the maximal ideal $\mathfrak{m}_i$ of each R_i which occurs in the construction of the (strict) henselization of R, it is clear that $\mathfrak{m}$ also generates the maximal ideal of the (strict) henselization. In particular, we see that the (strict) henselization of a discrete valuation ring is a discrete valuation ring, and that a uniformizing parameter of R yields a uniformizing parameter of the (strict) henselization. Furthermore, one can show that properties of local rings such as being noetherian or regular are preserved by the process of (strict) henselization. We state this for later reference:

Proposition 10. *If R is a reduced (resp. normal, resp. regular, resp. noetherian) local ring, the (strict) henselization is reduced (resp. normal, resp. regular, resp. noetherian) again. In particular, if R is a discrete valuation ring with uniformizing parameter π, then the (strict) henselization is a discrete valuation ring, and π gives rise to a uniformizing element there.*

Finally, we want to have a closer look at the ring extensions

$$R \longrightarrow R^h \longrightarrow R^{sh} .$$

Due to the local structure of étale morphisms (Proposition 3), these canonical homomorphisms are injective. Since R^{sh} can also be interpreted as the strict henselization of R^h, it follows from the construction of R^{sh} that the extension $R^h \hookrightarrow R^{sh}$ is integral, as can be seen by using the characterization of henselian rings mentioned in Proposition 4(e). If R is normal, the rings R^h and R^{sh} are normal and, hence, integral domains. Thus we can consider their fields of fractions

$$K \subset K^h \subset K^{sh} ,$$

which are separable algebraic over K. Moreover, K^{sh} is a Galois extension of K^h, the Galois group of K^{sh} over K^h acts on R^{sh}, and the fixed subring of R^{sh} is R^h. Due to Lemma 7, the Galois group is canonically isomorphic to the Galois group of k_s over k.

Proposition 11. *Let R be normal with field of fractions K. Let K_s be a separable closure of K, and let G be the Galois group of K_s over K. Let R_s be the integral closure of R in K_s, and let $\mathfrak{m}_s$ be a maximal ideal of R_s lying over the maximal ideal $\mathfrak{m}$ of R. Let*

$$D = \{\sigma \in G; \sigma(\mathfrak{m}_s) = \mathfrak{m}_s\}$$

be the decomposition group of $\mathfrak{m}_s$, and let

$$I = \{\sigma \in D; \sigma(\bar{x}) = \bar{x} \text{ for } \bar{x} \in R_s/\mathfrak{m}_s\}$$

be the inertia group of $\mathfrak{m}_s$. Then the following assertions hold:

(a) *The localization R' of the fixed ring R_s^D of R_s under D at the maximal ideal $\mathfrak{m}_s \cap R_s^D$ is the henselization of R.*

(b) *The localization R'' of the fixed ring R_s^I of R_s under I at the maximal ideal $\mathfrak{m}_s \cap R_s^I$ is the strict henselization of R.*

(c) *The extension $R^h \subset R^{sh}$ is Galois. Its Galois group D/I is canonically isomorphic to the Galois group of the residue field extension k_s over k.*

Proof. (a) Let $P(T) \in R'[T]$ be a monic polynomial whose reduction $\bar{P}(T)$ has a simple zero $\bar{\alpha}$ lying in the residue field of R'. Now $P(T)$ has a zero α lying in $(R_s)_{\mathfrak{m}_s}$ which induces $\bar{\alpha}$ if we regard $\bar{\alpha}$ as an element of $R_s/\mathfrak{m}_s$. Since $\bar{\alpha}$ is simple, there is only one zero α of this kind. Then it is easily seen that α is invariant under D. Hence α lies in R'. Thus we see R' is henselian. Moreover it is known that R' is a limit of étale extensions R_i of R which have the same residue fields as R; cf. Raynaud [5], Chap. X. So R' is a henselization of R.

(b) follows similarly as (a), one has only to replace the decomposition group by the inertia group. Assertion (c) follows from (a) and (b) by formal arguments. □

2.4 Flatness

Let R be a ring, and let M be an R-module. Then M is called *flat over R* (or a flat R-module) if

$$Mod_R \longrightarrow Mod_R , \qquad N \longmapsto N \otimes_R M ,$$

constitutes an exact functor on the category of R-modules Mod_R. If R is a field, flatness poses no condition, and if R is a Dedekind domain, the flatness of M means that M has no torsion. Flatness is a local property; i.e., an R-module M is flat over R if and only if, for each prime ideal $\mathfrak{p}$ of R, the localization $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$. For a local ring R, a finitely generated R-module is flat if and only if it is free; cf. Bourbaki [2], Chap. I, § 2, ex. 23. But, in general, flat modules do not need to be free or projective (in the sense of being a direct factor of a free module); for example, the field of fractions of a discrete valuation ring R is a flat R-module which cannot be free. Nevertheless, it can be shown that an R-module M is flat if and only if M is a direct limit of free R-modules of finite type; cf. Lazard [1], Thm. 1.2, or Bourbaki [1], Chap. X, § 1, n°6, Thm. 1. A flat R-module M is called *faithfully flat* if the tensor product by M is a faithful functor; i.e., if $N \otimes_R M \neq 0$ for all R-modules $N \neq 0$. Viewing R-algebras as R-modules, one has also the notion of flatness (resp. faithful flatness) for R-algebras. For example, localizations $S^{-1}R$ are flat R-algebras and polynomial rings $R[T_1, \ldots, T_n]$ are faithfully flat R-algebras. Furthermore, we want to mention that a local flat morphism $R \longrightarrow A$ of local rings is automatically faithfully flat.

Now, turning to schemes, a morphism $f: X \longrightarrow S$ of schemes is called *flat at a point* x of X if $\mathcal{O}_{S,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is flat, and f is called *flat* if it is flat at all points of X. Furthermore, a morphism $f: X \longrightarrow S$ is said to be *faithfully flat* if f is flat and surjective. If X and S are affine, say $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$, then f is flat (resp. faithfully flat) if and only if $f^*: R \longrightarrow A$ is flat (resp. faithfully flat). Obviously, open immersions are flat, and it is easy to see that the class of flat (resp. faithfully flat) morphisms is stable under composition, base change, and formation of products; cf. [EGA IV_2], 2.1 and 2.2. In the case where S is the spectrum of a discrete valuation ring, $f: X \longrightarrow S$ is flat if and only if $\mathcal{O}_X$ has no R-torsion. So there are no irreducible and no embedded components of X which are contained in the special fibre. Since the notion of flatness is quite transparent over valuation rings, it is useful to know that there is a valuative criterion for flatness which applies to the geometric case.

Proposition 1 ([EGA IV_3], 11.8.1). *Let $f: X \longrightarrow S$ be locally of finite presentation. Let x be a point of X, and set $s = f(x)$. Assume that $\mathcal{O}_{S,s}$ is reduced and noetherian. Then f is flat at x if and only if, for each scheme S' which is the spectrum of a discrete valuation ring, and each morphism $S' \longrightarrow S$ sending the special point s' of S' to s, the morphism $f': X' \longrightarrow S'$ obtained from f by the base change $S' \longrightarrow S$ is flat at all points $x' \in X'$ lying over x.*

It is much more difficult to understand the notion of flatness in the case where the base has nilpotent elements, for example, where the base is a non-trivial artinian ring. In this case there exists no criterion to test flatness by geometric properties.

Furthermore, we want to mention a criterion which allows to test the flatness of an S-morphism between flat S-schemes on fibres.

Proposition 2 ([EGA IV_3], 11.3.11). *Let $g: X \longrightarrow S$ and $h: Y \longrightarrow S$ be locally of finite presentation. Let $f: X \longrightarrow Y$ be an S-morphism. The following conditions are equivalent:*

(a) *f is flat, and h is flat at the points of $f(X)$.*

(b) *$f_s = f \times_S k(s)$ is flat for all $s \in S$, and g is flat.*

Now let us illustrate the meaning of flatness by some geometric properties of flat morphisms of finite presentation. In the following, let $f: X \longrightarrow Y$ always be a morphism of finite presentation. There are two general facts concerning the geometry of such morphisms. First, the image $f(C)$ of a constructible subset C of X is constructible in Y if Y is quasi-compact; a subset of a topological space is called constructible if it is a union of a finite collection of locally closed subsets; cf. [EGA IV_1], 1.8.4. Second, the function of relative dimension of f

$$X \longrightarrow \mathbb{N}, \qquad x \longmapsto \dim_x f^{-1}(f(x)),$$

is upper semi-continuous; i.e., for each $n \in \mathbb{N}$ the subset where the relative dimension is $\geq n$ is closed; cf. [EGA IV_3], 13.1.3. If we assume that, in addition, f is flat, the situation becomes much better.

Proposition 3 ([EGA IV_2], 2.4.6). *Let $f: X \longrightarrow Y$ be locally of finite presentation. If f is flat, then f is open.*

Proposition 4 ([EGA IV_3], 14.2.2). *Let $f: X \longrightarrow Y$ be locally of finite type and flat. Assume that X is irreducible and that Y is locally noetherian. Then the relative dimension of f is constant on X.*

Dropping the finiteness condition in Proposition 3, its assertion has to be weakened.

Proposition 5 ([EGA IV_2], 2.3.12). *Let $f: X \longrightarrow Y$ be faithfully flat and quasi-compact. Then the topology of Y is the quotient topology of X with respect to f; i.e., a subset $V \subset Y$ is open if and only if $f^{-1}(V)$ is open in X.*

It is impossible to characterize the flatness of an S-scheme X of finite type by geometric properties when the base S is not reduced. But under reducedness conditions on the base and on the fibres, flatness is equivalent to universal openness; cf. [EGA IV_3], 15.2.3. Moreover, if the base S is reduced and noetherian, each S-scheme X of finite type is generically flat.

Proposition 6 ([EGA IV_2], 6.9.1). *Let S be reduced and noetherian, and let X be an S-scheme of finite type. Then there exists a dense open subscheme S' of S such that $X \times_S S'$ is flat over S'.*

Anyway, the flat locus of an S-scheme which is locally of finite presentation is open.

Proposition 7 ([EGA IV_3], 11.3.1). *Let X be an S-scheme which is locally of finite presentation. Then the set of points $x \in X$ such that X is flat over S at x is open.*

Non-trivial examples of flat morphisms of finite presentation are the smooth ones; see below. Furthermore, there is a useful criterion which relates smoothness over a general base to flatness and smoothness of the fibres. The latter are schemes over fields; in this case one can apply the nice criterion 2.2/15 to test smoothness.

Proposition 8. *Let $f: X \longrightarrow S$ be locally of finite presentation. Let x be a point of X, and set $s = f(x)$. The following conditions are equivalent:*

(a) *f is smooth at x.*

(b) *f is flat at x and the fibre $X_s = X \times_S k(s)$ is smooth over $k(s)$ at x.*

In Section 2.2, we gave detailed proofs for all statements concerning smoothness. Proceeding similarly with Proposition 8, let us give its *proof.* For the implication (a) $\Longrightarrow$ (b), it is only necessary to explain that smooth morphisms are flat. Due to 2.2/11, it suffices to treat the étale case. But in this case the assertion follows easily by looking at the local structure of étale morphisms; cf. 2.3/3.

If one wants to verify this implication without using the local structure of étale morphisms (which involves Zariski's Main Theorem), one can proceed as follows. If Z is a smooth S-scheme which is flat over S, and if X is a subscheme of Z given by one equation, say $g = 0$, such that $d_{X/S}(g)$ does not vanish at a certain point $x \in X$, then X is flat over S at x. It suffices to prove this statement, since, in the general case, we can use an induction argument on the number of equations describing X locally at x as a subscheme of $\mathbb{A}^n_S$. In order to prove the assertion above, we may assume that S is noetherian. Then consider the exact sequence

$$\mathcal{O}_{Z,x} \xrightarrow{g} \mathcal{O}_{Z,x} \longrightarrow \mathcal{O}_{X,x} \longrightarrow 0$$

If S is the spectrum of a field, then $\mathcal{O}_{Z,x}$ is an integral domain and g must be a regular element, so the map on the left-hand side is injective in this case. Since smoothness is stable under any base change, we see that the map $g \otimes k(s)$ is injective, where $k(s)$ is the residue field at the image s of x. Because Z is flat over S, we get

$$\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, k(s)) = 0 .$$

Hence X is flat over S at x, cf. Bourbaki [2], Chap. III, § 5, n°2, Thm. 1.

For the implication (b) $\Longrightarrow$ (a), we may assume that X is a closed subscheme of a linear space $\mathbb{A}_S^n$ over an affine scheme $S = \operatorname{Spec} R$ which is defined by a finitely generated ideal $I \subset R[T_1, \ldots, T_n]$. Let r be the relative dimension of X_s at x. Since X_s is smooth over $k(s)$ at x, there exist sections $g_{r+1}, \ldots, g_n$ of I such that, locally at x, the induced functions $\bar{g}_{r+1}, \ldots, \bar{g}_n$ define X_s as a subscheme of $\mathbb{A}_s^n$ and such that $d\bar{g}_{r+1}(x), \ldots, d\bar{g}_n(x)$ are linearly independent in $\Omega^1_{\mathbb{A}_S^n/S} \otimes k(x)$; cf. 2.2/7. Now let Z be the S-scheme defined by $g_{r+1}, \ldots, g_n$. Notice that Z is smooth at x and that Z contains X as a closed subscheme. The fibres of X_s and Z_s coincide locally at x. Now let B be the algebra associated to Z, and let A be the algebra associated to X. Then A is a quotient B/J of B by a finitely generated ideal J of B. Since A is flat over R at x, the exact sequence

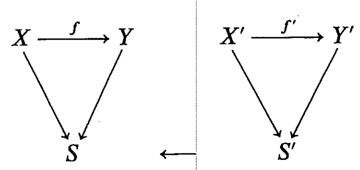
$$0 \longrightarrow J \longrightarrow B \longrightarrow A \longrightarrow 0$$

remains exact at x after tensoring with $k(s)$ over R. Since X_s coincides with Z_s locally at x, we see that $J \otimes_R k(s)$ vanishes at x. Nakayama's lemma yields $J_x = 0$. So X and Z coincide in a neighborhood of x and, hence, X is smooth over S at x. □

Since étale morphisms are flat, henselization and strict henselization are direct limits of flat ring extensions and, hence, they are flat extensions of the given ring.

Corollary 9. *Let R be a local ring. The ring extensions $R \longrightarrow R^h \longrightarrow R^{sh}$, where R^h is a henselization and R^{sh} a strict henselization of R, are faithfully flat.*

Apart from the nice geometric results for flat morphisms of finite presentation, the importance of flatness is expressed in the descent techniques for faithfully flat and quasi-compact morphisms. We want to mention here only the descent for properties of morphisms, the more involved program of the descent for modules or schemes will be explained in Section 6.1. Consider the following situation. Let



be a commutative diagram of morphisms, and assume that the triangle on the right-hand side is obtained from the one on the left by means of the base change $S' \longrightarrow S$. Frequently one wants to show that f enjoys a certain property provided it is known that f' has this property. So it is useful to know that quite a lot of properties descend under a faithfully flat and quasi-compact base change $S' \longrightarrow S$; for example, topological and set-theoretical properties (cf. [EGA IV$_2$], 2.6), finiteness properties (cf. [EGA IV$_2$], 2.7.1), and smoothness properties (cf. [EGA IV$_4$], 17.7.3). For precise statements, the reader is referred to the literature.

2.5 S-Rational Maps

A rational map $X \dashrightarrow Y$ between schemes X and Y is generally defined as an equivalence class of morphisms from dense open subschemes of X to Y; cf. [EGA I], 7. Two such morphisms $U \longrightarrow Y$ and $U' \longrightarrow Y$ are called equivalent if they coincide on a dense open part of $U \cap U'$. However, when working over a base scheme S, this notion does not behave well with respect to a base change $S' \longrightarrow S$. So we want to introduce a relative version of rational maps over a base scheme S which is compatible with base change. For our purposes, it is enough to consider S-rational maps between *smooth* S-schemes. So we will restrict ourselves to this case; for more general versions see [EGA IV$_4$], 20.

An open subscheme U of a smooth S-scheme X is called *S-dense* if, for each $s \in S$, the fibre $U_s = U \times_S \operatorname{Spec} k(s)$ is Zariski-dense in the fibre $X_s = X \times_S k(s)$. Clearly, finite intersections of S-dense open subschemes of X are S-dense in X again. Furthermore, if U is S-dense and open in X and if V is an open subscheme of X, then $U \cap V$ is S-dense in V. Considering a second smooth S-scheme Y, an *S-rational map* $\varphi : X \dashrightarrow Y$ is defined as an equivalence class of S-morphisms $U \longrightarrow Y$, where U is some S-dense open subscheme of X. Two such S-morphisms $U \longrightarrow Y$ and $U' \longrightarrow Y$ are called equivalent if they coincide on an S-dense open part of $U \cap U'$. We will say that $\varphi : X \dashrightarrow Y$ is *defined at a point* $x \in X$ if there is a morphism $U \longrightarrow Y$ representing φ with $x \in U$. The set of all points $x \in X$ where φ is defined constitutes an S-dense open subscheme of X. It is called the *domain of definition* of X; we denote it by $\operatorname{dom}(\varphi)$; but note that, without any further assumptions, there is no global morphism $\operatorname{dom}(\varphi) \longrightarrow Y$ defining φ. Furthermore, if $\varphi : X \dashrightarrow Y$ can be defined by an S-morphism $U \longrightarrow Y$ which induces an isomorphism from U onto an S-dense open subscheme of Y, then $\varphi : X \dashrightarrow Y$ is called *S-birational.* In this case we have an S-birational map $\varphi^{-1} : Y \dashrightarrow X$ which serves as an inverse of φ. It is clear that the notions S-dense, S-rational, and S-birational are preserved by any base change $S' \longrightarrow S$. In general, the same is not true for the domain of definition of S-rational maps. For example, set $S = \operatorname{Spec} \mathbb{Z}$, and consider the $\mathbb{Z}$-rational map $\varphi : \mathbb{A}^1_{\mathbb{Z}} \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$ given by the rational function $(T+1)/(T-1)$. Then the base change $\operatorname{Spec} \mathbb{Z}/2\mathbb{Z} \longrightarrow \operatorname{Spec} \mathbb{Z}$ transforms φ into a morphism $\mathbb{A}^1_{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathbb{A}^1_{\mathbb{Z}/2\mathbb{Z}}$.

Let $f : X \longrightarrow Y$ be a quasi-compact and quasi-separated morphism between arbitrary schemes X and Y. Then the direct image $f_*\mathcal{O}_X$ of the structure sheaf of X is a quasi-coherent $\mathcal{O}_Y$-module, cf. [EGA I], 9.2.1, and the kernel $\mathcal{I}$ of the canonical morphism $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a quasi-coherent sheaf of ideals in $\mathcal{O}_Y$. The *schematic image* of f is defined to be the subscheme of Y associated to $\mathcal{I}$; it is the smallest closed subscheme of Y that f factors through. If V is a subscheme of Y such that the inclusion $j : V \hookrightarrow Y$ is quasi-compact, the schematic image of j is also referred to as the *schematic closure* of V in Y. Furthermore, if the schematic closure of V in Y coincides with Y, we will say V is *schematically dense* in Y.

Lemma 1. *Let Y be a smooth S-scheme, and let V be an open quasi-compact subscheme of Y.*

(a) *If Y is of finite presentation, the set of points $s \in S$ such that V_s is not dense in Y_s is locally constructible in S (i.e. constructible if S is quasi-compact; cf.* [EGA 0_{III}], 9.1.12).

(b) *If V is S-dense in Y, it is schematically dense in Y.*

Proof. (a) We may assume that the base S is noetherian. Let A be the closed reduced subscheme $Y - V$, and denote by $p: A \longrightarrow S$ the structural morphism. Then consider the set

$$F = \{y \in A; \dim_y p^{-1}(p(y)) = \dim_y(Y/S)\} .$$

It is clear that V_s is not dense in Y_s if and only if $s \in p(F)$. Due to [EGA IV_3], 13.1.3, the set F is closed in Y and, due to [EGA IV_1], 1.8.5, the image $p(F)$ is locally constructible in S.

(b) follows from [EGA IV_3], 11.10.10. But, for the convenience of the reader, we will treat the case where the base is locally noetherian. It is enough to show that the restriction map $\mathcal{O}_Y(Y') \longrightarrow \mathcal{O}_Y(V \cap Y')$ is injective for each open subscheme Y' in a basis of the topology of Y; note that $V \cap Y'$ is S-dense in Y' for each open subscheme Y' of Y. So we may assume that S is an affine scheme $\operatorname{Spec} R$, and that Y is an affine scheme $\operatorname{Spec} A$. It suffices to show that $A \longrightarrow \mathcal{O}_Y(V)$ is injective.

Since A is flat over R, cf. 2.4/8, the associated prime ideals of A are just the associated prime ideals $\mathfrak{p}_{ij}$ of $\mathfrak{p}_i A$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the associated prime ideals of R; cf. [EGA IV_2], 3.3.1. Since A is smooth over R, the prime ideals $\mathfrak{p}_{ij}$ are the minimal prime ideals over $\mathfrak{p}_i A$. So V meets each component $V(\mathfrak{p}_{ij})$ and, hence, the restriction map $A \longrightarrow \mathcal{O}_Y(V)$ is injective. □

For later reference we state that the schematic image is compatible with flat base change.

Proposition 2. *Let $f: X \longrightarrow Y$ be an S-morphism which is quasi-compact and quasi-separated. Let $g: S' \longrightarrow S$ be a flat morphism, and denote by $f': X' \longrightarrow Y'$ the S'-morphism obtained from f by base change. Let Z (resp. Z') be the schematic image of f (resp. of f'). Then, $Z \times_S S'$ is canonically isomorphic to Z'.*

The assertion follows immediately from the fact that the pull-back of $\mathcal{O}_Y$-modules with respect to the projection $Y' \longrightarrow Y$ gives rise to an exact functor from the category of $\mathcal{O}_Y$-modules to the category of $\mathcal{O}_{Y'}$-modules; cf. [EGA IV_2], 2.3.2.

Next we want to define the *graph of an S-rational map* $\varphi: X \dashrightarrow Y$, where X and Y are smooth S-schemes of finite type. Let U be an S-dense open subscheme of X such that φ is given by an S-morphism $U \longrightarrow Y$. We need to know that we may assume U to be quasi-compact.

Lemma 3. *Let U be an S-dense open subscheme of a smooth and quasi-compact S-scheme X. Then U contains an S-dense open subscheme which is quasi-compact.*

Proof. Let $\{U_i\}_{i \in I}$ be an affine open covering of U and, for each $i \in I$, consider the second projection $\tau_i: X \times_S U_i \longrightarrow U_i$. It admits a section $\delta_i: U_i \longrightarrow X \times_S U_i$, namely the tautological one. Denote by V_i the union of all connected components of fibres of τ_i which meet the image of δ_i. Then, τ_i being smooth, V_i is open in $X \times_S U_i$ by [EGA IV_3], 15.6.5. Let $\operatorname{Sat}(U_i)$ be the image of V_i under the first projection $X \times_S U_i \longrightarrow X$. Since U_i is smooth and, hence, flat over S, the image $\operatorname{Sat}(U_i)$ is open in X and contains U_i; it may be viewed as a saturation of U_i with respect to the structural morphism $X \longrightarrow S$. Now $\{\operatorname{Sat}(U_i)\}_{i \in I}$ is an open covering of X because U is S-dense in X, and this covering contains a finite subcover $\{\operatorname{Sat}(U_{i_1}), \ldots, \operatorname{Sat}(U_{i_n})\}$ because X is quasi-compact. Thus $U' := U_{i_1} \cup \cdots \cup U_{i_n}$ is S-dense and quasi-compact in U. □

So we have seen that $\varphi: X \dashrightarrow Y$ can be represented by an S-morphism $U \longrightarrow Y$ where U is S-dense open and quasi-compact in X. Let Γ_U be the graph of this morphism; it is a locally closed subscheme of $U \times_S Y$ (closed if Y is separated over S). Since U is quasi-compact over S, one can define the graph Γ of φ as the schematic closure of $U \cong \Gamma_U$ in $X \times_S Y$. In order to see that the definition is independent of the choice of U, it suffices to mention the fact that any quasi-compact S-dense open subscheme $V \subset U$ is schematically dense in U due to Lemma 1; hence V and U have the same schematic closure Γ in $X \times_S Y$.

