.e., there does not exist an R-morphism $u : X \longrightarrow Y$ of proper normal curves over R which is an isomorphism over $Y - \{y\}$ and which satisfies $\tilde{E}_k = u^{-1}(y)$. However, if we pass from R to a henselization R^h and consider the curve $X' = X \otimes_R R^h$ over R^h, the special fibre of X remains unchanged, and we will be able to conclude from Proposition 4 below that $\tilde{E}_k$ can be contracted in X'.

Let $u' : X' \longrightarrow Y'$ be such a contraction. There are canonical descent data on X' and on Y' with respect to $R \longrightarrow R^h$; namely on X', since it is obtained from X by means of the base change $R \longrightarrow R^h$, and on Y' since u' is an isomorphism on generic fibres and since each descent datum on the generic fibre of Y' extends uniquely to a descent datum on Y' by 6.2/D3. Furthermore, u' is compatible with these data. So if the descent datum on Y' were effective, $u' : X' \longrightarrow Y'$ would descend to an R-morphism $u : X \longrightarrow Y$, where Y is a proper normal curve by [EGA IV_2], 2.7.1 and 6.5.4. Since u' coincides with u on special fibres, the latter morphism would be a contraction of $\tilde{E}_k$ in X. However such a contraction cannot exist by Lemma 6 and, consequently, the descent datum on Y' cannot be effective.

Now, after we have given the description of the curve Y' and the non-effective descent datum on it, let us fill in the results mentioned above which are needed to make the example work. We begin with the explanation of contractions; see also M. Artin [1], [2]. So consider an arbitrary discrete valuation ring R and an R-curve X where, as we have said at the beginning of this section, X is assumed to be proper and normal and to have a connected generic fibre. Let $(X_i)_{i \in I}$ be the family of irreducible components of the special fibre X_k, providing them with the canonical reduced structure. For a strict subset $J \subset I$, a contraction of the components X_j, $j \in J$, in X consists of an R-morphism $u : X \longrightarrow Y$ of proper normal curves over R such that

(a) for each $j \in J$, the image $u(X_j)$ consists of a single point $y_j \in Y$, and

(b) u defines an isomorphism $X - \bigcup_{j \in J} X_j \xrightarrow{\sim} Y - \bigcup_{j \in J} \{y_j\}$.

Then u is automatically proper since X is proper over R and since Y is separated over R. Furthermore, using the Stein factorization [EGA III_1], 4.3.1, it is easily seen that u depends uniquely on the subset $J \subset I$ and that the fibres of u are connected. In order to give a criterion for the existence of contractions, we use the notion of effective relative Cartier divisors; cf. Section 8.2, in particular 8.2/6.

Theorem 1. *Let X be a proper normal R-curve with connected generic fibre X_K, let $(X_i)_{i \in I}$ be the family of irreducible components of the special fibre X_k, and consider a non-trivial effective relative Cartier divisor D on X. Let J be the set of all indices $j \in I$ such that $\operatorname{supp}(D) \cap X_j = \varnothing$. Then the canonical morphism*

$$u : X \longrightarrow Y := \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty} \Gamma(X, \mathcal{O}_X(mD)) \right)$$

is a contraction of the components $X_j, j \in J$, and Y is a proper normal R-curve which is projective.

Before we give a proof, let us look at properties of Y which follow from its definition as a projective spectrum of a graded ring.

Lemma 2. *Let X be a proper scheme over a noetherian ring R and let $\mathscr{L}$ be an invertible sheaf on X such that, for some $n > 0$, the sheaf $\mathscr{L}^{\otimes n}$ is generated by its global sections. Then, for*

$$A = \bigoplus_{m=0}^{\infty} \Gamma(X, \mathscr{L}^{\otimes m}),$$

the scheme $Y = \mathrm{Proj}(A)$ is projective over R and the canonical morphism $u: X \longrightarrow Y$ has connected fibres. If, in addition, X is normal, Y is normal also.

Proof. Applying [EGA III_1], 3.3.1, we see that the ring A is of finite type over R. Thus $Y = \mathrm{Proj}(A)$ is projective over R; cf. [EGA II], 4.4.1.

For any section $l \in \Gamma(X, \mathscr{L}^{\otimes n})$, the morphism u gives rise to an isomorphism

$$A_{(l)} \xrightarrow{\sim} \Gamma(X_l, \mathcal{O}_X).$$

So $u_*(\mathcal{O}_X) = \mathcal{O}_Y$ and, since u is proper, it follows from [EGA III_1], 4.3.2, that the fibres of u are connected. Finally, if X is normal, the ring $\Gamma(X_l, \mathcal{O}_X)$ is seen to be integrally closed in its total ring of fractions. This implies that Y is normal. □