Now let Ω be the largest open subscheme of X such that the projection $p: X \times_S Y \longrightarrow X$ onto the first factor induces an isomorphism

$$\Gamma \cap p^{-1}(\Omega) \xrightarrow{\sim} \Omega .$$

Then $\Omega \subset \operatorname{dom}(\varphi)$. Furthermore, if Y is *separated over* S, each graph Γ_U as above is closed in $U \times_S Y$ so that $\Gamma \cap (U \times_S Y) = \Gamma_U$. Therefore we have an isomorphism

$$\Gamma \cap p^{-1}(U) \xrightarrow{\sim} U ,$$

which shows $U \subset \Omega$. This shows $\operatorname{dom}(\varphi) \subset \Omega$ and thus $\operatorname{dom}(\varphi) = \Omega$. In particular, there is a unique S-morphism $\operatorname{dom}(\varphi) \longrightarrow Y$ corresponding to the S-rational map $\varphi: X \dashrightarrow Y$; but note that, in general, $\operatorname{dom}(\varphi)$ is not necessarily quasi-compact.

Example 4. Let $\xi = (\xi_i)_{i \in I}$ and $\eta = (\eta_j)_{j \in J}$ be systems of variables, and let k be a field with $\operatorname{char}(k) \neq 2$. Let R denote the k-algebra $k[\xi, \eta]/(\xi\eta)$ where $(\xi\eta)$ is the ideal generated by all products $\xi_i \eta_j$, $i \in I$ and $j \in J$. Set $S = \operatorname{Spec} R$. Then we can view $X = \operatorname{Spec} k[\xi]$ and $Y = \operatorname{Spec} k[\eta]$ as closed subschemes of S, intersecting each other at a single point, namely, at the origin of X and Y. Furthermore, the union of X and Y is S. Now fix indices $i_0 \in I$ and $j_0 \in J$, and consider the S-rational map $\varphi: \mathbb{A}^1_S \dashrightarrow \mathbb{A}^1_S$ given by the rational function

$$\frac{T^2 - 1}{(T - \xi_{i_0} + 1)(T - \eta_{j_0} - 1)} ,$$

where T is a coordinate of $\mathbb{A}^1_S$. Let D be the complement in $\mathbb{A}^1_S$ of the domain of definition $\operatorname{dom}(\varphi)$. Then $D \cap \mathbb{A}^1_X$ is the union of two closed subsets of $\mathbb{A}^1_X$; namely, of the zero set of $(T - \xi_{i_0} + 1)$ and of the closed point $(\xi, T - 1)$ which lies over the origin of X. A similar assertion is true for $D \cap \mathbb{A}^1_Y$. Since $\operatorname{char}(k) \neq 2$, both parts are disjoint. Thus, if the system ξ contains infinitely many variables, the domain of definition $\mathrm{dom}(\varphi)$ cannot be quasi-compact, since a subset of $\mathbb{A}_X^1$ consisting of a single closed point cannot be described by finitely many equations.

Proposition 5. *Let X, X', Y be smooth S-schemes of finite type, and assume that Y is separated over S. Let $\varphi : X \dashrightarrow Y$ be an S-rational map, and consider a flat S-morphism $f : X' \longrightarrow X$. Then $f^{-1}(\mathrm{dom}(\varphi))$ is an S-dense open subscheme of X', and $\varphi \circ f$ is an S-rational map from X' to Y which satisfies*

$$\mathrm{dom}(\varphi \circ f) = f^{-1}(\mathrm{dom}(\varphi)) .$$

In particular, if f is faithfully flat and if $\varphi \circ f$ is defined everywhere on X', the map φ is defined everywhere on X.

Proof. Since f is flat and locally of finite presentation, cf. [EGA IV_1], 1.4.3, the map f is open. Using this fact, one shows $f^{-1}(\mathrm{dom}(\varphi))$ is S-dense in X'. So $\varphi \circ f$ is an S-rational map and $\mathrm{dom}(\varphi \circ f)$ contains $f^{-1}(\mathrm{dom}(\varphi))$. Denote by $\Gamma \subset X \times_S Y$ the graph of φ and by $\Gamma' \subset X' \times_S Y$ the graph of $\varphi \circ f$. Then we see from Proposition 2 that

$$X' \times_X \Gamma = \Gamma' .$$

Let $p : \Gamma \longrightarrow X$ and $p' : \Gamma' \longrightarrow X'$ be the projections onto the first factors. Set $U' := \mathrm{dom}(\varphi \circ f)$, and consider its image $U := f(U')$ which is an open subscheme of X. Since $U' \longrightarrow U$ is faithfully flat, the projection p is an isomorphism over U if and only if p' is an isomorphism over U'. Therefore $U \subset \mathrm{dom}(\varphi)$, and the assertion is clear. □

Finally we want to show that the domain of definition of S-rational maps is compatible with flat base change.

Proposition 6. *Let $\varphi : X \dashrightarrow Y$ be an S-rational map between smooth S-schemes of finite type where Y is separated over S. Let $S' \longrightarrow S$ be a flat morphism, and denote by $\varphi' : X' \dashrightarrow Y'$ the S'-rational map obtained from f by base change. Then*

$$\mathrm{dom}(\varphi') = \mathrm{dom}(\varphi) \times_S S' .$$

Proof. It is clear that $\mathrm{dom}(\varphi) \times_S S' \subset \mathrm{dom}(\varphi')$. To show the opposite inclusion, denote the graph of φ by $\Gamma \subset X \times_S Y$ and the graph of φ' by $\Gamma' \subset X' \times_{S'} Y'$. Since the schematic closure commutes with flat base change, we have

$$\Gamma \times_S S' = \Gamma' .$$

Let $p : \Gamma \longrightarrow X$ and $p' : \Gamma' \longrightarrow X'$ be the projections onto the first factors. Furthermore, consider a point $x' \in \mathrm{dom}(\varphi')$, and let x be its image in X. Then we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\, \mathcal{O}_{X',x'} & \longrightarrow & \mathrm{Spec}\, \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X , \end{array}$$

where the map in the first row is faithfully flat. Therefore, the fact that p' is an isomorphism over $\mathrm{Spec}\, \mathcal{O}_{X',x'}$ implies that p is an isomorphism over $\mathrm{Spec}\, \mathcal{O}_{X,x}$. Since Y is of finite type over S, we see that Γ is of finite type over X. Hence, there exists an affine open neighborhood W of x such that p induces a closed immersion $p^{-1}(W) \longrightarrow W$. Let Z be the schematic image in W of this map and let U be a quasi-compact S-dense open subscheme of X where φ is defined. Then the open subscheme $U \cap W$ of W is contained in Z. Since $U \cap W$ is S-dense in W, the scheme Z coincides with W. Thus $p^{-1}(W) \longrightarrow W$ is an isomorphism, and x is contained in $\mathrm{dom}(\varphi)$. □

Chapter 3. The Smoothening Process

The smoothening process, in the form needed in the construction of Néron models, is presented in Sections 3.1 to 3.4. After we have explained the main assertion, we discuss the technique of blowing-up which is basic for obtaining smoothenings. The actual proof of the existence of smoothenings is carried out in Sections 3.3 and 3.4. As an application, we construct weak Néron models under appropriate conditions.

Our version of the smoothening process differs from the one of Néron insofar as we have added a constructibility assertion, thereby avoiding the use of provarieties; for more details see Section 1.6. A generic form of Néron's smoothening process has also been explained by M. Artin in [4].

The chapter ends with a generalization of the smoothening along a section where the base is a polynomial ring over an excellent discrete valuation ring. This kind of smoothening technique is very close to that developed by M. Artin [4] for the proof of his approximation theorem; see also Artin and Rotthaus [1].

3.1 Statement of the Theorem

In the following let R be a discrete valuation ring with field of fractions K, with residue field k, and with uniformizing element π. We denote by R^h a henselization of R and by R^{sh} a strict henselization of R. Then R^h and R^{sh} are discrete valuation rings with uniformizing element π and the residue field of R^{sh} equals the separable closure k_s of k. For any R-scheme X, let $X_K = X \otimes_R K$ be its generic fibre and $X_k = X \otimes_R k$ its special fibre.

Definition 1. *Let X be an R-scheme of finite type whose generic fibre X_K is smooth over K. A smoothening of X is an R-morphism $f: X' \longrightarrow X$ which satisfies the following conditions:*

(i) f is proper and is an isomorphism on generic fibres.

(ii) For each étale R-algebra R', each R'-valued point of X lifts uniquely to an R'-valued point of X' which factors through the smooth locus X'_{smooth} of X'. More precisely, the canonical map $X'_{\text{smooth}}(R') \longrightarrow X(R')$ is bijective.

Each étale R-algebra R' is semi-local. So in order to test condition (ii), one may restrict oneself to local extensions R' of R which are étale. In particular, such rings are discrete valuation rings; they are flat over R. Due to the valuative criterion of properness [EGA II], 7.3.8, condition (i) implies that the map $X'(R') \longrightarrow X(R')$

deduced from f is bijective for any flat R-algebra R' which is a discrete valuation ring. Hence, if condition (i) is satisfied, condition (ii) says that, for each local étale extension R' of R, the R'-valued points of X' factor through the smooth locus of X'. As seen in Section 2.3, the strict henselization R^{sh} of R is the direct limit of all local étale extensions of R. So condition (ii) is fulfilled if and only if the canonical map $X'_{\text{smooth}}(R^{sh}) \longrightarrow X(R^{sh})$ is bijective.

In general, a smoothening $X' \longrightarrow X$ is not a desingularization of X (i.e., a proper morphism $X'' \longrightarrow X$ from a regular scheme X'' to X which is an isomorphism over the regular locus of X), because the points in the complement of the smooth locus of X' do not need to be regular. However, a desingularization of X is always a smoothening, as we will see by using the following fact from commutative algebra.

Proposition 2. *Let $\iota: R \longrightarrow A$ and $\varepsilon: A \longrightarrow R$ be morphisms of regular local rings such that $\varepsilon \circ \iota = \mathrm{id}_R$ (i.e., ε defines a section of the morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec} R$ associated to ι). Then the image of each regular system of parameters of R under ι is part of a regular system of parameters of A. If $\mathfrak{J}$ is the kernel of ε, then $\mathfrak{J}$ is generated by a part of a regular system of parameters. If $t_1, \ldots, t_n$ is a minimal system of generators of $\mathfrak{J}$, the completion of A with respect to $\mathfrak{J}$ is canonically isomorphic to $R[[t_1, \ldots, t_n]]$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R, and let $s_1, \ldots, s_m$ be a minimal system of generators of $\mathfrak{m}$. Let $\mathfrak{m}'$ be the maximal ideal of A. As $\varepsilon \circ \iota = \mathrm{id}_R$, the residue fields $R/\mathfrak{m}$ and $A/\mathfrak{m}'$ are canonically isomorphic, and $\mathfrak{m}/\mathfrak{m}^2$ may be viewed as a subspace of $\mathfrak{m}'/\mathfrak{m}'^2$. Hence $\iota(s_1), \ldots, \iota(s_m)$ is a part of a regular system of parameters of A. So there exist elements $t_1, \ldots, t_n$ in $\mathfrak{m}'$ such that $\iota(s_1), \ldots, \iota(s_m), t_1, \ldots, t_n$ is a regular system of parameters in A. After replacing t_i by $t_i - \iota(\varepsilon(t_i))$, we may assume that $t_1, \ldots, t_n$ are in the kernel $\mathfrak{J}$ of ε. An easy calculation shows $\mathfrak{J} = (t_1, \ldots, t_n)$ as required. The assertion concerning the $\mathfrak{J}$-adic completion of A follows immediately from the definition of a regular system of parameters. □

In order to show that a desingularization $X'' \longrightarrow X$ is a smoothening of X one has only to verify that, for any étale R-algebra R', each $a \in X''(R')$ factors through the smooth locus of X''. One knows that $X'' \otimes_R R'$ is a desingularization of $X \otimes_R R'$ (see 2.3/9) and, furthermore, that the image of $a: \operatorname{Spec} R' \longrightarrow X''$ factors through the smooth locus of X'' if the corresponding fact is true for $(a, \mathrm{id}): \operatorname{Spec} R' \longrightarrow X'' \otimes_R R'$ ([EGA IV_4], 17.7.4). So we may assume $R = R'$. Then it follows from Proposition 2 that X'' is smooth over R along a; cf. [EGA IV_4], 17.5.3.

Theorem 3 (Smoothening Process). *Let X be an R-scheme of finite type whose generic fibre X_K is smooth over K. Then X admits a smoothening $f: X' \longrightarrow X$.*

Moreover, one can construct f as a finite sequence of blowing-ups with centers in the special fibres. In particular, if X is quasi-projective over R, the same is true for X'.

Removing from X' the non-smooth locus, we see:

Corollary 4. *Let X be as before. Then there is an R-morphism $f: X'' \longrightarrow X$ from a smooth R-scheme X'' of finite type to X such that*

(i) *f is an isomorphism on generic fibres, and*
(ii) *the canonical map $X''(R^{sh}) \longrightarrow X(R^{sh})$ is bijective.*

Such schemes X'' are not unique, and they do not need to be proper over R, even if X is proper over R.

The smoothening process provides a first step towards the construction of Néron models. For example, if X_K is an abelian variety with a projective embedding $X_K \subset \mathbb{P}^n_K$, one can apply the smoothening process to the schematic closure X of X_K in $\mathbb{P}^n_R$. Restricting the resulting R-scheme to its smooth locus, we obtain a *smooth* R-model of X_K which, although it might not be proper over R, nevertheless satisfies the valuative criterion of properness for the special class of valuation rings which are étale over R.

3.2 Dilatation

We have claimed that a smoothening of X can be constructed by blowing up subschemes of the special fibre. First, let us explain what happens to the sections $X(R)$ when such a blowing-up is applied to X. Consider the following example. Set $X := \operatorname{Spec} R[T]$, where $T = (T_1, \ldots, T_n)$ is a set of variables, and let Y_k be the reduced subscheme of X which consists of the origin of the special fibre X_k of X. Then Y_k is defined by the ideal $\mathfrak{I} \subset R[T]$ which is generated by $\pi, T_1, \ldots, T_n$. Using an absolute value on K belonging to the valuation ring R, the R-valued points of X correspond bijectively to the rational points $x_K \in \mathbb{A}^n_K$ with $|T_i(x_K)| \leqq 1, i = 1, \ldots, n$. Furthermore, the R-valued points of X which specialize into Y_k correspond to the rational points $x_K \in \mathbb{A}^n_K$ with $|T_i(x_K)| \leqq |\pi|$. Now let $X' \longrightarrow X$ be the blowing-up of Y_k in X. Let $\mathscr{I}'$ be the sheaf of ideals of $\mathcal{O}_{X'}$ generated by $\mathfrak{I}$, and denote by X'_π the set of points of X' at which $\mathscr{I}'$ is generated by π. Then $X'_\pi = \operatorname{Spec} R[T']$, where $T' = (T'_1, \ldots, T'_n)$ is a second set of variables, and the morphism $X'_\pi \longrightarrow X$ corresponds to the morphism induced by sending T_i to $\pi T'_i$ for $i = 1, \ldots, n$. It is seen that $X'_\pi(R)$ is mapped bijectively onto the set of those R-valued points of X which specialize into Y_k; hence $X'_\pi(R)$ corresponds to the rational points $x_K \in \mathbb{A}^n_K$ which satisfy $|T_i(x_K)| \leqq |\pi|$. Furthermore, two points $x, y \in X'_\pi(R)$ have the same specialization over k if and only if $|T_i(x_K) - T_i(y_K)| \leqq |\pi^2|$ for all i. We will call X'_π the *dilatation* of Y_k in X.

In order to construct dilatations of more general type, consider an arbitrary R-scheme X of finite type and a closed subscheme Y_k of the special fibre X_k. Let $\mathscr{I}$ be the associated sheaf of ideals in $\mathcal{O}_X$; in particular, $\pi \in \mathscr{I}$. The blowing-up X' of $\mathscr{I}$ on X is defined as the homogeneous spectrum $\operatorname{Proj}(\mathscr{S})$ of the graded $\mathcal{O}_X$-algebra $\mathscr{S} = \bigoplus_{n \geqq 0} \mathscr{I}^n$ (cf. [EGA II], 3.1 and 8.1.3). Locally, it is defined as follows. If $X = \operatorname{Spec} A$, the sheaf of ideals $\mathscr{I}$ is associated to an ideal $\mathfrak{I}$ of A. Since A is noetherian, $\mathfrak{I}$ is generated by finitely many elements $g_0 = \pi, g_1, \ldots, g_n$ of A. Then X' is the closed subscheme of $\mathbb{P}^n_A$ which is given by the homogeneous ideal

$$\mathfrak{I}' = \ker\Big(A[T_0, \ldots, T_n] \longrightarrow \bigoplus_{n \geqq 0} \mathfrak{I}^n\Big),$$

where we consider the graded homomorphism sending the variable T_i to $g_i \in \mathfrak{I}^1$. Let U_i be the affine open subscheme of $\mathbb{P}^n_A$ where T_i does not vanish. Then $X' \cap U_i$ is affine, and the A-algebra of its global sections is given by

$$A\left[\frac{g_0}{g_i}, \ldots, \frac{g_n}{g_i}\right] \Big/ (g_i\text{-torsion})$$

where, suggestively, we have written

$$A\left[\frac{g_0}{g_i}, \ldots, \frac{g_n}{g_i}\right] := A[T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n]/(g_j - g_i T_j)_{j \neq i}.$$

That we have to divide by the g_i-torsion corresponds to the fact that the sheaf of ideals $\mathscr{I}' = \mathscr{I} \cdot \mathcal{O}_{X'}$ is invertible on X'. Furthermore, one shows that X' is R-flat (i.e., has no π-torsion) if the same is true for X.

Returning to the case of a global R-scheme X, we set

$$X'_\pi := \{x \in X';\ \mathscr{I}'_x \text{ is generated by } \pi\},$$

which is an open subscheme of X'. Over an affine open part $\operatorname{Spec} A$ of X, it consists of the affine A-scheme $\operatorname{Spec} A'_{(\pi)}$, where

$$A'_{(\pi)} = A\left[\frac{g_1}{\pi}, \ldots, \frac{g_n}{\pi}\right] \Big/ (\pi\text{-torsion}).$$

So X'_π is always flat over R, even if X is not. Let $u : X'_\pi \longrightarrow X$ be the canonical morphism, and denote by an index k restrictions to special fibres. The pair (X'_π, u) has the following universal property:

If Z is a flat R-scheme, and if $v : Z \longrightarrow X$ is an R-morphism such that its restriction v_k to special fibres factors through Y_k, then v factors uniquely through u.

Indeed, since the problem is local on X and Z, we may assume that both schemes are affine, say $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$. Using notations as before, the fact that v_k factors through Y_k implies that the ideal $\mathfrak{I} \cdot B$ is contained in πB. Hence there exist elements $h_i \in B$ with $v^* g_i = \pi h_i$; the elements h_i are unique, because B has no π-torsion. Thus, the A-morphism $A[T_0, \ldots, T_n] \longrightarrow X$ sending T_i to h_i yields a morphism $w^* : A'_{(\pi)} \longrightarrow B$ and hence a morphism $w : Z \longrightarrow X'_\pi$ such that $v = u \circ w$.

We summarize what we have shown.

Proposition 1. *Let X be an R-scheme of finite type, let Y_k be a closed subscheme of its special fibre X_k, and let $\mathscr{I}$ be the sheaf of ideals of $\mathcal{O}_X$ defining Y_k. Let $X' \longrightarrow X$ be the blowing-up of Y_k on X, and let $u : X'_\pi \longrightarrow X$ denote its restriction to the open subscheme of X' where $\mathscr{I} \cdot \mathcal{O}_{X'}$ is generated by π. Then*

(a) *X'_π is a flat R-scheme, and $u_k : (X'_\pi)_k \longrightarrow X_k$ factors through Y_k.*

(b) *For any flat R-scheme Z and for any R-morphism $v : Z \longrightarrow X$ such that $v_k : Z_k \longrightarrow X_k$ factors through Y_k, there exists a unique R-morphism $v' : Z \longrightarrow X'_\pi$ such that $v = u \circ v'$.*

Due to property (b), the couple (X'_π, u) is unique (up to canonical isomorphism) in the class of all couples (Z, v) satisfying property (a). We call X'_π the *dilatation of* Y_k *on* X. It is clear that one can construct dilatations also for locally closed subschemes of X_k. We want to mention some elementary properties of dilatations.

Proposition 2. (a) *All dilatations factor through the largest flat R-subscheme of X, which is given by the ideal of π-torsion in $\mathcal{O}_X$.*

(b) *Dilatations commute with flat base change $R \longrightarrow R'$ where R' is a discrete valuation ring such that π is also a uniformizing element of R'.*

(c) *Let X be a closed subscheme of an R-scheme Z, and let Y_k be a closed subscheme of X_k. Then the dilatation X'_π of Y_k on X can be realized as a closed subscheme of the dilatation Z'_π of Y_k on Z.*

(d) *Dilatations commute with products: Let X^i be R-schemes, and let Y^i_k be subschemes of X^i_k for $i = 1, 2$. Then the dilatation of $Y^1_k \times_k Y^2_k$ on $X^1 \times_R X^2$ is the product $(X^1)'_\pi \times_R (X^2)'_\pi$ of the dilatations of Y^i_k on X^i. In particular, if X is an R-group scheme, and if Y_k is a subgroup scheme of X_k, the dilatation X'_π of Y_k on X is an R-group scheme and the canonical map $X'_\pi \longrightarrow X$ is a group homomorphism.*

Finally we investigate how dilatations behave with respect to smoothness.

Proposition 3. *Let X be a smooth R-scheme, and let Y_k be a smooth k-subscheme of X_k. Then the dilatation X'_π of Y_k on X is smooth over R.*

Proof. Let $u : X'_\pi \longrightarrow X$ be the dilatation of Y_k on X, let x' be a point of the special fibre of X'_π, and set $x = u(x')$. Let n be the dimension of X_k at x, and let r be the dimension of Y_k at x. Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{O}_X$ defining Y_k, and let $\bar{\mathcal{I}} = \mathcal{I}/\pi\mathcal{O}_X$ denote the sheaf of ideals of $\mathcal{O}_{X_k}$ defining Y_k in X_k. Due to the Jacobi Criterion 2.2/7 there exist $\bar{f}_1, \ldots, \bar{f}_r \in \mathcal{O}_{X_k,x}$ and $\bar{g}_{r+1}, \ldots, \bar{g}_n \in \bar{\mathcal{I}}_x$ such that $\bar{f}_1, \ldots, \bar{f}_r, \bar{g}_{r+1}, \ldots, \bar{g}_n$ form a system of local coordinates of X_k at x (cf. 2.2/12), and such that $\bar{g}_{r+1}, \ldots, \bar{g}_n$ generate $\bar{\mathcal{I}}_x$. On an affine neighborhood U of x in X there exist liftings $f_i \in \mathcal{O}_X(U)$ of $\bar{f}_i$ and $g_j \in \mathcal{I}(U)$ of $\bar{g}_j$. Then $f_1, \ldots, f_r, g_{r+1}, \ldots, g_n$ form a system of local coordinates of X over R at x, and $\pi, g_{r+1}, \ldots, g_n$ generate $\mathcal{I}$ at x. From the local construction of X'_π we see that $df_1, \ldots, df_r, dg'_{r+1}, \ldots, dg'_n$ generate $\Omega^1_{X'_\pi/R}$ at x', where $g'_j \in \mathcal{O}_{X'_\pi,x'}$ satisfies $g_j = \pi g'_j$. Hence $\Omega^1_{X'_\pi/R}$ is generated by n elements at x'. Since the relative dimension of X'_π over R is at least n at x' (cf. [EGA IV_3], 13.1.3), it follows from 2.4/8 and 2.2/15 that X'_π is smooth over R at x'. □

3.3 Néron's Measure for the Defect of Smoothness

Throughout this section, let X be an R-scheme *of finite type* whose *generic fibre* X_K is *smooth* over K. Let a be an R^{sh}-valued point of X, and let a_K (resp. a_k) denote its generic (resp. special) fibre. Consider the pull-back $a^*\Omega^1_{X/R}$ of the $\mathcal{O}_X$-module of relative differential forms from X to $\operatorname{Spec} R^{sh}$. By abuse of notation, we will identify

it with its module of global sections. Thereby $a^*\Omega^1_{X/R}$ becomes an R^{sh}-module of finite type. Since R^{sh} is a discrete valuation ring, this module splits into a direct sum of a free part and of a torsion part. The rank of the free part is just the rank of $\Omega^1_{X/R}$ at a_K which is the dimension of X_K at a_K (since X_K is smooth at a_K). Looking at the torsion part, we define

$$\delta(a) := \text{length of the torsion part of } a^*\Omega^1_{X/R}$$

as *Néron's measure for the defect of smoothness at a*. First we want to show that, indeed, $\delta(a)$ provides a measure of how far X is from being smooth at a.

Lemma 1. *Let a be an R^{sh}-valued point of X. Then a factors through the smooth locus of X if and only if $\delta(a) = 0$.*

Proof. If a is contained in the smooth locus of X, then $\Omega^1_{X/R}$ is locally free at a_k and, hence, $a^*\Omega^1_{X/R}$ is free. Thus we have $\delta(a) = 0$. Conversely, if $\delta(a) = 0$, then $a^*\Omega^1_{X/R}$ can be generated by d elements, where d is the dimension of X_K at a_K. In particular, $\Omega^1_{X/R}$ and, hence, $\Omega^1_{X_k/k}$ can be generated by d elements at a_k. Since the relative dimension at a_k is at least d (cf. [EGA IV_3], 13.1.3), it follows from 2.2/15 that X_k is smooth over k at a_k of relative dimension d. Then X is smooth over R at a_k. This follows from 2.4/8, if it is known that X is R-flat at a_k. Avoiding the interference of flatness, one can proceed as follows. Choose a representation of a neighborhood $U \subset X$ of a_k as a closed subscheme of some $\mathbb{A}^n_R$. Due to the Jacobi Criterion 2.2/7(c), there exist local sections $g_{d+1}, \ldots, g_n$ on a neighborhood of $a_k \in \mathbb{A}^n_R$ which vanish on U, and which have the property that U_k is defined by $(\pi, g_{d+1}, \ldots, g_n)$ at a_k and that $dg_{d+1}, \ldots, dg_n$ generate a direct factor of $\Omega^1_{\mathbb{A}^n_R/R}$ at a_k. Then, in a neighborhood of a, the subscheme Z of $\mathbb{A}^n_R$ given by $g_{d+1}, \ldots, g_n$ is smooth of relative dimension d; furthermore locally at a, the scheme Z contains U as a closed subscheme. Thus, by reasons of dimension and of smoothness, the generic fibres U_K and Z_K coincide at a_K and, hence, U and Z coincide at a_k. □

The Jacobi Criterion provides a useful method to calculate $\delta(a)$. Namely, let $U \subset X$ be a neighborhood of a which can be realized as a closed subscheme of an R-scheme Z where Z is smooth over R and has constant relative dimension n. Assume that there exist $z_1, \ldots, z_n$ on Z such that $dz_1, \ldots, dz_n$ constitute a basis of $\Omega^1_{Z/R}$, and let $g_1, \ldots, g_m$ be functions on Z which generate the sheaf of ideals of $\mathcal{O}_Z$ defining U in Z. Representing the relative differentials dg_μ with respect to the basis $dz_1, \ldots, dz_n$, say

$$dg_\mu = \sum_{\nu=1}^{n} \frac{\partial g_\mu}{\partial z_\nu} dz_\nu ,$$

we define the Jacobi matrix of $g_1, \ldots, g_m$ by

$$J = \left(\frac{\partial g_\mu}{\partial z_\nu}\right)_{\substack{\mu=1,\ldots,m \\ \nu=1,\ldots,n}} .$$

If d is the relative dimension of X at a_K, we call Λ the set of all $(n-d)$-minors Δ of J. In this situation, Néron's measure for the defect of smoothness at a can be calculated from the minors $\Delta \in \Lambda$. To give a precise statement, let $v(r)$ denote the π-order of elements $r \in R$.

Lemma 2. $\delta(a) = \min\{v(a^*\Delta); \Delta \in \Lambda\}$.

Proof. Due to the Jacobi Criterion 2.2/7, there exists a minor $\Delta \in \Lambda$ with $a^*\Delta \neq 0$; any minor Δ' of J with more than $n-d$ rows will satisfy $a^*\Delta' = 0$. Furthermore, it follows from 2.1/2 that $a^*\Omega^1_{X/R}$ is representable as a quotient F/M, where $F := a^*\Omega^1_{Z/R}$ is a free R^{sh}-module of rank n, and where M is the submodule which is generated by $a^*dg_1, \ldots, a^*dg_m$. Since the rank of M is $(n-d)$, one can find a basis $e_1, \ldots, e_n$ of F such that M is generated by elements $r_{d+1}e_{d+1}, \ldots, r_ne_n$ where $r_i \in R^{sh}$ and $r_i \neq 0$; this follows from the theory of elementary divisors. Thus the length of the torsion part of F/M, which is $\delta(a)$ by definition, is given by the formula

$$\delta(a) = v(r_{d+1}) + \ldots + v(r_n)\ .$$

Now consider the ideal in R^{sh} which is generated by all elements $a^*\Delta$, $\Delta \in \Lambda$; it equals the ideal generated by all values which are assumed on M by alternating $(n-d)$-forms on F. Obviously, this ideal is generated by the product $r_{d+1} \cdots r_n$, and there exists a minor $\Delta \in \Lambda$ with $(a^*\Delta) = (r_{d+1} \cdots r_n)$. Thus the assertion is clear. □

The method we have just used can easily show that $\delta(a)$ is bounded when a varies over the set of R^{sh}-valued points of X.

Proposition 3. *There exists an integer c such that $\delta(a) \leqq c$ for all $a \in X(R^{sh})$.*

Proof. Since an R-scheme of finite type is quasi-compact by definition, we may assume that X is an affine R-scheme Spec A. Choose a representation

$$A = R[z_1, \ldots, z_n]/(g_1, \ldots, g_m)$$

of A as a quotient of a free polynomial ring $R[z_1, \ldots, z_n]$. For integers d, let $(X_K)_d$ be the union of all irreducible components of dimension d of X_K. Then $(X_K)_d$ is non-empty for at most finitely many d and, since X_K is smooth, X_K is the disjoint sum of the $(X_K)_d$. Let X_d be the schematic closure of $(X_K)_d$ in X; i.e., let X_d be the subscheme of X which is defined by the kernel of the homomorphism $A \longrightarrow \mathcal{O}_X((X_K)_d)$. Let A_d be its ring of global sections. Considering the Jacobi matrix

$$J = \left(\frac{\partial g_\mu}{\partial z_\nu}\right)_{\substack{\mu=1,\ldots,m \\ \nu=1,\ldots,n}},$$

let Λ be the set of all $(n-d)$-minors Δ of J. Then, due to the Jacobi Criterion 2.2/7, we see for each $x \in (X_K)_d$ that there exists a minor $\Delta \in \Lambda$ satisfying $\Delta(x) \neq 0$. Hence the family $(\Delta)_{\Delta \in \Lambda}$ generates the unit ideal in $A_d \otimes_R K$. After chasing denominators, one can find elements $f_1, \ldots, f_t \in A_d$, minors $\Delta_1, \ldots, \Delta_t \in \Lambda$, as well as an integer $c \geqq 0$ such that

$$\sum_{i=1}^{t} f_i \Delta_i|_{X_d} = \pi^c\ .$$

Hence, by Lemma 2, we have $\delta(a) \leqq c$ for all $a \in X(R^{sh})$ whose generic fibre a_K belongs to $(X_K)_d$. Since only finitely many of the schemes $(X_K)_d$ are non-empty, we see that δ is bounded on $X(R^{sh})$. □

It follows that the function δ assumes its maximum on $X(R^{sh})$. The maximum of δ can be viewed as a global measure of how far X is from being smooth at the points of $X(R^{sh})$. Since we want to construct a smoothening of X by blowing up subschemes of X_k, we have to define suitable centers Y_k in the special fibre such that the defect of smoothness, i.e., the maximum of δ, decreases. Smooth R^{sh}-schemes have many sections (cf. 2.3/5). So it is natural to look at subschemes $Y_k \subset X_k$ such that there exist enough R^{sh}-valued points of X whose special fibres factor through Y_k. More precisely, if k_s denotes the residue field of R^{sh}, we will consider the following property (N) for couples (X, Y_k) consisting of an R-scheme X of finite type and of a closed subscheme $Y_k \subset X_k$:

(N) *The family of those k_s-valued points of Y_k, which lift to R^{sh}-valued points of X, is schematically dense in Y_k.*

For the notion of schematic density (more precisely, of schematic dominance) see [EGA IV$_3$], 11.10.2. In our situation the condition just means that the sheaf of $\mathcal{O}_X$-ideals defining Y_k equals the intersection of all kernels of morphisms $a^* : \mathcal{O}_X \longrightarrow a_* \mathcal{O}_{\mathrm{Spec}\, k_s}$, where a varies over the set of R^{sh}-valued points of X whose special fibres factor through Y_k.

Since the strict henselization R^{sh} is the limit over all local étale extensions R' of R, condition (N) is equivalent to the following condition: the set of closed points of Y_k which lift to R'-valued points of X for some local étale extension R' of R is schematically dense in Y_k. For example, if X is smooth over R, and if Y_k is a geometrically reduced closed subscheme of X_k, then it follows from 2.2/16, 2.2/13, and 2.2/14 that (X, Y_k) has the property (N).

Lemma 4. *If the couple (X, Y_k) satisfies property (N), then Y_k is geometrically reduced, and the smooth locus of the k-scheme Y_k is open and dense in Y_k.*

Proof. Property (N) yields that the k_s-valued points of Y_k are schematically dense in Y_k. Since k_s is a geometrically reduced k-algebra, Y_k is also geometrically reduced (cf. [EGA IV$_3$], 11.10.7). So the assertion follows from 2.2/16. □

Next we want to establish the key tool which is needed for the proof of Theorem 3.1/3. It provides us with a means of lowering the defect of smoothness of X so that eventually X becomes smooth at the points we are interested in.

Proposition 5. *Let Y_k be a closed subscheme of X_k such that the couple (X, Y_k) satisfies property* (N). *Let U_k be an open subscheme of Y_k such that U_k is smooth over k and such that the pull-back $\Omega^1_{X/R}|_{U_k}$ of $\Omega^1_{X/R}$ to U_k is locally free. Let $X'_\pi \longrightarrow X$ be the dilatation of Y_k in X and, for each $a \in X(R^{sh})$ with $a_k \in Y_k$, denote by $a' \in X'_\pi(R^{sh})$ the unique lifting of a. Then if $a \in X(R^{sh})$ specializes into a point of U_k, we have*

$$\delta(a') \leq \max\{0, \delta(a) - 1\} .$$

In particular, $\delta(a') < \delta(a)$ for all R^{sh}-valued points a of X which specialize into points of U_k and which are not contained in the smooth locus of X.

First we want to look at an example which explains how the proposition works in a special situation. Let X be the closed subscheme of $\mathbb{A}^2_R = \operatorname{Spec} R[T_1, T_2]$ which is defined by the equation $T_1 T_2 = \pi^2$. Then X is affine, and its R-algebra of global sections is

$$A = R[T_1, T_2]/(T_1 T_2 - \pi^2) .$$

Let Y_k be the closed subscheme of X_k which is defined by (π, T_1, T_2); it consists of a single k-valued point. Using the R-morphism

$$A \longrightarrow R , \qquad T_1 \longmapsto \pi , \qquad T_2 \longmapsto \pi ,$$

this point lifts to an R-valued point of X. Hence (X, Y_k) satisfies property (N). Furthermore, an easy calculation shows $\delta(a) = 1$. The dilatation X'_π of Y_k in X is an affine A-scheme with coordinate ring

$$A' = A[T'_1, T'_2]/(T_1 - \pi T'_1, T_2 - \pi T'_2) = R[T'_1, T'_2]/(T'_1 T'_2 - 1) .$$

In particular, X'_π is smooth over R, and the lifting $a' \in X'_\pi(R^{sh})$ of a, which corresponds to the R-morphism

$$A' \longrightarrow R , \qquad T'_1 \longmapsto 1 , \qquad T'_2 \longmapsto 1 ,$$

fulfills $\delta(a') = 0$.

Proof of Proposition 5. Since the problem is local on X, it is enough to work in a neighborhood of a point $u \in U_k$. So we may assume that X is affine, say $X = \operatorname{Spec} A$, that U_k coincides with Y_k, and that the latter is irreducible. Let r be the dimension of Y_k. Then the sheaves $\Omega^1_{Y_k/k}$ and $\Omega^1_{X/R}|_{Y_k}$ are locally free and the first is obtained from the second one by dividing through the submodule which is generated by all differentials dg of functions $g \in A$ vanishing on Y_k (cf. 2.1/2). Shrinking X if necessary, we can assume that both sheaves are free and that there exist elements $\bar{y}_1, \ldots, \bar{y}_r$, $\bar{z}_1, \ldots, \bar{z}_n \in A$ having the following properties:

The differentials $d\bar{y}_1, \ldots, d\bar{y}_r$ give rise to a basis of $\Omega^1_{Y_k/k}$, the functions $\bar{z}_1, \ldots, \bar{z}_n$ vanish on Y_k, and $d\bar{y}_1, \ldots, d\bar{y}_r, d\bar{z}_1, \ldots, d\bar{z}_n$ give rise to a basis of $\Omega^1_{X/R}|_{Y_k}$.

It follows then from Nakayama's lemma that $\Omega^1_{X/R}$ is generated by $d\bar{y}_1, \ldots, d\bar{y}_r$, $d\bar{z}_1, \ldots, d\bar{z}_n$ at all points of Y_k. However, in general we will not have a basis, because $\Omega^1_{X/R}$ does not need to be locally free. Therefore we want to construct a closed embedding $X \hookrightarrow Z$ into a smooth R-scheme Z such that the above generators of $\Omega^1_{X/R}$ lift to a basis of $\Omega^1_{Z/R}$. This is possible after shrinking X.

Namely, represent A as a quotient of a free polynomial ring $R[T_1, \ldots, T_{r+n+m}]$ with respect to an ideal H and require that T_i is a lifting of $\bar{y}_i$ for $i = 1, \ldots, r$ and that T_{r+j} is a lifting of $\bar{z}_j$ for $j = 1, \ldots, n$. Since $\Omega^1_{X/R}|_{Y_k}$ is free of rank $r + n$, we know that $\Omega^1_{X/R} \otimes k(u)$ is of dimension $r + n$ over $k(u)$ where u is the point in Y_k around which we want to work. Hence there exist $h_1, \ldots, h_m \in H$ such that the Jacobi matrix

$$\left(\frac{\partial h_i}{\partial T_j}(u)\right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,r+n+m}}$$

at u is of rank m. Writing Z for the closed subscheme of $\mathbb{A}^{r+n+m}_R$ which is defined by $h_1, \ldots, h_m$, we have closed immersions

$$Y_k \hookrightarrow X \hookrightarrow Z ,$$

where Z is smooth at u of relative dimension $r + n$. Let C be the R-algebra of global sections of $\mathcal{O}_Z$, and represent the algebras of global sections on Y_k and X as quotients of C; say $A = C/I$ with $I = \operatorname{Id}(X)$ and $B = C/J$ with $J = \operatorname{Id}(Y_k)$. So we know $I \subset J$. Furthermore, let $y_i \in C$ be the image of T_i for $i = 1, \ldots, r$, and $z_j \in C$ the image of T_{r+j} for $j = 1, \ldots, n$. Then y_i is a lifting of $\bar{y}_i \in A$, and the same is true for z_j and $\bar{z}_j$. Replacing Z by an affine open neighborhood of u, we may assume that Z is smooth over R of relative dimension $r + n$ and that $dy_1, \ldots, dy_r, dz_1, \ldots, dz_n$ form a basis of $\Omega^1_{Z/R}$. Also we may assume that Y_k, as a subscheme of Z, is defined by π, $z_1, \ldots, z_n$; i.e., that $J = (\pi, z_1, \ldots, z_n)$. Namely, these functions define a smooth k-subscheme Y'_k of Z of dimension r. Since Y_k is contained in Y'_k and since Y_k is smooth of dimension r, we have $Y_k = Y'_k$ locally at u.

Now we come to the key point of the proof. We claim $I \subset J^2$. This relation will enable us to give the desired estimate for the function δ, when X is replaced by the dilatation X'_π. So consider an element $f \in I$. Since $I \subset J$, we can write

$$f = g\pi + \sum_{i=1}^{n} g_i z_i$$

where $g, g_i \in C$. The differential df vanishes on X and hence on Y_k. Therefore we have

$$\sum_{i=1}^{n} g_i dz_i|_{Y_k} = df|_{Y_k} = 0 .$$

Then $g_i|_{Y_k} = 0$, i.e., $g_1, \ldots, g_n \in J$, since $z_1, \ldots, z_n$ have been chosen in such a way that their differentials form part of a basis of $\Omega^1_{X/R}|_{Y_k}$. In particular, we can write f as

$$(*) \qquad f = g\pi + h$$

with

$$h = g_1 z_1 + \ldots + g_n z_n \in J^2$$

since $z_1, \ldots, z_n \in J$. For any $a \in X(R^{sh})$ with $a_k \in Y_k$, we know $h'(a) \equiv 0 \pmod{\pi}$ for all $h' \in J$. Therefore $h(a) \equiv 0 \pmod{\pi^2}$. On the other hand, we have $f(a) = 0$ for all $a \in X(R^{sh})$. Thus the equation $(*)$ implies $g(a) \equiv 0 \pmod{\pi}$ for all $a \in X(R^{sh})$ such that $a_k \in Y_k$. Since the couple (X, Y_k) satisfies property (N), this yields $g|_{Y_k} = 0$ and, hence, $g \in J$. So $I \subset J^2$ as claimed.

Next consider the dilatation X'_π of Y_k in X. It can be realized as a closed subscheme of the dilatation Z'_π of Y_k in Z. Giving a more precise description of these dilatations, we have $Z'_\pi = \operatorname{Spec} C'$ where

$$C' = C\left[\frac{z_1}{\pi}, \ldots, \frac{z_n}{\pi}\right],$$

and Z'_π is smooth over R, since Z is smooth over R (cf. 3.2/3). Writing $z'_j := \frac{z_j}{\pi}$, the differentials $dy_1, \ldots, dy_r, dz'_1, \ldots, dz'_n$ form a basis of $\Omega^1_{Z'_\pi/R}$. Then $X'_\pi = \operatorname{Spec} A'$ with $A' = C'/I'$, and the ideal $I' \subset C'$ is the smallest one such that I' contains the image of I and such that C'/I' has no π-torsion; i.e., I' consists of those elements $c' \in C'$ such that $\pi^\nu c' \in IC'$ for some $\nu \in \mathbb{N}$. Since $I \subset J^2$, any element $f \in I$ can be written as

$$(\dagger) \qquad f = \pi^2 f'$$

with $f' \in C'$; hence $f' \in I'$. The differential of f has a representation

$$df = \sum_{i=1}^{r} b_i dy_i + \sum_{j=1}^{n} c_j dz_j$$

in $\Omega^1_{Z/R}$, where $b_i, c_j \in C$. It implies the representation

$$df = \sum_{i=1}^{r} b_i dy_i + \sum_{j=1}^{n} \pi c_j dz'_j$$

in $\Omega^1_{Z'_\pi/R}$. Furthermore, we have a representation

$$df' = \sum_{i=1}^{r} b'_i dy_i + \sum_{j=1}^{n} c'_j dz'_j$$

in $\Omega^1_{Z'_\pi/R}$, where $b'_i, c'_j \in C'$. Then the relation $(\dagger)$ implies

$$(\dagger\dagger) \qquad b_i = \pi^2 b'_i\,, \qquad c_j = \pi c'_j\,,$$

since the dy_i, dz'_j form a basis of $\Omega^1_{Z'_\pi/R}$. Now choose a point $a \in X(R^{sh})$ with $a_k \in U_k = Y_k$ and let $a' \in X'_\pi(R^{sh})$ be the lifting of a. Let d be the dimension of X_K at a_K. In order to relate $\delta(a')$ to $\delta(a)$, we want to apply Lemma 2. So let $f_1, \ldots, f_l$ be generators of I. There exists an $(r+n-d)$-minor Δ of the Jacobi matrix

$$\left(\frac{\partial f_\lambda}{\partial y_i}, \frac{\partial f_\lambda}{\partial z_j}\right)_{\substack{\lambda=1,\ldots,l\\ i=1,\ldots,r;\, j=1,\ldots,n}}$$

such that $\delta(a) = \nu(\Delta(a_K))$. Then, using the equation $(\dagger)$, we can define elements $f'_\lambda \in I'$ by $f'_\lambda := \pi^{-2} f_\lambda$. Let Δ' be the minor of the Jacobi matrix

$$\left(\frac{\partial f'_\lambda}{\partial y_i}, \frac{\partial f'_\lambda}{\partial z'_j}\right)_{\substack{\lambda=1,\ldots,l\\ i=1,\ldots,r;\, j=1,\ldots,n}}$$

which corresponds to Δ. Then the relations $(\dagger\dagger)$ show that Δ' is obtained from Δ by multiplying each column of Δ with a factor π^{-1} or π^{-2}. Thus

$$\nu(\Delta'(a_K)) \leqq \nu(\Delta(a_K)) - (n + r - d)$$

and, hence,

$$\delta(a') \leqq \delta(a) - (n + r - d)\,.$$

If $n + r - d > 0$, the assertion of the proposition is clear. If $n + r = d$, the smooth R-scheme Z has relative dimension d, and this is just the dimension of X_K at a_K. So Z_K and X_K coincide on an open neighborhood of a_K. Since X is a closed subscheme of Z, and since Z_K is schematically dense in Z, we see that X coincides with Z locally at a. So a factors through the smooth locus of X, and $\delta(a) = 0$ in this case. □

We mention here that, as we have seen, the proof actually yields a better estimate for the defect of smoothness than the one stated in Proposition 5. For example, if X_K is equidimensional of dimension d, if Y_k is equidimensional of dimension r, and if $\Omega^1_{X/R}|_{Y_k}$ is locally free of rank $r + n$, then

$$\delta(a') \leqq \delta(a) - (n + r - d)\,.$$

3.4 Proof of the Theorem

In order to prove Theorem 3.1/3, let us fix the notation we will use. As in the preceding section, X is an R-scheme of finite type whose generic fibre X_K is smooth over K. Let E be a subset of $X(R^{sh})$. A closed subscheme Y_k of X_k is called *E-permissible* if the following conditions are satisfied:

(i) *The set of k_s-valued points of Y_k which lift to R^{sh}-valued points in E is schematically dense in Y_k; in particular, the couple (X, Y_k) has the property* (N).

(ii) *Let U_k be the largest open subscheme of Y_k which is smooth over k and where $\Omega^1_{X/R}|_{Y_k}$ is locally free. Then there is no k_s-valued point in $Y_k - U_k$ which lifts to a point in E.*

Note that the subscheme $U_k \subset Y_k$ of (ii) is always Zariski-dense in Y_k due to Lemma 3.3/4. Using the notion of E-permissible subschemes, we can formulate Proposition 3.3/5 in a more precise form.

Lemma 1. *Let Y_k be an E-permissible subscheme of X_k, and let $X' \longrightarrow X$ be the blowing-up of Y_k on X. For a point $a \in E$, denote by $a' \in X'(R^{sh})$ its (unique) lifting.*

(a) *If a does not specialize into a point of Y_k, then $\delta(a) = \delta(a')$.*

(b) *If a specializes into a point of Y_k, then $\delta(a') \leqq \max\{0, \delta(a) - 1\}$.*

Proof. If $a_k \notin Y_k$, there exists an open neighborhood of a over which the blowing-up is an isomorphism; hence $\delta(a) = \delta(a')$. If $a_k \in Y_k$, Proposition 3.2/1 shows that the point a' is necessarily contained in the dilatation X'_π of Y_k in X. Since X'_π is an open subscheme of X' and since Y_k is E-permissible in X, Proposition 3.3/5 yields the desired estimate for $\delta(a')$. □

If Y_k is E-permissible in X, the blowing-up $X' \longrightarrow X$ of Y_k on X is said to be *E-permissible*. For any blowing-up $X' \longrightarrow X$ of a subscheme of the special fibre X_k, one has a canonical bijection $X'(R^{sh}) \xrightarrow{\sim} X(R^{sh})$. So we may identify $E \subset X(R^{sh})$ with the corresponding subset of $X'(R^{sh})$. Hence we get the notion of E-permissible blowing-ups for X' again. This allows us to formulate a more precise version of Theorem 3.1/3.