Now we come to the *proof of Theorem* 1. Set $\mathscr{L} := \mathcal{O}_X(D)$. We claim that $\mathscr{L}^{\otimes n}$ is generated by its global sections if n is large enough. Then Y will be projective and normal by the preceding lemma. In order to justify the claim, it is enough to find global sections generating $\mathscr{L}^{\otimes n}$ at the points of $\mathrm{supp}(D)$; the constant 1, as a global section of $\mathcal{O}_X$, will generate $\mathscr{L}^{\otimes n}$ elsewhere. So consider the exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Taking the tensor product with $\mathscr{L}^{\otimes n}$ yields the exact sequence

$$0 \longrightarrow \mathscr{L}^{\otimes n-1} \longrightarrow \mathscr{L}^{\otimes n} \longrightarrow \mathcal{O}_D \otimes \mathscr{L}^{\otimes n} \longrightarrow 0,$$

and we can use the following part of the associated cohomology sequence:

$$(*) \quad H^0(X, \mathscr{L}^{\otimes n}) \longrightarrow H^0(X, \mathcal{O}_D \otimes \mathscr{L}^{\otimes n}) \longrightarrow H^1(X, \mathscr{L}^{\otimes n-1}) \longrightarrow H^1(X, \mathscr{L}^{\otimes n}) \longrightarrow 0.$$

Note that $H^1(X, \mathcal{O}_D \otimes \mathscr{L}^{\otimes n}) = 0$ since D defines a closed subscheme of X which is affine; the latter is due to the fact that D is quasi-finite, proper and, hence, finite over R.

Next, consider the restriction D_K of D to the generic fibre X_K. Then D_K has a positive degree on X_K since D is effective and non-trivial, and we see that D_K is ample since X_K is irreducible. Therefore $H^1(X_K, \mathscr{L}^{\otimes n}) = 0$ for n big enough, and it follows that $H^1(X, \mathscr{L}^{\otimes n})$ is an R-torsion module of finite length since it is of finite type. The exact sequence $(*)$ implies that the length is decreasing for ascending n. Hence the

length will become stationary and, for n big enough, the map

$$H^1(X, \mathscr{L}^{\otimes n-1}) \longrightarrow H^1(X, \mathscr{L}^{\otimes n})$$

is an isomorphism. But then

$$H^0(X, \mathscr{L}^{\otimes n}) \longrightarrow H^0(X, \mathcal{O}_D \otimes \mathscr{L}^{\otimes n})$$

is surjective. Thereby we see that $\mathscr{L}^{\otimes n}$ is generated by its global sections at the points of $\mathrm{supp}(D)$ and, hence, at all points of X, as claimed.

It remains to show that $u: X \longrightarrow Y$ is a contraction of the components $X_j, j \in J$. Fix such a component X_j. Then, since X_j is proper, each global section of $\mathcal{O}_X(nD)$ induces a constant function on X_j; i.e., an element of the finite extension $\Gamma(X_j, \mathcal{O}_{X_j})$ of k. Therefore the image $u(X_j)$ consists of a single point $y_j \in Y$. Next look at a component X_i with $i \in I - J$. Fix a point $x \in X_i \cap \mathrm{supp}(D)$ and, for some $n \in \mathbb{N}$ big enough, choose a global section l of $\mathcal{O}_X(nD)$ such that l generates $\mathcal{O}_X(nD)$ over a neighborhood U of x. Then $1/l$ may be viewed as a section in $\mathcal{O}_Y$ over Y_l or (by means of the pull-back under u) as a section in $\mathcal{O}_X$ over X_l. By its construction, $1/l$ vanishes on $U \cap \mathrm{supp}(D)$ and is non-zero on $U - \mathrm{supp}(D)$. Therefore the image $u(X_i)$ cannot consist of a single point so that u must be quasi-finite on X_i. Finally, using the facts that the fibres of $u: X \longrightarrow Y$ are connected and that Y is normal (see Lemma 2), one concludes with the help of Zariski's Main Theorem 2.3/2′ that u is a contraction of the components $X_j, j \in J$.

Corollary 3. *Let X be a proper normal R-curve with connected irreducible generic fibre X_K and let $X_i, i \in I$, be the irreducible components of the special fibre X_k. Let J be a strict subset of I. Then the following conditions are equivalent:*

(a) *There exists a contraction $X \longrightarrow Y$ of the components $X_j, j \in J$, where Y is projective over R.*

(b) *There exists a contraction $X \longrightarrow Y$ of the components $X_j, j \in J$, and there is a non-empty R-dense affine open subset $V \subset Y$ such that the images of the X_j as well as all singular points of Y are contained in V.*

(c) *There exists an effective relative Cartier divisor D on X with the property that $\mathrm{supp}(D) \cap X_j = \emptyset$ for all $j \in J$ and $\mathrm{supp}(D) \cap X_i \neq \emptyset$ for all $i \in I - J$.*

Proof. The implication (a) $\Longrightarrow$ (b) is clear since the set of singular points of Y is a finite subset of the special fibre Y_k and since Y is projective over R. To show the implication (b) $\Longrightarrow$ (c), choose an R-dense affine open subscheme $V \subset Y$ which contains the images of the components $X_j, j \in J$, as well as all singular points of Y. Then $Y - V$ gives rise to a relative Cartier divisor on Y whose inverse under $X \longrightarrow Y$ is a divisor on X as required in condition (c). Finally, the implication (c) $\Longrightarrow$ (a) follows from Theorem 1. □

Proposition 4. *In the situation of Corollary 3, assume that the valuation ring R is henselian. Then there exists an effective relative Cartier divisor D on X as required in condition (c) of Corollary 3. In particular, any strict subset of the set of irreducible components of X_k can be contracted in X.*