Theorem 2. *Let X be an R-scheme of finite type with a smooth generic fibre X_K, and let E be a subset of $X(R^{sh})$. Then there exists a proper morphism $X' \longrightarrow X$ which consists of a finite sequence of E-permissible blowing-ups with centers contained in the non-smooth parts of the corresponding schemes, such that each R^{sh}-valued point $a \in E$ factors through the smooth locus of X'. In particular, if X is quasi-projective over R, so is X'.*

Proof. For a subset $E \subset X(R^{sh})$, we introduce the defect of smoothness of X along E by

$$\delta(X,E) := \max\{\delta(a); a \in E\} .$$

Due to Proposition 3.3/3, we know $\delta(X,E)$ is finite. So we can proceed by induction on $\delta(X,E)$. If $\delta(X,E) = 0$, then each $a \in E$ factors through the smooth locus of X (cf. Lemma 3.3/1), and the assertion is trivial. So let $\delta(X,E) > 0$. Since we consider only blowing-ups with centers in the non-smooth locus, we can remove from E all points which factor through the smooth locus of X, and thereby we may assume $\delta(a) > 0$ for all $a \in E$.

For the induction step, we have to arrange things in such a way that Lemma 1 can be applied. We do this by introducing a canonical partition of the set $E \subset X(R^{sh})$. First let us fix some notations. For a subset $F \subset X(R^{sh})$, we denote by F_k the subset of $X(k_s)$ which is induced from F by specialization. Identifying points in F_k with their associated closed points in X_k, let $\overline{F}_k$ be the Zariski closure of F_k in X_k, provided with the canonical reduced structure. Then $(X, \overline{F}_k)$ satisfies property (N).

In order to obtain the desired partition of E, set $F^1 := E$ and $Y_k^1 := \overline{F_k^1}$. Let U_k^1 be the largest open subscheme of Y_k^1 which is smooth over k and where $\Omega^1_{X/R}|_{Y_k^1}$ is locally free, and define

$$E^1 := \{a \in F^1; a_k \in U_k^1\} .$$

Proceeding in the same way with $F^2 := F^1 - E^1$, and so on, we obtain

(i) a decreasing sequence $F^1 \supset F^2 \supset \ldots$ in $X(R^{sh})$,

(ii) subsets $E^1, E^2, \ldots \subset X(R^{sh})$ such that E decomposes into a disjoint union

$$E = E^1 \,\dot\cup\, \ldots \,\dot\cup\, E^i \,\dot\cup\, F^{i+1} ,$$

(iii) dense open subschemes $U_k^i \subset Y_k^i := \overline{F_k^i}$ such that $E_k^i \subset U_k^i$ and, moreover, $Y_k^{i+1} \subset Y_k^i - U_k^i$; in particular, $\dim Y_k^{i+1} < \dim Y_k^i$ if $Y_k^i \neq \emptyset$.

So we see that necessarily $Y_k^{t+1} = \emptyset$ for some $t \in \mathbb{N}$ big enough and, consequently, that $F^{t+1} = \emptyset$. Hence we have the partition

$$E = E^1 \,\dot\cup\, \ldots \,\dot\cup\, E^t .$$

Since each U_k^i is smooth over k, and since $\Omega^1_{X/R}|_{Y_k^i}$ is locally free on U_k^i, it follows that each Y_k^i is E^i-permissible, and that Y_k^t is, in fact, E-permissible. Furthermore, note that, in terms of subsets of X, each Y_k^i is disjoint from the smooth locus of X, since E_k and, hence, all F_k^i are disjoint from it, and since the non-smooth locus of X is a closed subset of X_k.

Now we can carry out the induction step. Let $X' \longrightarrow X$ be the blowing-up of Y_k^t on X. Then

$$\delta(X', E^t) < \delta(X, E^t)$$

by Lemma 1, because Y_k^t is E^t-permissible. Furthermore, due to the induction hypothesis, there exists a morphism $X'' \longrightarrow X'$ which consists of a sequence of E^t-permissible blowing-ups with centers contained in the non-smooth loci of the corresponding schemes, such that each $a \in E^t$, when viewed as an R^{sh}-valued point of X'', factors through the smooth locus of X''. Considering the composition $X'' \longrightarrow X' \longrightarrow X$, this modification does not affect the set $E - E^t$. So it is a sequence of E-permissible blowing-ups.

Writing $(E'')^i$ for the lifting of E^i to $X''(R^{sh})$, let us consider the partition

$$E'' = (E'')^1 \,\dot\cup\, \ldots \,\dot\cup\, (E'')^{t-1} ,$$

where E'' is obtained from the lifting of E by removing $(E'')^t$; i.e., by removing the set of points which factor through the smooth locus of X''. Then, obviously, this partition equals the canonical partition of E''. Since $\delta(X'', E'') \leqq \delta(X, E)$, a second induction on the length of such a partition yields a sequence of E''-permissible blowing-ups $X''' \longrightarrow X''$ with centers in non-smooth loci such that all points of E'', when viewed as R^{sh}-valued points of X''', factor through the smooth locus of X'''. Then

$$X''' \longrightarrow X'' \longrightarrow X' \longrightarrow X$$

is a sequence of E-permissible blowing-ups as desired. □

Remark 3. If in the situation of Theorem 2 it is not known that the generic fibre X_K is smooth, the assertion nevertheless remains true if the generic fibres of the points in E factor through the smooth locus of X_K and have a bounded defect of smoothness. Namely, these are the properties of E and X_K which are used in the proof.

3.5 Weak Néron Models

In the following let X_K be a smooth and separated K-scheme of finite type, and let K^{sh} be the field of fractions of a strict henselization R^{sh} of R. As a first step towards the construction of a Néron model of X_K, we want to look for a smooth and separated R-model of finite type, say X, such that each K^{sh}-valued point of X_K extends to an R^{sh}-valued point of X. We will see that such R-models X of X_K even satisfy certain aspects of the universal mapping property characterizing Néron models.

If X_K admits a separated R-model X of finite type such that the canonical map $X(R^{sh}) \longrightarrow X_K(K^{sh})$ is bijective, we can apply Corollary 3.1/4 to get a smooth R-model of the type we are looking for. For example, in the case of an abelian variety X_K we can proceed in this way, since there is a closed immersion $X_K \hookrightarrow \mathbb{P}^n_K$ into a projective space; we can take X to be the schematic closure of X_K in $\mathbb{P}^n_R$.

If it is only known that $X_K(K^{sh})$ is bounded in X_K, and if no separated R-model X of finite type such that $X(R^{sh}) \longrightarrow X_K(K^{sh})$ is bijective is given in an obvious way, we will consider a finite collection of separated R-models instead of a single one as before. Using the flattening techniques of Raynaud and Gruson [1], one can actually show that there exists a single separated R-model X of finite type such that each K^{sh}-valued point of X_K extends to an R^{sh}-valued point of X; we will give a sketch of proof in Proposition 6 below. But, for our purpose, it is not necessary to make use of this result, since we are mainly interested in group schemes X_K. Namely, in this case, it makes no difference if we start with a finite collection of R-models, since group arguments will help us later to reduce to a single R-model. As the second method is much more elementary, we will use it for our construction. We begin with a definition characterizing the collections of R-models of X_K we want to work with.

Definition 1. *A weak Néron model of X_K is a finite collection $(X_i)_{i \in I}$ of smooth and separated R-models of finite type such that each K^{sh}-valued point of X_K extends to an R^{sh}-valued point in at least one of these R-models.*

Theorem 2. *Let X_K be a smooth and separated K-scheme of finite type. If $X_K(K^{sh})$ is bounded in X_K, there exists a weak Néron model of X_K.*

Proof. Since $X_K(K^{sh})$ is bounded in X_K, it follows from 1.1/7 that there exists a finite family $(X_i)_{i \in I}$ of separated R-models of finite type such that each K^{sh}-valued point of X_K extends to an R^{sh}-valued point in at least one of these R-models. Applying Corollary 3.1/4 to each X_i, we obtain smooth and separated R-models X_i' of finite type such that the R^{sh}-valued points of X_i' and X_i correspond bijectively to each other. Hence $(X_i')_{i \in I}$ is a weak Néron model of X_K. □

Weak Néron models satisfy a certain mapping property which later leads to the universal mapping property characterizing Néron models.

Proposition 3 (Weak Néron Property). *Let $(X_i)_{i \in I}$ be a weak Néron model of X_K, and let Z be a smooth R-scheme with irreducible special fibre Z_k. Furthermore, let $u_K : Z_K \dashrightarrow X_K$ be a K-rational map. Then there exists an $i \in I$ such that u_K extends to an R-rational map $u : Z \dashrightarrow X_i$.*

Proof. There is an open dense subscheme $V_K \subset Z_K$ such that u_K is defined on V_K. Let F be the schematic closure of $F_K := Z_K - V_K$ in Z. Since we are working over a discrete valuation ring, F_k is nowhere dense in Z_k, and we may replace Z by $V := Z - F$ which is R-dense in Z. Thereby we may assume that u_K is defined on all of Z_K and thus is a K-morphism $Z_K \longrightarrow X_K$. Moreover, we may assume that Z is of finite type.

Consider the graph of u_K and denote its schematic closure in $Z \times_R X_i$ by Γ^i. Let $p_i : \Gamma^i \longrightarrow Z$ and $q_i : \Gamma^i \longrightarrow X_i$ be the projections. It is only necessary to show that, for some $i \in I$, the projection p_i is invertible on an R-dense open part of Z. Then $u := q_i \circ p_i^{-1} : Z \dashrightarrow X_i$ is a solution of our problem. One knows from Chevalley's theorem ([EGA IV$_1$], 1.8.4) that T^i_k, the image of Γ^i_k under p_i, is a constructible subset of Z_k, and we claim that, for some $i \in I$, the set T^i_k must contain a non-empty open part of Z_k. To verify this, we may assume $R = R^{sh}$, and hence, that k coincides with its separable algebraic closure. Then, by 2.2/13, the set of k-rational points is Zariski-dense in Z_k, and each $z_k \in Z_k(k)$ lifts to a point $z \in Z(R)$. Let $z_K \in Z(K)$ be the associated generic fibre, and set $x_K := u_K(z_K)$. By the definition of weak Néron models, there is an index $i \in I$ such that x_K extends to a point $x \in X_i(R)$. Consequently, we must have $(z, x) \in \Gamma^i(R)$ and thus $z_k \in T^i(k)$. This consideration shows that $\bigcup_{i \in I} T^i(k)$ is Zariski-dense in Z_k, and, since all T^i_k are constructible and I is finite, that there is some T^i_k containing a non-empty open part of Z_k.

Fixing such an index $i \in I$, let us consider the projection $p_i : \Gamma^i \longrightarrow Z$. The local ring $\mathcal{O}_{Z,\eta}$ at the generic point η of Z_k is a discrete valuation ring. Furthermore, as we have seen, there is a point $\xi \in \Gamma^i$ above η. Thus $\mathcal{O}_{\Gamma^i,\xi}$ dominates $\mathcal{O}_{Z,\eta}$. Since p_i is an isomorphism on generic fibres and since Γ^i is flat over R, both local rings give rise to the same field as total ring of fractions so that $\mathcal{O}_{Z,\eta} \longrightarrow \mathcal{O}_{\Gamma^i,\xi}$ is an isomorphism. Since Z and Γ^i are of finite type over R, there exist open neighborhoods U of η in Z and V of ξ in Γ^i such that p_i induces an isomorphism between U and V. Hence p_i is invertible over an R-dense open part of Z. □

Corollary 4. *Let Z be a smooth R-scheme, and let ζ be a generic point of the special fibre of Z. Denote by R' the local ring $\mathcal{O}_{Z,\zeta}$ of Z at ζ and by K' the field of fractions of R'. If $(X_i)_{i \in I}$ is a weak Néron model of X_K, then $(X_i \otimes_R R')_{i \in I}$ is a weak Néron model of $X_K \otimes_K K'$.*

Proof. Since the strict henselization of R' is a direct limit of étale extensions of R', it suffices to show that, for any étale Z-scheme Z', for any point ζ' of Z' above ζ, and for any K'-rational map $u'_{K'}$ from $Z'_{K'}$ to X_K, there exists an index $i \in I$ such that $u'_{K'}$ extends to a rational map $u' : Z' \dashrightarrow X_i$ which is defined at ζ'. Since ζ' is a generic point of the special fibre of Z', the assertion follows from Proposition 3. □

In the situation of Proposition 3, one cannot expect, in general, that the R-rational map $Z \dashrightarrow X$ is a morphism if $Z_K \dashrightarrow X_K$ is a morphism, even if the weak Néron model $(X_i)_{i \in I}$ of X_K consists of a single proper R-model of X_K. In particular, weak Néron models fail to be unique, even if one restricts to the class of weak Néron models consisting of a single R-model of X_K.

Example 5. Set $Z = X = \mathbb{P}^r_R$, the r-dimensional projective space over R, and consider a K-isomorphism $u_K : Z_K \xrightarrow{\sim} X_K$; i.e., a K-automorphism $u_K : \mathbb{P}^r_K \xrightarrow{\sim} \mathbb{P}^r_K$. Using a set of homogeneous coordinates $x_0, \dots, x_r$ of $\mathbb{P}^r_R$, we can describe u_K by

$$x_i \longmapsto \sum_{j=0}^{r} a_{ij} x_j \,, \qquad i = 0, \dots, r \,,$$

where $A := (a_{ij})$ is a matrix in $\mathrm{Gl}_{r+1}(K)$. We may assume that all coefficients a_{ij} belong to R. Then, by the theory of elementary divisors, there are matrices $S, T \in \mathrm{Gl}_{r+1}(R)$ and integers $0 \le n_0 \le \ldots \le n_r$ such that

$$SAT = \begin{pmatrix} \pi^{n_0} & & 0 \\ & \ddots & \\ 0 & & \pi^{n_r} \end{pmatrix}.$$

Hence there exist sets of homogeneous coordinates $x_0, \ldots, x_r$ and $x'_0, \ldots, x'_r$ of $\mathbb{P}^r_R$ such that u_K is described by

$$x_i \longmapsto \pi^{n_i} x'_i\,, \qquad i = 0, \ldots, r\,,$$

where we may assume $n_0 = 0$.

If $n_0 = \ldots = n_r = 0$, it is clear that $u_K : \mathbb{P}^r_K \xrightarrow{\sim} \mathbb{P}^r_K$ extends to an automorphism $u : \mathbb{P}^r_R \xrightarrow{\sim} \mathbb{P}^r_R$. However, if $n_0 = \ldots = n_s = 0$ and $n_{s+1}, \ldots, n_r > 0$ for some $s < r$, then u_K extends only to an R-rational map $u : \mathbb{P}^r_R \dashrightarrow \mathbb{P}^r_R$. Namely, u is defined on the R-dense open subscheme $V \subset \mathbb{P}^r_R$ which consists of the generic fibre $\mathbb{P}^r_K$ and of the open part $V_k \subset \mathbb{P}^r_k$ complementary to the linear subspace Q_k where $x_0, \ldots, x_s$ vanish. In fact, if Q'_k is the linear subspace in $\mathbb{P}^r_k$ where $x'_{s+1}, \ldots, x'_r$ vanish, we can view u_k as a projection of $\mathbb{P}^r_k$ onto Q'_k with center Q_k.

Finally, as indicated at the beginning of this section, we want to show how, for a separated K-scheme X_K of finite type, one can always find a single separated R-model X of finite type such that $X(R^{sh}) \longrightarrow X_K(K^{sh})$ is bijective. The key fact which has to be established is the following result:

Proposition 6. *Let X_K be a separated (not necessarily smooth) K-scheme of finite type. Let $X_1, \ldots, X_n$ be separated R-models of X_K which are of finite type. Then there exist a separated R-model X of finite type of X_K and proper morphisms $X'_i \longrightarrow X_i$, $i = 1, \ldots, n$, consisting of finite sequences of blowing-ups with centers in the special fibres such that the given isomorphisms*

$$X'_i \otimes K \xrightarrow{\sim} X \otimes K$$

extend to open immersions $X'_i \hookrightarrow X$.

Thus, using the valuative criterion of properness, we obtain the desired characterization of boundedness.

Corollary 7. *$X_K(K^{sh})$ is bounded in X_K if and only if X_K admits a separated R-model X of finite type such that each K^{sh}-valued point of X_K extends to an R^{sh}-valued point of X.*

Before starting the proof, let us list some elementary facts we will need. Let U, U', V, V' be separated and flat R-schemes of finite type and, for shortness, let us refer here to an R-morphism $W \longrightarrow U$ as a *blowing-up* if it is a finite sequence of blowing-ups with centers in the special fibres; note that W is separated, flat, and of finite type if U is.

(a) Let $U' \longrightarrow U$ be a blowing-up, and let $U \hookrightarrow V$ be an open immersion. Then there exists a blowing-up $V' \longrightarrow V$ such that $U' \longrightarrow U$ is obtained from $V' \longrightarrow V$ by the base change $U \hookrightarrow V$.

Just extend the center of the blowing-up $U' \longrightarrow U$ to a subscheme of V and define V' by blowing up this subscheme in V.

(b) If $U'_i \longrightarrow U$, $i = 1, 2$, are blowing-ups, then there exists a commutative diagram of blowing-ups

$$\begin{array}{ccc} U' & \longrightarrow & U'_1 \\ \downarrow & & \downarrow \\ U'_2 & \longrightarrow & U\,. \end{array}$$

Namely, if $U'_i \longrightarrow U$ is the blowing-up of the ideal $\mathscr{I}_i$ of $\mathcal{O}_U$, $i = 1, 2$, then define U' as the blowing-up of $\mathscr{I}_1 \cdot \mathscr{I}_2$ on U. Note that U' is isomorphic to the blowing-up on U'_2 of the pull-back of $\mathscr{I}_1$ under $U'_2 \longrightarrow U$ and to the blowing-up on U'_1 of the pull-back of $\mathscr{I}_2$ under $U'_1 \longrightarrow U$.

(c) Let $f : U \longrightarrow V$ be a flat R-morphism such that f_K is an open immersion. Then f is an open immersion.

Let us justify the latter statement. Since f is open, we may assume f faithfully flat. Furthermore, it is enough to show that f is an open immersion after faithfully flat base change. So we may perform the base change $U \longrightarrow V$ and thereby assume that f has a section ε. Then it is to verify that ε is an isomorphism. We know already that ε is a closed immersion, since f is separated. Thus we have the canonical surjective map

$$\alpha : \mathcal{O}_U \longrightarrow \varepsilon_* \mathcal{O}_V\,.$$

Since f_K is an isomorphism, the kernel of $\alpha \otimes_R K$ vanishes. But $\mathcal{O}_U$ is flat over R, so the kernel of α must vanish identically. Then α is an isomorphism and, hence, ε is an isomorphism.

Finally we mention the technique of flattening by blowing up which will serve as a key point in the proof of Proposition 6; cf. Raynaud and Gruson [1], Thm. 5.2.2.

Let $f : U \longrightarrow V$ be an R-morphism such that f_K is flat. Then there exists a blowing-up $V' \longrightarrow V$ such that the strict transform $f' : U' \longrightarrow V'$ of f is flat.

Here U' is the schematic closure of U_K in $U \times_V V'$ (the strict transform of U), and f' is the restriction of $f \times_V \mathrm{id}_{V'}$ to U'.

Now let us give the *proof of Proposition* 6. By an induction argument, one reduces to the case where only two R-models X_1 and X_2 are given. Denote by Γ the schematic closure of the graph of the isomorphism $X_1 \otimes K \xrightarrow{\sim} X_2 \otimes K$ in $X_1 \times_R X_2$. Applying the flattening by blowing up, there exist blowing-ups $X'_i \longrightarrow X_i$, $i = 1, 2$, such that the strict transform $p'_i : \Gamma'_i \longrightarrow X'_i$ of the i-th projection $p_i : \Gamma \longrightarrow X_i$ is flat. Notice that the canonical map $\Gamma'_i \longrightarrow \Gamma$ is a blowing-up, too. Then, by (c), the map p'_i is an open immersion and, by (b), there is a commutative diagram of blowing-ups

$$\begin{array}{ccc} \Gamma'' & \longrightarrow & \Gamma'_1 \\ \downarrow & & \downarrow \\ \Gamma'_2 & \longrightarrow & \Gamma\,. \end{array}$$

Furthermore, since $p'_i : \Gamma'_i \longrightarrow X'_i$ is an open immersion, there exists a blowing-up $X''_i \longrightarrow X'_i$ such that $\Gamma'' \longrightarrow \Gamma'_i$ is obtained from $X''_i \longrightarrow X'_i$ by restriction to Γ'_i; see (a). Then $\Gamma'' \longrightarrow X''_i$ is an open immersion, and we can glue X''_1 and X''_2 along Γ''. Thereby we obtain an R-model X of X_K which is of finite type, and which contains X''_1 and X''_2 as open subschemes. Moreover, X is separated. Namely, let Γ^* be the schematic closure of the graph of the isomorphism $X''_1 \otimes K \xrightarrow{\sim} X''_2 \otimes K$ in $X''_1 \times_R X''_2$. Since Γ'' is flat over R, the canonical isomorphism $\Gamma'' \otimes K \longrightarrow \Gamma^* \otimes K$ extends by continuity to a morphism $\Gamma'' \longrightarrow \Gamma^*$. Similar arguments show that the canonical morphism $\Gamma^* \otimes K \longrightarrow \Gamma \otimes K$ extends to a morphism $\Gamma^* \longrightarrow \Gamma$. Then, due to its construction, the morphism $\Gamma'' \longrightarrow \Gamma$ is proper, and it follows from [EGA II], 5.4.3, that $\Gamma'' \longrightarrow \Gamma^*$ is proper. Thus Γ'' is closed in Γ^* and hence closed in $X''_1 \times_R X''_2$. Thereby it is seen that X is separated over R. □

3.6 Algebraic Approximation of Formal Points

Apart from its importance for the construction of Néron models, the smoothening process is also a necessary tool for the proof of M. Artin's approximation theorem, which will be the subject of this section. As a first step, we have to show that a smoothening $X' \longrightarrow X$ of an R-scheme X satisfies the lifting property not only for R'-valued points, where R' is étale over R, but even for a larger class of extensions R'/R. For example, we are concerned with the case where R' is the π-adic completion $\hat{R}$ of R.

Definition 1. *A flat local extension $R \longrightarrow R'$ of discrete valuation rings is said to have ramification index 1 if a uniformizing element π of R induces a uniformizing element of R', and if the extension of the residue fields $k' = R'/\pi R'$ over $k = R/\pi R$ is separable.*

Recall that an extension of fields k'/k is separable if and only if $k' \otimes_k l$ is reduced for all fields l over k; cf. Bourbaki [1], Chap. VIII, § 7, n°3.

To illustrate the definition, we mention that the π-adic completion $\hat{R}$ of R has ramification index 1 over R. Furthermore, if R' is essentially of finite type over R, it has ramification index 1 over R if and only if R' is a local ring of a smooth R-scheme at a generic point of the special fibre. In this case, $R \longrightarrow R'$ or, better, the morphism $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$ is regular in the sense of [EGA IV_2], 6.8.1. The class of ring extensions of ramification index 1 is stable under the formation of direct limits and completions.

If $R \longrightarrow R'$ has ramification index 1 and if, in addition, the extension of fields of fractions K'/K is separable, the extension R'/R is regular. For example, the extension $\hat{R}/R$ is regular or, equivalently, the extension of fields of fractions $Q(\hat{R})/Q(R)$ is separable, if and only if R is excellent (cf. [EGA IV_2], 7.8.2).

Lemma 2. *Let R be an excellent discrete valuation ring. If $R \longrightarrow R'$ has ramification index 1, then $R \longrightarrow R'$ is regular. In particular, since the completion of R' is of ramification index 1 over R, it follows that R' is excellent.*

Proof. Let K (resp. K') be the field of fractions of R (resp. R'). We have only to prove that K' is separable over K. So we may assume $p = \operatorname{char} K > 0$. It suffices to show that $L \otimes_K K'$ is reduced for each finite radicial extension L of K; cf. [EGA IV_2], 6.7.7. Let us first consider the case where the extension L/K is radicial of degree p. Since R is excellent, the integral closure $\tilde{R}$ of R in L is an R-module of finite type (cf. [EGA IV_2], 7.8.3) and, hence, a free R-module of rank p. Moreover, $\tilde{R}$ is a discrete valuation ring. So let $\tilde{k}$ be the residue field of $\tilde{R}$. If the degree of $\tilde{k}$ over k is p, then π is a uniformizing element of $\tilde{R}$, and $\tilde{R} \otimes_R R'/(\pi)$ is isomorphic to $\tilde{k} \otimes_k k'$. The latter is a field, since k' is separable over k and since $\tilde{k}$ is radicial over k; hence $\tilde{R} \otimes_R R'$ is a discrete valuation ring with uniformizing element π. If $\tilde{k} = k$, the p-th power of a uniformizing element $\tilde{\pi}$ of $\tilde{R}$ gives rise to a uniformizing element of R, and $\tilde{R} \otimes_R R'$ is a discrete valuation ring with uniformizing element $\tilde{\pi} \otimes 1$. In both cases, $\tilde{R} \otimes_R R'$ is a discrete valuation ring. Considering its field of fractions, it follows that $L \otimes_K K'$ is reduced. Since a finite radicial extension can be broken up into radicial subextensions of degree p, the same assertion remains true for arbitrary radicial extensions L of K. □

We mention that the ring of integers $\mathbb{Z}$ as well as all fields are excellent and that any R-algebra which is essentially of finite type over an excellent ring R is excellent; see [EGA IV_2], 7.8.3 and 7.8.6.

We want to show that smoothenings are compatible with ring extensions R'/R of ramification index 1. In order to do this, certain parts of the smoothening process have to be generalized. So let X be an R-scheme of finite type, and let R'/R be a ring extension of ramification index 1. Let a be an R'-valued point of X such that its generic fibre a_K factors through the smooth locus of the generic fibre X_K. Then, as in 3.3, we set

$$\delta(a) := \text{length of the torsion part of } a^*\Omega^1_{X/R} .$$

Without changes, the proof of 3.3/1 shows that $\delta(a) = 0$ if and only if a factors through the smooth locus of X. Furthermore, the key proposition of the smoothening process remains valid:

Proposition 3. *Let Y_k be the schematic closure of a_k in X_k. Let $X'_\pi \longrightarrow X$ be the dilatation of Y_k in X, and denote by a' the (unique) lifting of a to an R'-valued point of X'_π. Then $\delta(a') \le \max\{0, \delta(a) - 1\}$.*

Literally the same proof as the one of 3.3/5 works in this case; namely, one has only to observe the fact that a_k factors through the smooth locus of the k-scheme Y_k. Since Y_k is geometrically reduced, the generic point of Y_k, which is a_k, is contained in the smooth locus of the k-scheme Y_k; cf. 2.2/16. Applying Proposition 3 finitely many times, one obtains an analogue of 3.1/3.

Proposition 4. *Let X be an R-scheme of finite type, and consider an extension R'/R of ramification index 1. Let a be an R'-valued point of X such that a_K factors through the smooth locus of X_K. Then there exists an R-morphism $X' \longrightarrow X$, which consists of a finite sequence of dilatations with centers in special fibres, such that a lifts to an R'-valued point of X' which factors through the smooth locus of X'.*

Proposition 4 enables us to show that smoothenings are compatible with ring extensions R'/R of ramification index 1. One has only to justify the following fact.

Lemma 5. *Let X be an R-scheme of finite type with smooth generic fibre, let $X' \longrightarrow X$ be a smoothening of X, and consider an extension R'/R of ramification index 1. Then each R'-valued point a of X lifts to an R'-valued point a' of X' which factors through the smooth locus of X'.*

Proof. Due to the properness of $X' \longrightarrow X$, the point $a \in X(R')$ lifts to a point $a' \in X'(R')$. Due to Proposition 4, there exists a finite sequence of dilatations $\sigma : X'' \longrightarrow X'$ such that σ is an isomorphism on generic fibres and such that the (unique) lifting a'' of a' factors through the smooth locus of X''. Since the schematic closure A''_k of a''_k in X''_k is geometrically reduced and, hence, generically smooth over k by 2.2/16, the set of those closed points $x \in A''_k \cap X''_{\text{smooth}}$ which have a separable residue field $k(x)$ over k is dense in A''_k; cf. 2.2/13. Since all these points lift to R^{sh}-valued points of X'', the image of a''_k in X', which equals a'_k, is contained in the smooth locus of X' (because X' is a smoothening of X). □

Corollary 6. *Let X be an R-scheme of finite type with a smooth generic fibre, let $X' \longrightarrow X$ be a smoothening of the R-scheme X, and consider an extension R'/R of ramification index 1. Then $X' \otimes_R R' \longrightarrow X \otimes_R R'$ is a smoothening of the R'-scheme $X \otimes_R R'$.*

Proof. Since $R \longrightarrow (R')^{sh}$ has ramification index 1, the assertion follows from Lemma 5. □

Using the preceding result and the existence of Nagata compactifications (Nagata [1] and [2]) for separated schemes of finite type over R, we can generalize 3.5/4 and show that weak Néron models are stable under extensions R'/R of ramification index 1. As usual, fields of fractions are denoted by K, residue fields by k, and strict henselizations by an index "sh".

Proposition 7. *Let X_K be a smooth K-scheme of finite type admitting a weak Néron model $(X_i)_{i\in I}$ over R. Let R'/R be of ramification index 1. Then $(X_i \otimes_R R')_{i\in I}$ is a weak Néron model of $X_{K'}$ over R'.*

Proof. Using 3.5/6, one easily reduces to the case where the index set I consists of a single element. So let X be a smooth and separated R-model of finite type of X_K such that the canonical map $X(R^{sh}) \longrightarrow X(K^{sh})$ is bijective, and consider a K'-valued point of X_K; i.e., a K-morphism $a_K : \operatorname{Spec} K' \longrightarrow X_K$. We have to show that a_K extends to an R-morphism $a : \operatorname{Spec} R' \longrightarrow X$. In order to do this, let $\bar{X}$ be a Nagata compactification of X. The latter is a proper R-scheme containing X as a dense open subscheme. Since X is flat over R, we see that X_K is dense in X and, hence, that X_K is dense in $\bar{X}_K$.

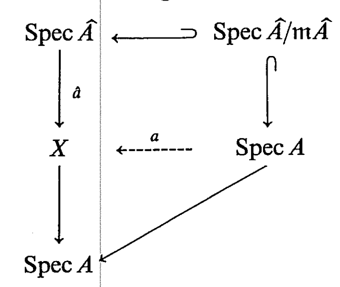
By the properness of $\bar{X}$, the morphism a_K extends to an R-morphism $\bar{a} : \operatorname{Spec} R' \longrightarrow \bar{X}$ such that the image of the generic point of $\operatorname{Spec} R'$ is contained in X_K and, thus, in the smooth locus of $\bar{X}_K$. So we can apply Proposition 4 and thereby find a finite sequence of dilatations $\bar{X}' \longrightarrow \bar{X}$ with centers in special fibres such that $\bar{a}$ lifts to an R'-valued point $\bar{a}'$ of the smooth locus of $\bar{X}'$. Similarly as in the proof of Lemma 5, let A_k be the schematic image of the special fibre of $\bar{a}'$ in the special fibre of $\bar{X}'$. Since A_k is generically smooth over k, the set E_k of its closed points x_k which have separable residue field $k(x_k)$ and which belong to the smooth part of $\bar{X}'$ is dense in A_k.

All points $x_k \in E_k$ lift to R^{sh}-valued points of $\bar{X}'$ by 2.2/14, and we claim that the liftings can be chosen in such a way that their generic fibres factor through X_K. Namely, as in the proof of 2.2/14, one uses the Jacobi Criterion 2.2/7 in order to construct local coordinates $g_1, \ldots, g_n$ in a neighborhood $U \subset \bar{X}'$ of x_k which, on the special fibre, generate the ideal of x_k. The g_i give rise to an étale morphism $g : U \longrightarrow \mathbb{A}^n_K$. Since the image of $\bar{X}_K - X_K$ under g is thin in $\mathbb{A}^n_K$, it follows that x_k can be lifted to a point $x \in \bar{X}'(R^{sh})$ whose generic fibre belongs to $X_K(K^{sh})$ as claimed.

Now, composing each such $x \in \bar{X}'(R^{sh})$ with the morphism $\bar{X}' \longrightarrow \bar{X}$, we obtain a set of points $F \subset \bar{X}(R^{sh})$ whose generic fibres belong to X_K and whose special fibres are dense in A_k. But then, since X is a weak Néron model of X_K, we must have $F \subset X(R^{sh})$, and it follows that the generic point of A_k belongs to X. Consequently, the R-morphism $\bar{a} : \operatorname{Spec} R' \longrightarrow \bar{X}$ factors through X giving rise to the desired extension of $a_K : \operatorname{Spec} K' \longrightarrow X_K$. □

For the remainder of this section, we will be concerned with approximation theory. Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$, and denote by $\hat{A}$ its $\mathfrak{m}$-adic completion. We say A satisfies the *approximation property* if, for each

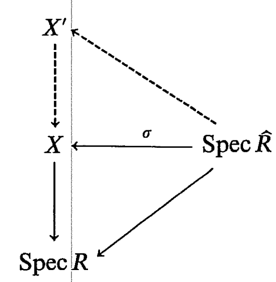
A-scheme X of finite type and for each $\hat{A}$-valued point $\hat{a}$ of X, there exists an A-valued point a of X such that the diagram



is commutative. Since $\hat{A}$ is henselian, it is clear by Definition 2.3/1′ that A is henselian if it satisfies the approximation property. Morever, if A is henselian, we see from 2.3/5 that, for each $\hat{A}$-valued point $\hat{a}$ of X which factors through the smooth locus of X, there exists an A-valued point of X which coincides with $\hat{a}$ on $\operatorname{Spec} \hat{A}/\mathfrak{m}\hat{A}$.

Using the smoothening process, it is easy to verify the approximation property for discrete valuation rings which are henselian and excellent, as can be seen from the following proposition.

Proposition 8. *Let R be an excellent discrete valuation ring, and let $\hat{R}$ be its completion. Furthermore, let X be an R-scheme of finite type, and let σ be an $\hat{R}$-valued point of X. Then there exists a commutative diagram of R-morphism*



where X' is smooth over R.

Proof. We may assume that σ is schematically dense in X. Since R is excellent, the generic fibre X_K is geometrically reduced and, hence, smooth at the generic point; cf. 2.2/16. So σ_K factors through the smooth locus of X_K and the assertion follows from Proposition 4. □

Corollary 9. *Let R be a discrete valuation ring which is henselian and excellent. Then R satisfies the approximation property.*

In the following we denote by $\hat{K}$ the field of fractions of $\hat{R}$. If X_K is a K-scheme which is locally of finite type, we can provide $X_K(\hat{K})$ with the canonical topology, which is induced by the valuation on K. We claim that this topology coincides with the one generated by all images of maps $X(\hat{R}) \longrightarrow X_K(\hat{K})$, where X varies over all R-models of X_K which are locally of finite type over R. Namely, each R-model U of an open subset $U_K \subset X_K$ induces an R-model X of X_K by gluing U and X_K over U_K. Since $X(\hat{R}) = U(\hat{R})$, it is enough to check the equality of the topologies for an affine K-scheme X_K, say $X_K = \operatorname{Spec} A_K$. In this case, a basis of the topology of $X_K(\hat{K})$ induced by the valuation of K is given by the family of subsets of type

$$U(g_1,\ldots,g_r) = \{x \in X_K(\hat{K})\,;\quad x^*(g_i) \in \hat{R} \text{ for } i = 1,\ldots,r\}$$

where $g_1, \ldots, g_r \in A_K$. Without loss of generality, we may assume that $g_1, \ldots, g_r$ generate A_K as a K-algebra. Then consider the R-model $X = \operatorname{Spec} A$ of X_K, where A is the image of the R-morphism

$$R[T_1,\ldots,T_r] \longrightarrow A_K\,, \qquad T_i \longmapsto g_i\,.$$

It follows that $U(g_1,\ldots,g_r)$ is the image of $X(\hat{R}) \longrightarrow X_K(\hat{K})$. Conversely, let X be an R-model of locally finite type of X_K. It remains to show that the image of $X(\hat{R}) \longrightarrow X_K(\hat{K})$ is open in $X_K(\hat{K})$. We may assume that X is affine, say $X = \operatorname{Spec} A$. Let $h_1, \ldots, h_r$ generate A as an R-algebra and denote by g_i the pull-back of h_i to X_K. Then the image of $X(\hat{R}) \longrightarrow X_K(\hat{K})$ coincides with the set $U(g_1,\ldots,g_r)$ (as defined above) and, hence, is open in $X_K(\hat{K})$.

Corollary 10. *Let R be a henselian discrete valuation ring and let X_K be a K-scheme which is locally of finite type. Assume either that R is excellent or that X_K is smooth. Then $X_K(K)$ is dense in $X_K(\hat{K})$ with respect to the topology induced by the valuation of K.*

Proof. It suffices to show that each R-model X of X_K which admits an $\hat{R}$-valued point admits an R-valued point. But this follows from Corollary 9 if R is excellent, and from Proposition 4 if X_K is smooth. □

There are examples of discrete valuation rings which are henselian, but which do not satisfy the approximation property; see the example below. Such rings cannot be excellent. In fact, it is easy to show that a discrete valuation ring R is excellent if it satisfies the approximation property. Thus, the approximation property for R is equivalent to the fact that R is henselian and excellent.

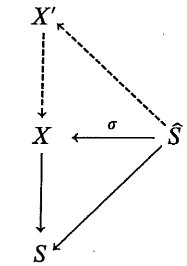
Example 11. Let $k = \mathbb{F}_p$ be the prime field of characteristic $p > 0$, and let Λ be the localization of the polynomial ring $k[T]$ at the maximal ideal generated by T. The completion $\hat{\Lambda}$ of Λ with respect to T is the ring $k[[T]]$ of formal power series. Looking at the cardinality of $k[[T]]$ (resp. of $k[T]$), it is clear that the extension $k((T))/k(T)$ of the fields of fractions is not algebraic. So pick an element $\xi \in \hat{\Lambda}$ which is not algebraic over $k(T)$. Set $U = \xi^p$, and let L be the field generated by T and U over k. Now define R as the intersection of L with $\hat{\Lambda}$. Then R is a discrete valuation ring whose completion $\hat{R}$ coincides with $k[[T]]$. Furthermore, $\hat{K} = Q(\hat{R})$ is not separable over $K = Q(R)$ since $\xi \in \hat{K} - K$. So R is not excellent. The henselization R^h of R can be viewed as the set of all elements of $k[[T]]$ which are separably algebraic over K. In particular, ξ is not contained in R^h, and it is easily verified that R^h does not satisfy the approximation property.

Next we want to generalize Proposition 8 to the case where the base consists of a polynomial ring over an excellent discrete valuation ring. The resulting assertion will be crucial in the proof of M. Artin's approximation theorem.

Theorem 12. *Let R be an excellent discrete valuation ring, and denote by $\hat{R}$ its π-adic completion. Let $T_1, \ldots, T_n$ be variables, and set*

$$S = \operatorname{Spec} R[T_1,\ldots,T_n]\,,$$
$$\hat{S} = \operatorname{Spec} \hat{R}[[T_1,\ldots,T_n]]\,.$$

Let X be an S-scheme of finite type, and let σ be an $\hat{S}$-valued point of X. Then there exists a commutative diagram of S-morphisms



where X' is smooth over S.

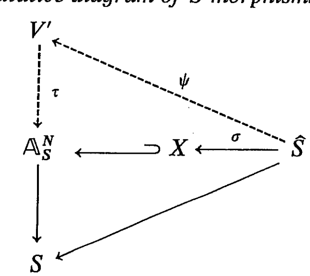
The *proof* is done by induction on the number n of variables $T_1, \ldots, T_n$. The case $n = 0$ is settled by Proposition 8. So let $n > 0$. We may assume that X is a closed subscheme of $\mathbb{A}_S^N$ and that X is defined by global sections of $\mathcal{O}_{\mathbb{A}_S^N}$, say

$$X = V(f_1,\ldots,f_r) \subset \mathbb{A}_S^N\,;$$

the coordinate functions of $\mathbb{A}_S^N$ will be denoted by $Y_1, \ldots, Y_N$. Let η (resp. $\hat{\eta}$) be the generic point of the special fibre of S (resp. $\hat{S}$), let $\hat{s}$ be the closed point of $\hat{S}$, and let s be its image in S.

In order to carry out the induction step, we will establish three lemmata, the first and the third one under the assumption of the induction hypothesis; i.e., under the assumption that Theorem 12 is true for less than n variables.

Lemma 13. *Let f_0 be a global section of $\mathcal{O}_{\mathbb{A}_S^N}$ such that $\sigma^* f_0$ does not vanish at $\hat{\eta}$. Then there exists a commutative diagram of S-morphisms*



such that V' is smooth over S and such that τ^*f_0 divides each τ^*f_i, $i = 1, \ldots, r$, in $\Gamma(V', \mathcal{O}_{V'})$.

In the proof of the lemma, we will use Weierstraß division for the formal power series ring $\hat{R}[[T_1, \ldots, T_n]]$; cf. Bourbaki [2], Chap. VII, §3, n°8. Let us first recall some basic facts of this theory. An element $f \in \hat{R}[[T_1, \ldots, T_n]]$ is called a *Weierstraß divisor in T_n of degree $d \geq 0$* if the coefficients $a_\nu \in \hat{R}[[T_1, \ldots, T_{n-1}]]$ of the power series expansion

$$f = \sum_{\nu=0}^{\infty} a_\nu T_n^\nu$$

satisfy the conditions

(1) a_d is a unit in $\hat{R}[[T_1, \ldots, T_{n-1}]]$,

(2) $a_\delta \in (\pi, T_1, \ldots, T_{n-1})$ for $\delta = 0, \ldots, d-1$.

An element of $\hat{R}[[T_1, \ldots, T_n]]$ is called a *Weierstraß polynomial in T_n of degree d* if it is a monic polynomial in T_n of degree d with coefficients in $\hat{R}[[T_1, \ldots, T_{n-1}]]$ and if it is a Weierstraß divisor in T_n of degree d. Note that an element $f \in \hat{R}[[T_1, \ldots, T_n]]$ is a Weierstraß divisor in T_n of degree d if and only if the reduction of f modulo π, as an element of $k[[T_1, \ldots, T_n]]$, is a Weierstraß divisor in T_n of degree d. Since $\hat{R}$ is complete, the Weierstraß division theorem for $k[[T_1, \ldots, T_n]]$ lifts to a division theorem for $\hat{R}[[T_1, \ldots, T_n]]$:

If $f \in \hat{R}[[T_1, \ldots, T_n]]$ is a Weierstraß divisor in T_n of degree d, then $\hat{R}[[T_1, \ldots, T_n]]$ decomposes into a direct sum

$$(*) \qquad \hat{R}[[T_1, \ldots, T_n]] = \bigoplus_{\delta=0}^{d-1} \hat{R}[[T_1, \ldots, T_{n-1}]]T_n^\delta \oplus \hat{R}[[T_1, \ldots, T_n]] \cdot f$$

of $\hat{R}[[T_1, \ldots, T_{n-1}]]$-modules. Furthermore, f can be written as a product of a unit in $\hat{R}[[T_1, \ldots, T_n]]$ and a Weierstraß polynomial of degree d.

The last assertion follows easily if one applies the decomposition $(*)$ to the element T_n^d, say

$$T_n^d = \sum_{\delta=0}^{d-1} a_\delta T_n^\delta + u \cdot f .$$

Then u is a unit, and

$$p = T_n^d - \sum_{\delta=0}^{d-1} a_\delta T_n^\delta$$

is the Weierstraß polynomial we are looking for. Further, we want to mention that, for each element $f \in \hat{R}[[T_1, \ldots, T_n]]$ which does not vanish identically modulo π, there exists an $\hat{R}$-automorphism φ of $\hat{R}[[T_1, \ldots, T_n]]$ of type

$$T_n \longmapsto T_n$$
$$T_i \longmapsto T_i + T_n^{b_i}, \qquad i = 1, \ldots, n-1,$$

such that $\varphi(f)$ is a Weierstraß divisor in T_n of some degree $d \geq 0$.

Proof of Lemma 13. If σ^*f_0 is a unit, then f_0 is invertible in a neighborhood of $\sigma(\hat{s})$ and, hence, the assertion is obvious. So we may assume that σ^*f_0 is not a unit. Since σ^*f_0 does not vanish at $\hat{\eta}$, there exists an $\hat{R}$-automorphism of $\hat{R}[[T_1, \ldots, T_n]]$ of type

$$T_n \longmapsto T_n , \qquad T_i \longmapsto T_i + T_n^{b_i} , \qquad i = 1, \ldots, n-1 ,$$

such that σ^*f_0 will be transformed by this automorphism into a Weierstraß divisor of degree $d \geq 1$. So we may assume that σ^*f_0 is a Weierstraß divisor of degree $d \geq 1$. Then σ^*f_0 can uniquely be written as

$$\sigma^*f_0 = \hat{u} \cdot \hat{p}$$

with a Weierstraß polynomial

$$\hat{p} = T_n^d + a'_{d-1} T_n^{d-1} + \ldots + a'_0 \in \hat{R}[[T_1, \ldots, T_{n-1}]][T_n]$$

of degree d and a unit $\hat{u}$ in $\hat{R}[[T_1, \ldots, T_n]]$. The Weierstraß division theorem yields a decomposition of $\hat{R}[[T_1, \ldots, T_n]]$ into a direct sum

$$(*) \qquad \hat{R}[[T_1, \ldots, T_n]] = \bigoplus_{\delta=0}^{d-1} \hat{R}[[T_1, \ldots, T_{n-1}]]T_n^\delta \oplus \hat{R}[[T_1, \ldots, T_n]] \cdot \hat{p}$$

of $\hat{R}[[T_1, \ldots, T_{n-1}]]$-modules. We will use the decomposition $(*)$ in order to make the application of the induction hypothesis possible. First we want to construct an auxiliary S-scheme V as a subscheme of $\mathbb{A}_S^{N'}$, where

$$N' = N \cdot d + d + N .$$

Let

$$Y_{\nu\delta}; \qquad \nu = 1, \ldots, N, \qquad \delta = 0, \ldots, d-1,$$
$$A_\delta; \qquad \delta = 0, \ldots, d-1,$$
$$Z_\nu; \qquad \nu = 1, \ldots, N,$$

be the coordinate functions of $\mathbb{A}_S^{N'}$ so that $\mathbb{A}_S^{N'} = \operatorname{Spec} R[T_\mu, Y_{\nu\delta}, A_\delta, Z_\nu]$. Consider the polynomial

$$p = T_n^d + A_{d-1} T_n^{d-1} + \ldots + A_0$$

and define an S-morphism $\tau: \mathbb{A}_S^{N'} \longrightarrow \mathbb{A}_S^N$ by setting

$$\tau^* Y_\nu = \sum_{\delta=0}^{d-1} Y_{\nu\delta} T_n^\delta + Z_\nu p$$

for $\nu = 1, \ldots, N$. Then Euclid's division yields unique decompositions

$$(**) \qquad \tau^* f_i = \sum_{\delta=0}^{d-1} f_{i\delta} T_n^\delta + q_i \cdot p , \qquad i = 0, \ldots, r ,$$

in $\mathcal{O}_{\mathbb{A}_S^{N'}}$ where $f_{i\delta}$ is independent of T_n for all i and δ. Furthermore, each $f_{i\delta}$ is independent of $Z_1, \ldots, Z_N$ by the definition of τ. Thus we have

$$f_{i\delta} \in R[T_\mu, Y_{\nu\delta'}, A_{\delta'}]_{\mu=1,\ldots,n-1;\, \nu=1,\ldots,N;\, \delta'=0,\ldots,d-1} .$$

Denote by S' (resp. $\hat{S}'$) the spectrum of $R[T_1, \ldots, T_{n-1}]$ (resp. $\hat{R}[[T_1, \ldots, T_{n-1}]]$), set

$$N'' = d \cdot N + N\,,$$

and regard the above ring $R[T_\mu, Y_{\nu\delta'}, A_{\delta'}]$ as the ring of global sections of $\mathcal{O}_{\mathbb{A}^{N''}_{S'}}$. Then the inclusion

$$R[T_\mu, Y_{\nu\delta'}, A_{\delta'}] \hookrightarrow R[T_\mu, Y_{\nu\delta'}, A_{\delta'}, Z_\nu]\,,$$

where on the left-hand side μ runs from 1 to $n-1$ and on the right-hand side from 1 to n, defines a projection

$$\rho : \mathbb{A}^{N'}_S \longrightarrow \mathbb{A}^{N''}_{S'}\,.$$

Consider now the closed subschemes

$$W = V(f_{i\delta})_{\substack{i=0,\ldots,r\\ \delta=0,\ldots,d-1}} \subset \mathbb{A}^{N''}_{S'}\,, \qquad \text{and}$$

$$V = V(f_{i\delta})_{\substack{i=0,\ldots,r\\ \delta=0,\ldots,d-1}} \subset \mathbb{A}^{N'}_S\,.$$

Then V is the pull-back of W by the map ρ. So V is isomorphic to $\mathbb{A}^{N+1}_W$, and $T_n, Z_1, \ldots, Z_N$ can be viewed as coordinate functions of $\mathbb{A}^{N+1}_W$. Due to the decomposition $(*)$, for each ν we obtain a representation

$$\hat{y}_\nu := \sigma^* Y_\nu = y'_\nu + \hat{z}_\nu \cdot \hat{p}\,,$$

where

$$y'_\nu = \sum_{\delta=0}^{d-1} y'_{\nu\delta} T_n^\delta$$

with $y'_{\nu\delta} \in \hat{R}[[T_1, \ldots, T_{n-1}]]$ and $\hat{z}_\nu \in \hat{R}[[T_1, \ldots, T_n]]$. Then define an S'-morphism

$$\varphi' : \hat{S}' \longrightarrow \mathbb{A}^{N''}_{S'}$$

by setting

$$(\varphi')^* Y_{\nu\delta} = y'_{\nu\delta} \quad \text{for} \quad \nu = 1, \ldots, N\,, \quad \delta = 0, \ldots, d-1\,,$$

$$(\varphi')^* A_\delta = a'_\delta \quad \text{for} \quad \delta = 0, \ldots, d-1\,.$$

Furthermore, consider the S-morphism

$$\varphi : \hat{S} \longrightarrow \mathbb{A}^{N'}_S$$

defined by

$$\varphi^* Y_{\nu\delta} = y'_{\nu\delta}\,; \qquad \nu = 1, \ldots, N\,, \qquad \delta = 0, \ldots, d-1\,,$$

$$\varphi^* A_\delta = a'_\delta\,; \qquad \delta = 0, \ldots, d-1\,,$$

$$\varphi^* Z_\nu = \hat{z}_\nu\,; \qquad \nu = 1, \ldots, N\,.$$

Then we have $\sigma = \tau \circ \varphi$, $\varphi^* p = \hat{p}$, and $\varphi^* f_{i\delta} = (\varphi')^* f_{i\delta}$ for all i and δ. In order to see that φ' factors through W, one considers Taylor expansions of

$$\sigma^* f_i = f_i(\hat{y}) = f_i(y' + \hat{z} \cdot \hat{p})\,,$$

thereby obtaining

$$\sigma^* f_i \equiv f_i(y') \mod \hat{p} \cdot \hat{R}[[T_1, \ldots, T_n]]\,, \qquad i = 0, \ldots, r\,.$$

Since $\sigma^* f_i = 0$ for $i = 1, \ldots, r$, it follows

$$f_i(y') \equiv 0 \mod \hat{p} \cdot \hat{R}[[T_1, \ldots, T_n]]$$

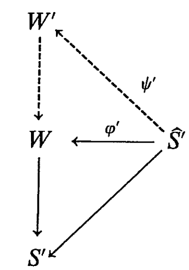
for $i > 0$. Moreover, since $\hat{p}$ and $\sigma^* f_0$ differ by a unit in $\hat{R}[[T_1, \ldots, T_n]]$, we have

$$f_i(y') \equiv 0 \mod \hat{p} \cdot \hat{R}[[T_1, \ldots, T_n]]$$

for $i = 0$, too. On the other hand, using $(**)$ we get relations

$$\sigma^* f_i = \varphi^* \tau^* f_i = \varphi^* \left(\sum_{\delta=0}^{d-1} f_{i\delta} \cdot T_n^\delta + q_i \cdot p \right) = \sum_{\delta=0}^{d-1} (\varphi')^* f_{i\delta} \cdot T_n^\delta + \hat{q}_i \cdot \hat{p}$$

for $i = 0, \ldots, r$, where $\hat{q}_i \in \hat{R}[[T_1, \ldots, T_n]]$. Then, since $\sigma^* f_i \equiv 0 \mod \hat{p}$, the direct sum decomposition $(*)$ implies $(\varphi')^* f_{i\delta} = 0$ for all i and all δ. So φ' factors through W, and the induction hypothesis can be applied. Thus there exists a factorization of φ' into S'-morphisms



where W' is a smooth S'-scheme. By base change we obtain from W' the smooth S-scheme $W'' = W' \times_{S'} S$ and, hence, the smooth S-scheme

$$V' = \mathbb{A}^N_{W''} = \mathbb{A}^{N+1}_{W'}\,,$$

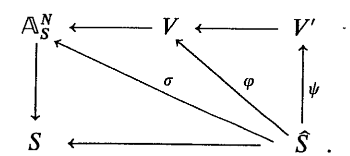
where $Z_1, \ldots, Z_N$ give rise to a set of coordinates of $\mathbb{A}^N_{W''}$. Furthermore, we can define an S-morphism

$$\psi : \hat{S} \longrightarrow V'$$

(over $\hat{S}' \longrightarrow W'$) by setting

$$\psi^* Z_\nu = \hat{z}_\nu \qquad \text{for} \qquad \nu = 1, \ldots, N\,.$$

Then there is a commutative diagram of S-morphisms



The map $V \longrightarrow \mathbb{A}^N_S$ is induced by τ; let us call it τ, too. It remains to show that $\tau^* f_0$

divides $\tau^* f_i$, $i = 1, \dots, r$, at least locally at $\varphi(\hat{s})$. Due to the definition of V, it suffices to know that the factor q_0 defined by the relation (**) is invertible at $\varphi(\hat{s})$. But this is clear. Namely, the equation

$$\hat{u} \cdot \hat{p} = \sigma^* f_0 = \varphi^* \tau^* f_0 = \varphi^*(q_0) \cdot \hat{p}$$

shows that $\varphi^*(q_0) = \hat{u}$ is a unit in $\hat{R}[[T_1, \dots, T_n]]$. □

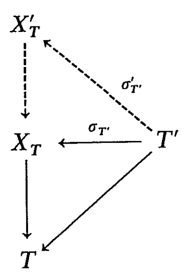
We will apply the preceding lemma in the situation where f_0 is the square of a maximal minor of the Jacobi matrix

$$J = \left(\frac{\partial f_i}{\partial Y_\nu}\right)_{\substack{i=1,\dots,r \\ \nu=1,\dots,N}} .$$

Before this can be done, however, we have to justify the following reduction step.

Lemma 14. *It suffices to prove Theorem 12 in the case where X, at the point $\sigma(\hat{\eta})$, is smooth over S of relative dimension $N - m$ and where X, as a closed subscheme of $\mathbb{A}_S^N$, is defined by m global sections $f_1, \dots, f_m$ of $\mathcal{O}_{\mathbb{A}_S^N}$.*

Proof. Replacing X by the schematic image of σ, one may assume σ to be schematically dense in X. Since the fields of fractions of $\hat{R}[[T_1, \dots, T_n]]$ and of $R[T_1, \dots, T_n]$ are separable over each other (cf. [EGA IV_2], 7.8.3), the generic fibre of X is geometrically reduced and, hence, generically smooth over S. Denote by Λ the local ring of S at η and by Λ' the local ring of $\hat{S}$ at $\hat{\eta}$. The extension $\Lambda \longrightarrow \Lambda'$ is regular, and π is a uniformizing element of Λ and of Λ'. Set $T = \operatorname{Spec} \Lambda$ and $T' = \operatorname{Spec} \Lambda'$. Then σ induces a T'-valued point $\sigma_{T'}$ of $X_T = X \times_S T$. Since the generic point t' of T' is mapped to the generic point of X_T and since the generic fibre of X_T is generically smooth over T, Proposition 4 shows the existence of a commutative diagram



where X'_T is smooth over T and where $X'_T \longrightarrow X_T$ is constructed as a finite sequence of dilatations with centers in the special fibres. Using a limit argument, we may assume that $X'_T \longrightarrow X_T$ is induced by the base change $T \longrightarrow S$ from an S-morphism $X' \longrightarrow X$ which is constructed in the same way; namely, we can extend the centers of the blowing-ups to closed subschemes which do not meet generic fibres. Due to the construction of X', Proposition 3.2/1 implies that σ lifts (uniquely) to an R-morphism $\sigma' : \hat{S} \longrightarrow X'$ which induces $\sigma'_{T'} : T' \longrightarrow X'_T$. Obviously, σ' is an S-morphism. Thus we may assume that X is smooth over S at $\sigma(\hat{\eta})$, say of relative dimension $N - m$. Due to 2.2/7, we may assume that $f_1, \dots, f_m$ define X as a subscheme of $\mathbb{A}_S^N$ at $\sigma(\hat{\eta})$. Now consider the closed subscheme $V \subset \mathbb{A}_S^N$ given by $f_1, \dots, f_m$. Then $X \subset V$, and both coincide in a neighborhood of $\sigma(\hat{\eta})$. In particular, the morphism $\hat{S} \longrightarrow X$ factors through V. Since smooth S-schemes are locally integral, we may replace X by V. Namely, if $V' \longrightarrow V$ is an S-morphism from a smooth S-scheme V' to V such that $\hat{S} \longrightarrow V$ factors through $V' \longrightarrow V$, we can assume that V' is integral. Then there is an open dense subscheme $V'' \subset V'$ which is mapped into X, and it follows that the map $V' \longrightarrow V$ must factor through X because V' is integral and because X is closed in V. □

Thus we may assume that X, as a closed subscheme of $\mathbb{A}_S^N$, is defined by m global sections, say

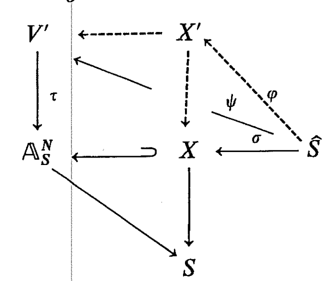
$$X = V(f_1, \dots, f_m) \subset \mathbb{A}_S^N ,$$

and that the determinant

$$\Delta = \det \left(\frac{\partial f_i}{\partial Y_j}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,m}}$$

does not vanish at $\sigma(\hat{\eta})$; cf. 2.2/7. We will now finish the proof of Theorem 12 by establishing a third lemma; see Bourbaki [2], Chap. III, §4, n°5, for a similar statement.

Lemma 15. *Consider a situation as in Lemma 13. Assume that X is as above and that $f_0 = \Delta^2$. Then there exists a diagram*



where $X' \longrightarrow V'$ is étale; in particular, X' is smooth over S. Except for the square in the upper left corner, the diagram is commutative.

Proof. In the following, we write f for the column vector $(f_1, \dots, f_m)^t$; the index t indicates the transpose. On V' we have a relation

$$(*) \qquad \tau^* f = \tau^* \Delta^2 \cdot a'$$

with a column vector $a' = (a'_1, \dots, a'_m)^t$ of global sections of $\mathcal{O}_{V'}$. Denote by $\Delta_1 = \Delta$, $\Delta_2, \dots, \Delta_l$ the $(m \times m)$-minors of

$$J = \left(\frac{\partial f_i}{\partial Y_\nu}\right)_{\substack{i=1,\dots,m \\ \nu=1,\dots,N}} .$$

Due to Cramer's rule, there exist $(N \times m)$-matrices M_λ, $\lambda = 1, \ldots, l$, with global sections of $\mathcal{O}_{\mathbb{A}_S^N}$ as entries such that

$$(**) \qquad J \cdot M_\lambda = \Delta_\lambda \cdot I_m .$$

I_m is the $(m \times m)$-unit matrix. We will construct X' as a subscheme of $\mathbb{A}_{V'}^{l\cdot N}$. So denote by $Z_{\lambda\nu}$, $\lambda = 1, \ldots, l$, $\nu = 1, \ldots, N$, the coordinate functions of $\mathbb{A}_S^{l\cdot N}$. Let $Z_{(\lambda)}$ be the column vector $(Z_{\lambda 1}, \ldots, Z_{\lambda N})^t$, $\lambda = 1, \ldots, l$. Now consider the S-morphism

$$\rho : \mathbb{A}_{V'}^{l\cdot N} \longrightarrow \mathbb{A}_S^N$$

given by

$$\rho^* Y = \sum_{\lambda=1}^{l} \tau^*\Delta_\lambda \cdot Z_{(\lambda)} + \tau^* Y$$

where Y is the column vector $(Y_1, \ldots, Y_N)^t$. By Taylor expansion we get an equation

$$\rho^* f = \tau^* f + \sum_{\lambda=1}^{l} \tau^*\Delta_\lambda \cdot \tau^* J \cdot Z_{(\lambda)} + \sum_{\lambda,\mu=1}^{l} \tau^*\Delta_\lambda \cdot \tau^*\Delta_\mu \cdot q_{(\lambda,\mu)}$$

with certain column vectors $q_{(\lambda,\mu)} = (q_{\lambda\mu 1}, \ldots, q_{\lambda\mu m})^t$. Each $q_{\lambda\mu i}$ is a polynomial in the variables $Z_{\lambda\mu}$ with global sections of $\mathcal{O}_{V'}$ as coefficients, and each monomial of $q_{\lambda\mu i}$ has degree ≥ 2. Using (*) and (**), we can write

$$\tau^* f = \tau^*\Delta \cdot (\tau^*\Delta \cdot I_m) \cdot a' = \tau^*\Delta \cdot \tau^* J \cdot a'_{(1)}$$

with

$$a'_{(1)} = \tau^* M_1 \cdot a' .$$

Furthermore, we have

$$\sum_{\mu=1}^{l} \tau^*\Delta_\mu \cdot q_{(\lambda,\mu)} = \tau^* J \cdot q_{(\lambda)}$$

with

$$q_{(\lambda)} = \sum_{\mu=1}^{l} \tau^* M_\mu \cdot q_{(\lambda,\mu)} .$$

Setting $a'_{(\lambda)} = 0$ for $\lambda = 2, \ldots, l$, we see

$$\rho^* f = \sum_{\lambda=1}^{l} \tau^*\Delta_\lambda \cdot \tau^* J \cdot [a'_{(\lambda)} + Z_{(\lambda)} + q_{(\lambda)}] .$$

Then let X' be the closed subscheme of $\mathbb{A}_{V'}^{l\cdot N}$ which is defined by the global sections

$$a'_{(\lambda)} + Z_{(\lambda)} + q_{(\lambda)}, \quad \lambda = 1, \ldots, l .$$

Due to 2.2/10, the projection $X' \longrightarrow V'$ is étale along the zero section of $\mathbb{A}_{V'}^{l\cdot N} \longrightarrow V'$. Obviously, the morphism $X' \longrightarrow \mathbb{A}_S^N$ induced by ρ factors through X. Since $\sigma^*\Delta$ is not a zero divisor, the relation

$$0 = \sigma^* f = \sigma^*\Delta^2 \cdot \psi^* a'$$

implies $\psi^* a' = 0$ and, hence, $\psi^* a'_{(\lambda)} = 0$ for $\lambda = 1, \ldots, l$. Thus, the zero section of

$\mathbb{A}_{V'}^{l\cdot N}$ induces a lifting φ of ψ. Replacing X' by the étale locus of $X' \longrightarrow V'$, the assertion of the lemma is clear. □

Thereby we have finished the proof of Theorem 12. The statement of Theorem 12 was announced by M. Artin in [8]. Its proof, given in terms of commutative algebra, has been published recently by M. Artin and C. Rotthaus; cf. Artin and Rotthaus [1]. The method of proof is similar to the one used in Artin [4], where it is shown that the henselization of $R[T_1, \ldots, T_n]$ at $(\pi, T_1, \ldots, T_n)$ satisfies the approximation property. In fact, the latter result can be obtained as a consequence of Theorem 12.

Theorem 16. (M. Artin). *Let R be a field or an excellent discrete valuation ring, and let A be a henselization of a local R-algebra A_0 which is essentially of finite type over R. Let $\mathfrak{m}$ be a proper ideal of A, and let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. Then, given a system of polynomial equations*

$$f(Y) = 0$$

where $Y = (Y_1, \ldots, Y_N)$ are variables and $f = (f_1, \ldots, f_r)$ are polynomials in Y with coefficients in A, given a solution $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_N) \in \hat{A}^N$ and an integer c, there exists a solution $y = (y_1, \ldots, y_N) \in A^N$ such that

$$y_\nu \equiv \hat{y}_\nu \mod \mathfrak{m}^c \cdot \hat{A}$$

for $\nu = 1, \ldots, N$.

Proof. Following M. Artin, we will reduce the assertion to the special case where A_0 is the localization of $R[T_1, \ldots, T_n]$ at the point $(\pi, T_1, \ldots, T_n)$ of $\operatorname{Spec} R[T_1, \ldots, T_n]$, where the integer c is 1, and where the ideal $\mathfrak{m}$ is the maximal ideal of A. In this case, the assertion is an easy consequence of Theorem 12. So let us start with the reductions.

One may assume that $\mathfrak{m}$ is the maximal ideal of A and that the integer c is 1. Namely, there exist elements $a_\nu \in A$ such that

$$\hat{y}_\nu \equiv a_\nu \mod \mathfrak{m}^c \cdot \hat{A}$$

for $\nu = 1, \ldots, N$. Let $m_1, \ldots, m_t$ be a system of generators of $\mathfrak{m}^c$. Then there exist elements $\hat{y}_{\nu j}$ of $\hat{A}$ such that

$$\hat{y}_\nu - a_\nu - \sum_{j=1}^{t} \hat{y}_{\nu j} m_j = 0 .$$

Let

$$g_\nu = Y_\nu - a_\nu - \sum_{j=1}^{t} Y_{\nu j} m_j \in A[Y_{\nu'}, Y_{\nu' j'}]_{\substack{\nu'=1,\ldots,N \\ j'=1,\ldots,t}}$$

and consider the system of polynomial equations given by $f_1, \ldots, f_r, g_1, \ldots, g_N$ in the variables $(Y_{\nu'})$ and $(Y_{\nu' j'})$. This system has the solution $((\hat{y}_\nu), (\hat{y}_{\nu j}))$ over $\hat{A}$, and any solution of this system lying in A gives rise to a solution of the required type of the system we started with.

We may assume that R is a discrete valuation ring and that the maximal ideal $\mathfrak{m}$ of A lies over the closed point of $\operatorname{Spec} R$. Namely, if R is a discrete valuation ring and if $\mathfrak{m}$ lies over the generic point of R, we can replace R by its field of fractions. If R is a field, we can replace it by the power series ring $R[[T]]$, and view A as an $R[[T]]$-algebra by sending T to zero. Since $R[[T]]$ is excellent, this reduction is permissible.

We may assume that the residue field $k' = A/\mathfrak{m}$ is finite over $k = R/\pi R$. Since A_0 is essentially of finite type over R, the field k' is finitely generated over k. Let d be its degree of transcendence. Then there exist elements $z_1, \ldots, z_d \in A_0$ such that k' is finite over $k(\bar{z}_1, \ldots, \bar{z}_d)$, where $\bar{z}_\delta$ denotes the residue class of $z_\delta \bmod \mathfrak{m}$. Let R' be the localization of $R[Z_1, \ldots, Z_d]$ at the prime ideal (π). The R-morphism

$$R[Z_1, \ldots, Z_d] \longrightarrow A_0$$

sending Z_δ to z_δ for $\delta = 1, \ldots, d$ factors through R', since $\bar{z}_1, \ldots, \bar{z}_d$ are transcendental over k. Furthermore, R' is an excellent discrete valuation ring, see [EGA IV_2], 7.8.3, and A_0 is essentially of finite type over R'.

We may assume that A is a finite S-algebra where S is a henselization of the localization S_0 of a polynomial ring $R[T_1, \ldots, T_n]$ at $(\pi, T_1, \ldots, T_n)$. Namely, let $t_1, \ldots, t_n$ be a system of generators of the maximal ideal of A_0. The R-morphism

$$\varphi : R[T_1, \ldots, T_n] \longrightarrow A_0$$

sending T_i to t_i for $i = 1, \ldots, n$ induces a morphism $S_0 \longrightarrow A_0$. Since A_0 is essentially of finite type and since the residue field $A/\mathfrak{m}$ is finite over k, it is easily seen that $\operatorname{Spec} A_0 \longrightarrow \operatorname{Spec} S_0$ is quasi-finite at the maximal ideal of A_0. Then the extension $S \longrightarrow A_0 \otimes_{S_0} S$ is finite (cf. 2.3/4); so $A_0 \otimes_{S_0} S$ is a direct sum of local henselian rings. Since A is among them, the extension $S \longrightarrow A$ is finite.

It suffices to prove the theorem for a henselization S of the localization S_0 of a polynomial ring $R[T_1, \ldots, T_n]$ at $(\pi, T_1, \ldots, T_n)$. Since we may assume that A is finite over S, the $\mathfrak{m}$-adic completion $\hat{A}$ of A is isomorphic to $A \otimes_S \hat{S}$. Write A as a quotient of a polynomial ring over S, say

$$0 \longrightarrow \mathfrak{a} \longrightarrow S[X_1, \ldots, X_m] \longrightarrow A \longrightarrow 0 .$$

Then let $a_1, \ldots, a_l$ be a finite system of generators of $\mathfrak{a}$. Lifting the system $f(Y)$ over A to a system $g(Y)$ over $S[X]$ and lifting the given solution $\hat{y}$ of $f(Y)$ to $\hat{y}' = (\hat{y}'_1, \ldots, \hat{y}'_N)$ with $\hat{y}'_1, \ldots, \hat{y}'_N \in S[X] \otimes_S \hat{S}$, we get a relation

$$(*) \qquad g(\hat{y}') = \sum_{\lambda=1}^{l} a_\lambda \hat{z}_{(\lambda)}$$

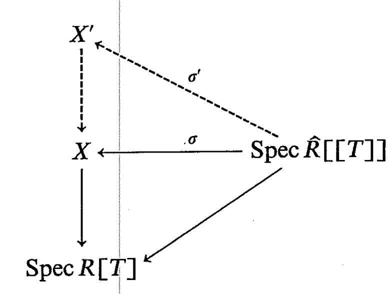
where $\hat{z}_{(\lambda)} = (\hat{z}_{\lambda 1}, \ldots, \hat{z}_{\lambda r})^t$ is a column vector of elements of $S[X] \otimes_S \hat{S}$. Then consider the system of equations

$$(**) \qquad g(Y) - \sum_{\lambda=1}^{l} a_\lambda Z_{(\lambda)} = 0$$

over $S[X]$, where $Y = (Y_1, \ldots, Y_N)$ and $Z = (Z_{\lambda i})$, for $\lambda = 1, \ldots, l$, $i = 1, \ldots, r$, are variables. Due to $(*)$, the system $(**)$ has a solution in $\hat{S}[X]$. Looking at the coefficients of the polynomials in $X_1, \ldots, X_m$ appearing in $(*)$, we can rewrite $(**)$

as a finite system of polynomial equations over S which has a solution over $\hat{S}$. Clearly, a solution over S of this system induces a solution over A of the system we started with.

Now let us show how, in this situation, the proof of the theorem follows from Theorem 12. The polynomials $f_1, \ldots, f_r \in S[Y_1, \ldots, Y_N]$ define a closed subscheme X of $\mathbb{A}^N_S$. Since only finitely many coefficients occur in $f_1, \ldots, f_r$, the scheme X is actually defined over an $R[T_1, \ldots, T_n]$-algebra of finite type. So we may view X as an $R[T_1, \ldots, T_n]$-scheme of finite type. The solution $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_N) \in \hat{S}^N$ gives rise to an $\hat{R}[[T_1, \ldots, T_n]]$-valued point σ of X. Then Theorem 12 yields a commutative diagram



where X' is smooth over $R[T]$. The closed point $\hat{s}$ of $\operatorname{Spec} \hat{R}[[T]]$ induces a k-rational point $x' = \sigma'(\hat{s})$ of X'. Due to 2.3/5, the k-valued point x' lifts to an S-valued point of X' and, hence, to an S-valued point x of X. Then, x gives rise to a solution y over S of $f(Y) = 0$, the one we are looking for. □

Let us conclude with some remarks on the history of the approximation property. Corollary 9 was first established in Greenberg [2], where the author actually proves a much stronger result, the so-called strong approximation property for discrete valuation rings. Theorem 16 is due to M. Artin, cf. Artin [4]; he even shows the strong approximation property for polynomial rings $k[T_1, \ldots, T_n]$, where k is a field. By methods of model theory, it can also be seen from Artin's result (Theorem 16) that all rings $R[T_1, \ldots, T_n]$ satisfy that property whenever R is an excellent discrete valuation ring; cf. Becker, Denef, Lipshitz, van den Dries [1]. Artin's conjecture that the approximation property holds for every excellent ring A was recently verified by C. Rotthaus for the case where A contains the rational numbers; see Rotthaus [1].

The importance of the approximation theorem is based on the applications to moduli problems; there it is a powerful tool to show that certain functors are representable by algebraic spaces; cf. Artin [5] and [6]. We will come back to this point in Section 8.3.

Chapter 4. Construction of Birational Group Laws

In the previous chapter, we discussed the smoothening process and, as an application, proved the existence of weak Néron models. The next step towards the construction of Néron models requires the use of group arguments.

For the convenience of the reader, we start with a general section on group schemes where we explain the functorial point of view. Then we discuss the existence of invariant differential forms and their properties. They are used in order to define the so-called minimal components of weak Néron models, which are unique up to R-birational isomorphism. The actual construction of Néron models is continued in Section 4.3. Starting with a smooth K-group scheme X_K of finite type and a weak Néron model $(X_i)_{i\in I}$, we select the minimal components from the X_i. After a possible shrinking, we glue them along the generic fibre to produce a smooth and separated R-model X of X_K and we show that the group structure on X_K extends to an R-birational group law on X. Admitting the fact (to be obtained in Chapters 5 and 6) that X with its R-birational group law can uniquely be enlarged to an R-group scheme $\bar{X}$, we show in Section 4.4 that $\bar{X}$ will be a Néron model of X_K. This is done by employing an argument of A. Weil, saying that a rational map from a smooth scheme to a separated group scheme is defined everywhere if it is defined in codimension 1.

4.1 Group Schemes

Let C be a category; for example, let C be the category (Sch/S) of schemes over a fixed scheme S. Each object $X \in C$ gives rise to its functor of points

$$h_X : C \longrightarrow (\text{Sets})$$

which associates to any $T \in C$ the set

$$h_X(T) := X(T) := \operatorname{Hom}(T, X)$$

of T-valued points of X. Each morphism $X \longrightarrow X'$ in C induces a morphism $h_X \longrightarrow h_{X'}$ of functors by the composition of morphisms in C. In this way one gets a covariant functor

$$h : C \longrightarrow \operatorname{Hom}(C^0, (\text{Sets}))$$

of C to the category of covariant functors from C^0 (the dual of C) to the category of sets; the category $\operatorname{Hom}(C^0, (\text{Sets}))$ is denoted by $\hat{C}$; it is called the category of contravariant functors from C to (Sets).

Proposition 1. *The functor $h : C \longrightarrow \hat{C}$ is fully faithful; i.e., for any two objects X, $X' \in C$, the canonical map*

$$\operatorname{Hom}_C(X, X') \longrightarrow \operatorname{Hom}_{\hat{C}}(h_X, h_{X'})$$

is bijective. More generally, for all objects $X \in C$ and $\mathscr{F} \in \hat{C}$, there is a canonical bijection

$$\mathscr{F}(X) \xrightarrow{\sim} \operatorname{Hom}_{\hat{C}}(h_X, \mathscr{F})$$

mapping $u \in \mathscr{F}(X)$ to the morphism $h_X \longrightarrow \mathscr{F}$ which to a T-valued point $g \in h_X(T)$, where T is an object of C, associates the element $\mathscr{F}(g)(u) \in \mathscr{F}(T)$. The bijection coincides with the above one if $\mathscr{F} = h_{X'}$ and is functorial in X and $\mathscr{F}$ in the sense that $\mathscr{F} \longmapsto \operatorname{Hom}_{\hat{C}}(h(\cdot), \mathscr{F})$ defines an isomorphism $\hat{C} \longrightarrow \hat{C}$.

Proof. Consider an element $u \in \mathscr{F}(X)$. We have only to show that there is a unique functorial morphism $h_X \longrightarrow \mathscr{F}$ mapping the so-called *universal point* $\mathrm{id}_X \in h_X(X)$ onto $u \in \mathscr{F}(X)$ and that it is as stated. Then all assertions of the proposition are immediately clear. So let us show how to justify this claim. For any object $T \in C$, each T-valued point $g : T \longrightarrow X$ factors through the universal point of X. Thus, if $h_X \longrightarrow \mathscr{F}$ exists as claimed, the image of g under $h_X(T) \longrightarrow \mathscr{F}(T)$ must coincide with the image of u under $\mathscr{F}(g) : \mathscr{F}(X) \longrightarrow \mathscr{F}(T)$. Conversely, taking the latter as a definition, we see that $h_X \longrightarrow \mathscr{F}$ can be constructed as required. □

In particular, if a functor $\mathscr{F} \in \operatorname{Hom}(C^0, (\text{Sets}))$ is isomorphic to a functor h_X, then X is uniquely determined by $\mathscr{F}$ up to an isomorphism in the category C. In this case, the functor $\mathscr{F}$ is said to be *representable*. Thus Proposition 1 says that the functor h defines an equivalence between the category C and the full subcategory of $\operatorname{Hom}(C^0, (\text{Sets}))$ consisting of all representable functors.

In order to define group objects in the category C, it is necessary to introduce the notion of a law of composition on an object X of C. By the latter we mean a functorial morphism

$$\gamma : h_X \times h_X \longrightarrow h_X .$$

Thus, a law of composition on X consists of a collection of maps

$$\gamma_T : h_X(T) \times h_X(T) \longrightarrow h_X(T)$$

(laws of composition on the sets of T-valued points of X) where T varies over the objects in C. The functoriality of γ means that all maps γ_T are compatible with canonical maps between points of X, i.e., for any morphism $u : T' \longrightarrow T$ in C, the diagram

$$\begin{array}{ccc} h_X(T) \times h_X(T) & \xrightarrow{\gamma_T} & h_X(T) \\ \downarrow{\scriptstyle h_X(u) \times h_X(u)} & & \downarrow{\scriptstyle h_X(u)} \\ h_X(T') \times h_X(T') & \xrightarrow{\gamma_{T'}} & h_X(T') \end{array}$$

is commutative. If the law of composition has the property that $h_X(T)$ is a group under γ_T for all T, then γ defines on h_X the structure of a group functor, i.e., of a contravariant functor from C to the category of groups. In this case, γ is called a group law on X.

Definition 2. *A group object in C is an object X together with a law of composition $\gamma : h_X \times h_X \longrightarrow h_X$ which is a group law.*

It follows that a group object in C is equivalent to a group functor which, as a functor to the category of sets, is representable.

When dealing with group objects, it is convenient to know that the category in question contains direct products and a final object, say S. The latter means that, for each object T of C, there is a uniqe morphism $T \longrightarrow S$. So, in the following, assume that C is of this type, and consider a group object X of C with group law γ. Then, since the product $X \times X$ exists in C and since the functor $h : C \longrightarrow \mathrm{Hom}(C^0, (\mathrm{Sets}))$ commutes with direct products, the law of composition $\gamma : h_X \times h_X \longrightarrow h_X$ corresponds to a morphism $m : X \times X \longrightarrow X$, as is seen by using Proposition 1. Furthermore, the injection of the unit element into each group $h_X(T)$ yields a natural transformation from h_S to h_X, hence it corresponds to a morphism

$$\varepsilon : S \longrightarrow X\,,$$

called the *unit section of* X, which is a section of the unique morphism $X \longrightarrow S$. Finally, the formation of the inverse in each $h_X(T)$ defines a natural transformation $h_X \longrightarrow h_X$ and hence a morphism

$$\iota : X \longrightarrow X\,,$$

called the *inverse map on* X. The group axioms which are satisfied by the groups $h_X(T)$, and hence by the functor h_X, correspond to certain properties of the maps m, ε and ι. Namely, the following diagrams are commutative:

(a) *associativity*

$$\begin{array}{ccc} X\times X\times X & \xrightarrow{m\times \mathrm{id}_X} & X\times X \\ \downarrow{\scriptstyle \mathrm{id}_X\times m} & & \downarrow{\scriptstyle m} \\ X\times X & \xrightarrow{m} & X \end{array}$$

(b) *existence of a left-identity*

$$\begin{array}{ccccc} X & \xrightarrow{(p,\mathrm{id}_X)} & S\times X & \xrightarrow{\varepsilon\times\mathrm{id}_X} & X\times X \\ & \searrow^{\mathrm{id}_X} & & & \downarrow{\scriptstyle m} \\ & & & & X \end{array}\quad ,$$

where $p : X \longrightarrow S$ is the morphism from X to the final object S.

(c) *existence of a left-inverse*

$$\begin{array}{ccc} X & \xrightarrow{(\iota,\mathrm{id}_X)} & X\times X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle m} \\ S & \xrightarrow{\varepsilon} & X \end{array}$$

(d) *commutativity* (only if all groups $h_X(T)$ are commutative)

$$\begin{array}{ccc} X\times X & \xrightarrow{\tau} & X\times X \\ & \searrow^{m} & \downarrow{\scriptstyle m} \\ & & X \end{array}$$

where τ commutes the factors.

Note that a left-identity is also a right-identity and that a left-inverse is also a right-inverse. It is clear that once we have an object X and morphisms m, ε, and ι with the above properties, we can construct a group object in the given category from these data, and furthermore, that group objects in C and data $(X, m, \varepsilon, \iota)$ correspond bijectively to each other.

Proposition 3. *The group objects in a category C correspond one-to-one to data $(X, m, \varepsilon, \iota)$ where X is an object of C and where*

$$m : X \times X \longrightarrow X\,, \qquad \varepsilon : S \longrightarrow X\,, \qquad \iota : X \longrightarrow X$$

are morphisms in C such that the diagrams (a), (b), (c) *above are commutative. Furthermore, a group object in C is commutative if and only if, in addition, the corresponding diagram* (d) *is commutative.*

In the following we restrict ourselves to the category (Sch/S) of S-schemes where S is a fixed base scheme. Then the direct product in (Sch/S) is given by the fibred product of schemes over S, and the S-scheme S is a final object in (Sch/S).

Definition 4. *An S-group scheme is a group object in the category of S-schemes* (Sch/S).

Due to Proposition 3, an S-group scheme G can be viewed as an S-scheme X together with appropriate morphisms m, ε, and ι. When no confusion about the group structure is possible, we will not mention these morphisms explicitly. In particular, in our notation we will make no difference between the group object G and the associated representing scheme X. Also we want to point out that there exist group functors on (Sch/S) which are not representable and thus do not correspond to S-group schemes. For example, let X be a smooth S-scheme and, for any S-scheme T, let $\mathscr{R}_{X/S}(T)$ be the set of all T-birational automorphisms of $X_T = X \times_S T$. Then, in general, the group functor $\mathscr{R}_{X/S}$ is not representable by a scheme, even if X is the projective line over a field.

It follows immediately from Definition 4 that the technique of base change can be applied to group schemes. Thus, for any base change $S' \longrightarrow S$, one obtains from an S-group scheme G an S'-group scheme $G_{S'} := G \times_S S'$. If $S = \operatorname{Spec} R$ for some ring R, we talk also about R-group schemes instead of S-group schemes. Furthermore, if $K = R$ is a field, an algebraic K-group is meant to be a K-group scheme of finite type (not necessarily smooth).

There are many notions for ordinary groups which have a natural analogue for group functors and thus for group schemes. For example, a homomorphism of group functors $\mathscr{G}' \longrightarrow \mathscr{G}$ is a natural transformation between $\mathscr{G}'$ and $\mathscr{G}$ (viewed as functors from (Sch/S) to (Groups)). If $\mathscr{G}'$ and $\mathscr{G}$ are represented by S-schemes G' and G, respectively, such a homomorphism corresponds to a morphism $G' \longrightarrow G$ which is compatible with the group law on G' and on G. We also have the notions of subgroup, kernel of a homorphism, monomorphism, etc., for group functors. However, when dealing with S-group schemes G, we reserve the notion of *subgroup schemes* to such representable subgroup functors which are represented by *subschemes* of G (the latter is not automatic). A subscheme Y of G defines a subgroup scheme of G if and only if the following conditions are satisfied:

(i) the unit-section $\varepsilon : S \longrightarrow G$ factors through Y,
(ii) the group law $m : G \times_S G \longrightarrow G$ restricts to a morphism $Y \times_S Y$, and
(iii) the inverse map $\iota : G \longrightarrow G$ restricts to a morphism $Y \longrightarrow Y$.

Let us look at some examples of S-group schemes. We start with the classical groups $\mathbb{G}_a$ (the additive group), $\mathbb{G}_m$ (the multiplicative group), GL_n (the general linear group), and PGL_n (the projective general linear group). In terms of group functors, these groups are defined as follows. For any S-scheme T set

$$\mathbb{G}_a(T) := \text{the additive group } \mathcal{O}_T(T)$$

$$\mathbb{G}_m(T) := \text{the group of units in } \mathcal{O}_T(T)$$

$$\mathrm{GL}_n(T) := \text{the group of } \mathcal{O}_T(T)\text{-linear automorphisms of } (\mathcal{O}_T(T))^n$$

$$\mathrm{PGL}_n(T) := \mathrm{Aut}_T\,(\mathbb{P}(\mathcal{O}_T^n))\ .$$

All these group functors are representable by affine schemes over $\mathbb{Z}$. Working over $S := \operatorname{Spec} \mathbb{Z}$, the additive group is represented by the scheme

$$X := \operatorname{Spec} \mathbb{Z}[\zeta]$$

(ζ is an indeterminate), where the group law $m : X \times X \longrightarrow X$ corresponds to the algebra homomorphism

$$\mathbb{Z}[\zeta] \longrightarrow \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta]\ , \qquad \zeta \longmapsto \zeta \otimes 1 + 1 \otimes \zeta\ .$$

Similarly, for $\mathbb{G}_m$, the representing object is $\operatorname{Spec} \mathbb{Z}[\zeta, \zeta^{-1}]$ with the group law given by $\zeta \longmapsto \zeta \otimes \zeta$. In the case of GL_n we consider a set ζ_{ij} of n^2 indeterminates. Then

$$X := \operatorname{Spec} \mathbb{Z}[\zeta_{ij}, \det(\zeta_{ij})^{-1}]$$

is a representing object; the group law is defined by the multiplication of matrices. Finally, PGL_n is represented by the open subscheme

$$X \subset \operatorname{Proj} \mathbb{Z}[\zeta_{ij}]$$

where $\det \zeta_{ij}$ does not vanish. For a general base S, the representing objects are obtained from the ones over $\operatorname{Spec} \mathbb{Z}$ by base extension. It is clear that the above procedure works as well for further classical groups ($\mathrm{SL}_n, \mathrm{Sp}_n, \mathrm{O}_n, \ldots$). Also it should be mentioned that one can define GL_V,PGL_V, ... for any vector bundle V over S. Just replace $\mathcal{O}_T^n$ in the above definitions by the pull-back of V with respect to $T \longrightarrow S$.

All the above group schemes are *affine*, i.e., the representing schemes are affine over the base S. Another important class of group schemes consists of the so-called *abelian schemes* over S. Thereby we mean smooth proper S-group schemes with connected fibres (the latter are abelian varieties in the usual sense). They are always commutative. As examples one may consider elliptic curves over fields which have a rational point or, more generally, Jacobians of smooth complete curves.

4.2 Invariant Differential Forms

Throughout this section, let G be a group scheme over a fixed scheme S. First we want to introduce the notion of translations on G. In order to do this, consider a T-valued point

$$g : T \longrightarrow G$$

of G; i.e., an S-morphism from an S-scheme T to G. Then g gives rise to the T-valued point

$$g_T := (g, \mathrm{id}_T) : T \longrightarrow G_T := G \times_S T$$

of the T-scheme $G_T := G \times_S T$. If $p_1 : G_T \longrightarrow G$ denotes the first projection, we have $g = p_1 \circ g_T$. In the special case where $T := G$ and $g := \mathrm{id}_G$ is the universal point of G, the morphism g_T equals the diagonal morphism Δ of G. For any other T-valued point g of G, the morphism g_T is obtained from Δ by performing the base change $g : T \longrightarrow G$.

As usual, let $m : G \times_S G \longrightarrow G$ be the group law of G and write m_T for its extension when a base change $T \longrightarrow S$ is applied to G. Then, for any T-valued point g of G, we define the left translation by

$$\tau_g : G_T \xrightarrow{\sim} T \times_T G_T \xrightarrow{g_T \times \mathrm{id}} G_T \times_T G_T \xrightarrow{m_T} G_T$$

and the right translation by

$$\tau'_g : G_T \xrightarrow{\sim} G_T \times_T T \xrightarrow{\mathrm{id} \times g_T} G_T \times_T G_T \xrightarrow{m_T} G_T\ .$$

Both morphisms are isomorphisms. Quite often we will drop the index T and characterize the map τ_g by writing

$$\tau_g : G \longrightarrow G\ , \qquad x \longmapsto gx\ ;$$

the same procedure will be applied for τ'_g and for similar morphisms. In the special case where $T := G$ and $g := \mathrm{id}_G$ is the universal point, τ_g is the so-called *universal left translation*, namely the morphism

$$\Phi : T \times_S G \longrightarrow T \times_S G\,, \qquad (x,y) \longmapsto (x, xy).$$

Similarly, τ'_g gives rise to the *universal right translation*

$$\Psi : G \times_S T \longrightarrow G \times_S T\,, \qquad (x,y) \longmapsto (xy, y)\,.$$

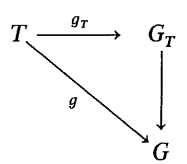
Each left translation by a T-valued point $g : T \longrightarrow G$ is obtained from the universal left translation Φ by performing the base change $g : T \longrightarrow G$; in a similar way one can proceed with right translations.

Now let us consider the sheaf $\Omega^i_{G/S}$ of relative differential forms of some degree $i \geqq 0$ on G; it is defined as the i-th exterior power of $\Omega^1_{G/S}$. For any S-scheme T and any T-valued point $g \in G(T)$, the left translation $\tau_g : G_T \longrightarrow G_T$ gives rise to an isomorphism

$$\tau_g^* \Omega^i_{G_T/T} \xrightarrow{\ \sim\ } \Omega^i_{G_T/T}\,.$$

A global section ω in $\Omega^i_{G/S}$ is called *left-invariant* if $\tau_g^*\omega_T = \omega_T$ in $\Omega^i_{G_T/T}$ for all $g \in G(T)$ and all T, where ω_T is the pull-back of ω with respect to the projection $p_1 : G_T \longrightarrow G$ (see 2.1/3 for the canonical isomorphism $p_1^*\Omega^i_{G/S} \xrightarrow{\sim} \Omega^i_{G_T/T}$; see also Section 2.1 for our notational convention on the pull-back of differential forms). Using right translations τ'_g, one defines *right-invariant* differential forms in the same way. Since each translation on the group scheme G_T is obtained by base change from the universal translation, it is clear that one has to check the invariance under translations only for the universal translation. Generally, in connection with translations, we will drop the index T and write ω instead of ω_T if no confusion is possible.

In the following we will frequently use the fact that two global sections ω and ω' of a sheaf $\mathscr{F}$ on G are equal provided they coincide on every T-valued point g of G; i.e., provided $g_T^*\omega_T = g_T^*\omega'_T$ in $g_T^*\mathscr{F}_T$, where $\mathscr{F}_T$ is the pull-back of $\mathscr{F}$ to G_T. This is easily verified by using the universal point $g := \mathrm{id}_G$ of G; namely, for $T = G$, we have the commutative diagram



where $G_T \longrightarrow G$ is the projection. Similarly, one shows that two sheaves $\mathscr{F}$ and $\mathscr{G}$ are isomorphic if their restrictions to each T-valued point of G are isomorphic.

Proposition 1. *Let G be an S-group scheme with unit section $\varepsilon : S \longrightarrow G$. Then, for each $\omega_0 \in \Gamma(S, \varepsilon^*\Omega^i_{G/S})$, there exists a unique left-invariant differential form $\omega \in \Gamma(G, \Omega^i_{G/S})$ such that $\varepsilon^*\omega = \omega_0$ in $\varepsilon^*\Omega^i_{G/S}$. The same assertion is true for right-invariant differential forms.*

Proof. It is only necessary to consider left-invariant differential forms since the inverse map $G \longrightarrow G$, $x \longmapsto x^{-1}$, transforms left-invariant forms into right-invariant ones.

The uniqueness assertion is easy to obtain. Consider two global left-invariant sections ω, ω' of $\Omega^i_{G/S}$ such that $\varepsilon^*\omega = \varepsilon^*\omega' = \omega_0$ in $\varepsilon^*\Omega^i_{G/S}$. Then we have $g^*\omega = g^*\omega'$ in $g^*\Omega^i_{G/S}$ for each point $g \in G(S)$, since $g = \tau_g \circ \varepsilon$. Hence ω and ω' coincide at all points of $G(S)$. This fact remains true after base change. So ω and ω' coincide at the universal point of G and we have $\omega = \omega'$.

In order to settle the existence part, it is only necessary to consider the case where $i = 1$. Furthermore, the problem is local on S; so we may assume that ω_0 lifts to a section ω' of $\Omega^1_{G/S}$ which is defined over a neighborhood U of the unit section $\varepsilon : S \longrightarrow G$. Then the decomposition

$$(*) \qquad \Omega^1_{G \times_S G/S} \cong p_1^*\Omega^1_{G/S} \oplus p_2^*\Omega^1_{G/S}$$

of 2.1/4 gives a decomposition

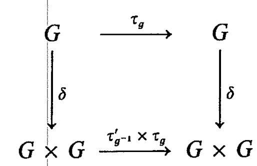
$$m^*\omega' = \omega_1 \oplus \omega_2$$

over $m^{-1}(U)$, where $m : G \times_S G \longrightarrow G$ is the group law. If

$$\delta : G \longrightarrow G \times_S G\,, \qquad x \longmapsto (x^{-1}, x)$$

denotes the twisted diagonal morphism, $m^*\omega'$ is defined in a neighborhood of the image of δ so that $\delta^*\omega_2$ gives rise to a global section ω of $\Omega^1_{G/S}$. We claim that ω is left-invariant and satisfies $\varepsilon^*\omega = \omega_0$ in $\varepsilon^*\Omega^1_{G/S}$.

For an arbitrary T-valued point $g \in G(T)$, the commutative diagram



gives $\tau_g^*\delta^*\omega_2 = \delta^*(\tau'_{g^{-1}} \times \tau_g)^*\omega_2$ in $\Omega^1_{G_T/T}$. So ω will be left-invariant if we can show $(\tau'_{g^{-1}} \times \tau_g)^*\omega_2 = \omega_2$. Since the product map $\tau'_{g^{-1}} \times \tau_g$ respects the decomposition $(*)$ over $m^{-1}(U)$, we see

$$\tilde{\omega}_j := (\tau'_{g^{-1}} \times \tau_g)^*\omega_j \in \Gamma(m^{-1}(U), p_j^*(\Omega^1_{G_S/S}))\,, \qquad j = 1, 2\,.$$

However $m \circ (\tau'_{g^{-1}} \times \tau_g) = m$ so that

$$m^*\omega' = \omega_1 \oplus \omega_2 = \tilde{\omega}_1 \oplus \tilde{\omega}_2\,.$$

The two decompositions must coincide. Hence $\tilde{\omega}_2 = \omega_2$, and ω is left-invariant.

It remains to show $\varepsilon^*\omega = \omega_0$ in $\varepsilon^*\Omega^1_{G/S}$. Consider the morphism

$$\varepsilon_T : T := G \longrightarrow G \times_S T = G \times_S G$$

obtained from the unit section $\varepsilon : S \longrightarrow G$ by the base change $T \longrightarrow S$. Since $\varepsilon_T^* p_1^*\Omega^1_{G/S}$ vanishes in $\Omega^1_{G/S}$ and since $m \circ \varepsilon_T = \mathrm{id}_G$, we have

$$\varepsilon_T^*\omega_2 = \varepsilon_T^*(\omega_1 + \omega_2) = \varepsilon_T^* m^*\omega' = \omega' \quad \text{in} \quad \Omega^1_{G/S}\,.$$

Since $p_2 \circ \varepsilon_T = id_G = p_2 \circ \delta$, there is a canonical identification

$$\varepsilon_T^* p_2^* \Omega^1_{G/S} = \Omega^1_{G/S} = \delta^* p_2^* \Omega^1_{G/S} .$$

Then $\delta \circ \varepsilon = \varepsilon_T \circ \varepsilon$ implies

$$\varepsilon^*\delta^*\omega_2 = \varepsilon^*\varepsilon_T^*\omega_2 \quad \text{in} \quad \varepsilon^*\Omega^1_{G/S} .$$

Furthermore, we know $\delta^*\omega_2 = \omega$. So we get

$$\varepsilon^*\omega = \varepsilon^*\delta^*\omega_2 = \varepsilon^*\varepsilon_T^*\omega_2 = \varepsilon^*\omega' = \omega_0 \quad \text{in} \quad \varepsilon^*\Omega^1_{G/S} .$$

Thus ω is as desired. □

Using the structural morphism $p: G \longrightarrow S$, we can state the result of Proposition 1 more elegantly in the following form:

Proposition 2. *There are canonical isomorphisms*

$$p^*\varepsilon^*\Omega^i_{G/S} \xrightarrow{\sim} \Omega^i_{G/S} , \qquad i \in \mathbb{N} ,$$

which are obtained by extending sections in $\varepsilon^*\Omega^i_{G/S}$ to left-invariant sections in $\Omega^i_{G/S}$. Similar isomorphisms are obtained by using right-invariant differential forms.

Actually, Proposition 1 provides only an $\mathcal{O}_G$-module homomorphism $p^*\varepsilon^*\Omega^i_{G/S} \longrightarrow \Omega^i_{G/S}$ which, under the pull-back by ε, becomes an isomorphism. However, applying translations, the same assertion is true for any S-valued point of G. In particular, after base change $T := G \longrightarrow S$, the above homomorphism is an isomorphism at the point $g_T \in G_T(T)$ which is induced by the universal point g of G. Hence, the above homomorphism is an isomorphism already over G. □

We are specially interested in the case where G is a smooth group scheme over a local scheme S. Then each $\mathcal{O}_G$-module $\Omega^i_{G/S}$ is locally free, and $\varepsilon^*\Omega^i_{G/S}$ is a free $\mathcal{O}_S$-module. Thus we see:

Corollary 3. *Let G be a smooth group scheme of relative dimension d over a local scheme S. Then each $\Omega^i_{G/S}$, $0 \le i \le d$, is a free $\mathcal{O}_G$-module generated by $\binom{d}{i}$ left-invariant differential forms of degree i. The same is true for right-invariant differential forms.*

For the rest of this section, let us assume that G is a *smooth S-group scheme of relative dimension d*, and that there is a *left-invariant differential form $\omega \in \Omega^d_{G/S}(G)$ generating $\Omega^d_{G/S}$* as an $\mathcal{O}_G$-module. For an arbitrary T-valued point g of G we can consider the interior automorphism

$$\mathrm{int}_g = \tau_g \circ \tau'_{g^{-1}} : G \longrightarrow G , \qquad x \longmapsto gxg^{-1} ,$$

given by g.

Proposition 4. *There exists a unique group homomorphism $\chi: G \longrightarrow \mathbb{G}_m$ (a character on G) such that*

$$\mathrm{int}_g^*\omega = \tau'^*_{g^{-1}}\omega = \chi(g)\omega$$

for each T-valued point g of G.

Proof. Since left translations commute with right translations, we see immediately that

$$\mathrm{int}_g^*\,\omega = \tau'^*_{g^{-1}}\tau_g^*\omega = \tau'^*_{g^{-1}}\omega$$

is left-invariant (on G_T) for any T-valued point g of G. Hence, since ω and $\mathrm{int}_g^*\,\omega$ generate $\Omega^d_{G_T/T}$, there exists a well-defined unit $\chi(g) \in \mathcal{O}_T(T)^*$ such that

$$\mathrm{int}_g^*\,\omega = \chi(g)\omega ;$$

recalling the functorial definition of the multiplicative group $\mathbb{G}_m$ and of group homomorphisms, one easily shows that $g \longmapsto \chi(g)$ defines a group homomorphism $\chi: G \longrightarrow \mathbb{G}_m$. □

Now let us consider the case where $S = \operatorname{Spec} K$ and where K is the field of fractions of a discrete valuation ring R. As usual, let R^{sh} denote a strict henselization of R and K^{sh} the field of fractions of R^{sh}. Let $|\ |$ be an absolute value on K and K^{sh}, which corresponds to R and R^{sh}. We want to look a little bit closer at the character χ occurring in the above lemma.

Proposition 5. *Let G be a smooth K-group scheme of relative dimension d, and assume that $G(K)$ (resp. $G(K^{sh})$) is bounded in G. Then the character χ of Proposition 4 satisfies $|\chi(g)| = 1$ for each $g \in G(K)$ (resp. each $g \in G(K^{sh})$).*

Proof. The character χ is bounded on $G(K)$; hence we may view $\chi(G(K))$ as a bounded subgroup of K^*. Such a subgroup consists of units in R. □

Remark 6. If, in the situation of Proposition 5, the group G is connected, one can actually show that the character χ is trivial. To see this, one uses the fact that G contains a maximal torus T defined over K, [SGA 3_{II}], Exp. XIV, 1.1. If χ is non-trivial, it induces a surjective map $T \longrightarrow \mathbb{G}_m$, and T must contain a subtorus isogenous to $\mathbb{G}_m$. But then $G(K)$ cannot be bounded.

4.3 R-Extensions of K-Group Laws

Let R be a discrete valuation ring with uniformizing element π, with field of fractions K, and with residue field k. As usual, R^{sh} denotes a strict henselization of R, and K^{sh} denotes the field of fractions of R^{sh}. Let X_K be a *smooth K-group scheme of dimension d*, assume that X_K is *of finite type*, and that $X_K(K^{sh})$ is *bounded* in X_K. As a group scheme over a field, X_K is automatically separated. The purpose of this section is to construct a smooth and separated R-scheme X of finite type with generic fibre X_K such that the group law of X_K extends to an R-birational group law on X and such that each translation on X_K by a K^{sh}-valued point extends to an R^{sh}-birational morphism of X. Later, it will turn out that X is already an R-dense open subscheme of the Néron model of X_K.

We start our construction by choosing a weak Néron model $(X_i)_{i \in I}$ of X_K; for the existence see Theorem 3.5/2. There is no restriction in assuming that the special fibres $X_i \otimes_R k$ are (non-empty and) irreducible for all $i \in I$. We will pick certain "minimal components" of this collection and glue them along the generic fibre to obtain the R-model X of X_K we are looking for.

In order to define minimal components, consider a left-invariant differential form ω of degree d on X_K which generates $\Omega^d_{X_K/K}$; for the existence see 4.2/1 and 4.2/3. It follows that ω is unique up to a constant in K^*. We want to define the order of ω on smooth R-models X of X_K which have an irreducible special fibre X_k, always assuming that X is separated and of finite type over R.

To do this, consider a general situation where $\mathscr{L}$ is a line bundle on a smooth R-scheme Z and where ζ is a generic point of the special fibre Z_k. Then the local ring $\mathcal{O}_{Z,\zeta}$ is a discrete valuation ring with uniformizing element π and, for any section f of $\mathscr{L}$ over the generic fibre Z_K which does not vanish at the generic point of Z_K lying over ζ, there is a unique integer n such that $\pi^{-n}f$ extends to a generator of $\mathscr{L}$ at ζ. The integer n is called the order of f at ζ, denoted by $\mathrm{ord}_\zeta f$.

Going back to the situation where we considered the section ω over the generic fibre of X, there is a unique generic point ξ of the special fibre X_k, since the latter has been assumed to be irreducible. We call $\mathrm{ord}_\xi\omega$ the order of ω at X and we denote it by $\mathrm{ord}_X\omega$. If $n = \mathrm{ord}_X\omega$, then $\pi^{-n}\omega$ generates $\Omega^d_{X/R}$ over X. Namely, $\pi^{-n}\omega$ is defined on X up to points of codimension $\geqq 2$, and X being normal, $\pi^{-n}\omega$ extends to a global section of X. Furthermore, since the zero set of a non-zero section in a line bundle is of pure codimension 1 on an irreducible normal scheme, it is seen that $\pi^{-n}\omega$ extends to a generator of $\Omega^d_{X/R}$ over X. Similarly, for sections $a \in \Gamma(X_K, \mathcal{O}_{X_K})$ (provided a is non-zero at the generic point of X_K lying over X_k), the order $\mathrm{ord}_X a$ can be defined. If $m = \mathrm{ord}_X a$, it follows that $\pi^{-m}a$ extends to a global section of $\mathcal{O}_X$. The latter is invertible if a is invertible over X_K. In this case, we have $|a(x)| = |\pi^m|$ for each K^{sh}-valued point x of X which extends to an R^{sh}-valued point of X.

Lemma 1. *Let X' and X'' be smooth and separated R-models of X_K which as above have irreducible special fibre each. Consider an R-rational map $u: X' \dashrightarrow X''$ which is an isomorphism on generic fibres; in particular, there is a unit $a \in \Gamma(X_K, \mathcal{O}^*_{X_K})$ satisfying $u_K^*\omega = a\omega$. Assume that $|a(x)| = 1$ for some $x \in X_K(K^{sh})$ such that x extends to a point in $X'(R^{sh})$. Then:*

(i) *$n' := \mathrm{ord}_{X'}\omega \geqq n'' := \mathrm{ord}_{X''}\omega$.*

(ii) *If U is the domain of definition of u, the morphism $u: U \longrightarrow X''$ is an open immersion if and only if $n' = n''$.*

Proof. Since $\pi^{-n'}\omega$ (resp. $\pi^{-n''}\omega$) generates $\Omega^d_{X'/R}$ (resp. $\Omega^d_{X''/R}$), there is a section $b \in \Gamma(X', \mathcal{O}_{X'})$ such that

$$u^*(\pi^{-n''}\omega) = b\pi^{-n'}\omega$$

over X'. Actually, b is only defined over U; however $X' - U$ is of codimension $\geqq 2$ in X' so that b extends to a section over X'. The preceding equation gives $a = \pi^{n''-n'}b$ over X_K. Since $\mathrm{ord}_{X'}a = 0$ by our assumption on a, we see

$$n' - n'' = \mathrm{ord}_{X'}b \geqq 0 \,.$$

This verifies the first assertion.

To obtain the second one, we see from 2.2/10 that u is étale on U if and only if $u^*\Omega^d_{X''/R} \longrightarrow \Omega^d_{U/R}$ is bijective; i.e., if and only if b is invertible over U and hence over X'. The latter is equivalent to $n' - n'' = 0$. Furthermore, since u_K is an isomorphism, Zariski's Main Theorem 2.3/2' implies that u is étale on U if and only if it is an open immersion. □

Let X' and X'' be smooth, separated R-models of X_K which are of finite type over R and which have irreducible special fibres. Then X' and X'' are called *equivalent* if the identity on X_K extends to an R-birational map $X' \dashrightarrow X''$.

Proposition 2. *Let X_K be a smooth K-group scheme of finite type such that $X_K(K^{sh})$ is bounded in X_K.*

(i) *There exists a largest integer n_0 such that $\mathrm{ord}_X\omega \geqq n_0$ for all R-models X of X_K which are smooth, separated, and of finite type over R, and which have an irreducible special fibre X_k. All such R-models X with $\mathrm{ord}_X\omega = n_0$ are called ω-minimal.*

(ii) *Up to equivalence there exist only finitely many R-models $X_1, \ldots, X_n$ of X_K which are ω-minimal.*

Proof. (i) Let $(X_i)_{i \in I}$ be a weak Néron model of X_K; for the existence see 3.5/2. We may assume that the special fibre of each X_i is irreducible. So the order of ω is defined with respect to each X_i. Let n_0 be the minimum of the finite set $\{\mathrm{ord}_{X_i}\omega; i \in I\}$. We claim that n_0 satisfies assertion (i). Namely, consider a smooth R-model X of X_K which is separated and of finite type over R and which has an irreducible special fibre. Due to the weak Néron property 3.5/3, the identity on X_K extends to an R-rational map $u: X \dashrightarrow X_i$ for some $i \in I$. Then $\mathrm{ord}_X\omega \geqq n_0$ by Lemma 1. In a similar way, assertion (ii) is deduced from Lemma 1 (ii). □

Since ω, as a left-invariant differential form of degree d, is unique up to a constant in K^*, it is clear that the notion of ω-minimality does not depend on the choice of ω. One has to interpret the ω-minimal R-models as those smooth R-models with irreducible special fibre, which are of "biggest" size, just as can be seen from the two R-models

$$\mathrm{Spec}\, R[\zeta, \zeta^{-1}] \qquad \text{and} \qquad \mathrm{Spec}\, R[\zeta, \zeta^{-1}, (\zeta - 1)/\pi]$$

of the multiplicative group $\mathbb{G}_m$ over K, and from the left-invariant differential form $\omega := \zeta^{-1}d\zeta$. Furthermore, we leave it to the reader to verify that, for the additive group $\mathbb{G}_a$ over K and for the left-invariant differential form $\omega := d\zeta$, there does not exist any ω-minimal R-model.

Lemma 3. *Let Z be a smooth R-scheme, and let η be a generic point of the special fibre of Z. Denote by R' the local ring $\mathcal{O}_{Z,\eta}$ of Z at η, and by K' the field of fractions of R'. If $X_1, \ldots, X_n$ is a set of representatives of the ω-minimal R-models of X_K, then, up to a splitting of special fibres into connected components, $X_1 \otimes_R R', \ldots, X_n \otimes_R R'$ represent the ω'-minimal R'-models of $X_K \otimes_K K'$, where ω' is the pull-back of ω to $X_K \otimes K'$.*

Proof. Due to 3.5/4, weak Néron models are compatible with the base change $R \longrightarrow R'$. Furthermore, each generic point ξ' of the special fibre of $X_i \otimes_R R'$ lies over a generic point ξ of the special fibre of X_i. Thus, we have $\operatorname{ord}_\xi \omega = \operatorname{ord}_{\xi'} \omega'$. Hence the R'-extension of an ω-minimal R-model of X_K decomposes into a union of ω'-minimal R'-models of $X_{K'}$. □

Now we are able to construct the R-model X of X_K we are looking for.

Proposition 4. *Let X_K be a smooth K-group scheme of finite type such that the set of K^{sh}-valued points of X_K is bounded in X_K. Then there exists an R-model X of X_K which is smooth, separated, faithfully flat, and of finite type over R and which satisfies the following conditions:*

(i) *Each open subscheme of X which is an R-model of X_K with irreducible special fibre is ω-minimal.*

(ii) *For each ω-minimal R-model X' of X_K, the identity on X_K extends to an R-rational map $X' \dashrightarrow X$ which is an open immersion on its domain of definition.*

(iii) *Let R' be the local ring $\mathcal{O}_{Z,\zeta}$ of a smooth R-scheme Z at a generic point ζ of the special fibre, and let K' be the field of fractions of R'. Then each translation on $X_{K'}$ by a K'-valued point of $X_{K'}$ extends to an R'-birational morphism of $X \otimes_R R'$, which is an open immersion on its domain of definition.*

Proof. Let $X_1, \ldots, X_n$ be a set of representatives of the ω-minimal R-models of X_K. By shrinking the special fibre of each X_i, we may assume that the following condition is satisfied:

(*) For each pair of indices $i, j \in \{1, \ldots, n\}$ with $i \neq j$, the diagonal of $X_K \times_K X_K$ constitutes a Zariski-closed subset in $X_i \times_R X_j$.

Namely, let Δ_K be the diagonal in $X_K \times_K X_K$, and consider its schematic closure Δ in $X_i \times_R X_j$. Let $p_h : \Delta \longrightarrow X_h$ for $h = i$ or j be the projection onto the first or second factor. It is enough to know that the image of Δ_k under p_i is nowhere dense in $(X_i)_k$. Assume the contrary. Then the image of Δ_k contains a non-empty open part of $(X_i)_k$ and, hence, there is a point $\eta \in \Delta$ above the generic point ξ of the special fibre of X_i. Thus the local ring $\mathcal{O}_{\Delta,\eta}$ dominates $\mathcal{O}_{X_i,\xi}$. Since p_i is an isomorphism on generic fibres and since Δ is flat over R, both local rings give rise to the same field of fractions. But then, $\mathcal{O}_{X_i,\xi}$ being a discrete valuation ring, the map $\mathcal{O}_{X_i,\xi} \longrightarrow \mathcal{O}_{\Delta,\eta}$ is an isomorphism. Since Δ is of finite type over X_i, there exist open neighborhoods U of ξ in X_i and V of η in Δ such that p_i induces an isomorphism between V and U; cf. [EGA I], 6.5.4. Hence p_i is invertible over an R-dense open part of X_i, and

$$p_j \circ (p_i|_\Delta)^{-1} : X_i \dashrightarrow X_j$$

constitutes an R-birational map, as is seen by Lemma 1. However, this contradicts the choice of $X_1, \ldots, X_n$.

Now we can construct the desired R-model X of X_K by gluing all models $X_1, \ldots, X_n$ along generic fibres. Then X is separated due to condition (*), and it satisfies conditions (i) and (ii) by construction.

To verify condition (iii), assume first $R = R'$, and consider a translation $\tau_K : X_K \longrightarrow X_K$ on X_K by a K-valued point. Fix an open subscheme U of X consisting of the generic fibre X_K and of an irreducible component of the special fibre X_k. Furthermore, let $(X_i)_{i \in I}$ be a weak Néron model of X_K, where the special fibre of X_i is irreducible for each $i \in I$. Then, due to the weak Néron property 3.5/3, there exists an index $i \in I$ such that τ_K extends to an R-rational map $\tau : U \dashrightarrow X_i$. Since U is ω-minimal, the map τ is R-birational; it is an open immersion on its domain of definition by Lemma 1 (note that, for right translations, the assumption of Lemma 1 is satisfied by 4.2/5). Moreover, X_i is ω-minimal. Then it is clear that τ_K extends to an R-rational map

$$\tau : X \dashrightarrow X.$$

Likewise, one can construct an R-rational extension

$$\tau' : X \dashrightarrow X$$

of the inverse translation $(\tau_K)^{-1}$ on X_K. Since τ and τ' are composable with each other in terms of R-rational maps, it is easily seen that they are, in fact, R-birational. Finally, Lemma 1 shows that τ is an open immersion on its domain of definition. So, if $R = R'$, condition (iii) is satisfied. In the general case, we can perform the base change $R \longrightarrow R'$, and thereby reduce to the above special case by using 3.5/4 and Lemma 3. □

Applying assertion (iii) of the preceding proposition, we want to show next that we can extend the K-group law on X_K to an R-birational group law on the R-scheme X we have just constructed.

Proposition 5. *Let X_K be a smooth K-group scheme of finite type such that the set of K^{sh}-valued points of X_K is bounded in X_K. Let X be the R-model obtained in Proposition 4 by gluing a set of representatives of ω-minimal R-models. Then the group law m_K on X_K extends to an R-birational group law on X.*

More precisely, m_K extends to an R-rational map

$$m : X \times_R X \dashrightarrow X$$

such that the universal translations

$$\Phi : X \times_R X \dashrightarrow X \times_R X, \qquad (x, y) \longmapsto (x, m(x, y))$$

$$\Psi : X \times_R X \dashrightarrow X \times_R X, \qquad (x, y) \longmapsto (m(x, y), y)$$

are R-birational. Furthermore, m is associative; i.e., the usual diagram for testing associativity is commutative as far as the occurring rational maps are defined.

Proof. Let ξ be a generic point of the special fibre X_k *of* X, and denote by R' the local ring $\mathcal{O}_{X,\xi}$ of X at ξ. Let S' be the spectrum of R'; it can be viewed as an X-scheme and as an R-scheme. Due to Proposition 4, the translation τ_K obtained from Φ_K by the base change $S'_K \longrightarrow X_K$ extends to an S'-birational map

$$\tau_\xi : S' \times_R X \dashrightarrow S' \times_R X .$$

Now consider the commutative diagram of rational maps

$$\begin{array}{ccc} \tau_\xi : S' \times_R X & \dashrightarrow & S' \times_R X \\ \downarrow & & \downarrow \\ \Phi : X \times_R X & \dashrightarrow & X \times_R X . \end{array}$$

It follows from 2.5/5 or by a simple direct argument that Φ is defined at all generic points of the special fibre of $X \times_R X$ which project to ξ under the first projection. As we can apply this reasoning to any generic point of the special fibre X_k, we see that Φ is R-rational. Since each τ_ξ is S'-birational, it follows that Φ is R-birational.

Dealing with Ψ_K in the same way as with Φ_K yields an R-birational extension Ψ of Ψ_K. Choose an R-dense open part $W \subset X \times_R X$ containing the generic fibre such that Φ and Ψ are defined on W. Then, composing Φ with the projection onto the second factor of $X \times_R X$, and Ψ with the projection onto the first factor, we obtain two extensions $W \longrightarrow X$ of the group law m_K of X_K, which must coincide. Thus, m_K extends to an R-rational map

$$m : X \times_R X \dashrightarrow X ,$$

and we see that Φ and Ψ can be described by m as stated. In particular, the associativity is a consequence of the uniqueness of R-rational extensions of K-morphisms. □

It is a general fact that an R-birational group law on X, as obtained in the preceding proposition, always determines an R-group scheme $\bar{X}$; cf. 5.1/5.

Theorem 6. *Let X_K be a smooth K-group scheme of finite type. Let X be a smooth and separated R-model of X_K which is of finite type, and assume that the group law m_K of X_K extends to an R-birational group law $m : X \times_R X \dashrightarrow X$. Then there is a smooth and separated R-group scheme $\bar{X}$ of finite type, containing X as an R-dense open subscheme, and having X_K as generic fibre such that the group law on $\bar{X}$ extends the R-birational group law m on X. Up to canonical isomorphism, $\bar{X}$ is unique.*

This result which, for the case of birational group laws over a field, goes back to A. Weil [2], § V, n°33, Thm. 15, will be proved in Chapter 5 for a strictly henselian base ring R. The generalization for an arbitrary discrete valuation ring will follow in Chapter 6 by means of descent theory. That $\bar{X}$ satisfies the Néron mapping property will be shown in the next section by using an extension theorem for morphisms into group schemes; cf. 4.4/4.

4.4 Rational Maps into Group Schemes

In order to establish the Néron mapping property for the R-group scheme $\bar{X}$ which has been introduced in the last section, we want to make use of an extension argument of Weil for rational maps into group schemes; cf. Weil [2], § II, n°15, Prop. 1.

Theorem 1. *Let S be a normal noetherian base scheme, and let $u : Z \dashrightarrow G$ be an S-rational map from a smooth S-scheme Z to a smooth and separated S-group scheme G. Then, if u is defined in codimension $\leqq 1$, it is defined everywhere.*

As in Weil's proof, which deals with the case where the base consists of a field, we will proceed by reducing to the diagonal; the following basic fact is needed:

Lemma 2. *Let $u : Z \dashrightarrow$* Spec A *be a rational map from a normal noetherian scheme Z into an affine scheme* Spec A. *Then the set of points in Z, where u is not defined, is of pure codimension 1. In particular, if u is defined in codimension $\leqq 1$, it is defined everywhere.*

The assertion (cf. [EGA IV_4], 20.4.12) is due to the fact that a rational function on Z, which is defined in codimension $\leqq 1$, is defined everywhere or, equivalently, that any noetherian normal integral domain equals the intersection over all its localizations at prime ideals of height 1.

Now let us start the *proof of Theorem* 1. Consider the rational map

$$v : Z \times_S Z \dashrightarrow G , \qquad (z_1, z_2) \longmapsto u(z_1)u(z_2)^{-1},$$

and let U (resp. V) denote the domain of definition of u (resp. v). Then $U \times_S U$ is contained in V. First we want to show that V contains the diagonal Δ of $Z \times_S Z$. Since

$$V \cap \Delta \supset (U \times_R U) \cap \Delta = U$$

(where we have identified Z with Δ), we see that $v|_{V \cap \Delta}$ factors through the unit section $\varepsilon : S \longrightarrow G$. Set $F := (Z \times_S Z) - V$. We have to show $F \cap \Delta = \emptyset$. Consider a point x of $F \cap \Delta$, and let $s \in S$ be the image of x in S. Let H be an affine open neighborhood of $\varepsilon(s)$ in G. Then there exists an open neighborhood W of x in $Z \times_S Z$ such that v induces a rational map

$$v' := v|_W : W \dashrightarrow H .$$

Let V' be the domain of definition of v'; we have $V' \subset V$. Since $v|_{V \cap \Delta}$ factors through H, we see $V' \cap \Delta = V \cap \Delta$. Furthermore, set $F' := W - V'$. Since H is affine and $Z \times_S Z$ is normal (cf. 2.3/9), it follows from Lemma 2 that F' is of pure codimension 1 in W. Since

$$F' \cap \Delta = F \cap \Delta \subset Z - U$$

(where we have identified Z with Δ again), we know that $F' \cap \Delta$ is of codimension $\geqq 2$ in Δ if u is defined in codimension $\leqq 1$. Let d be the relative dimension of Z over S at x. Since F' is of pure codimension 1 in $W \subset Z \times_S Z$, and, since $\Delta \subset Z \times_S Z$ is defined locally by d equations, due to the smoothness of Z, we get

$$\dim_x(F' \cap \Delta) \geqq \dim_x F' - d = \dim_x(Z \times_S Z) - 1 - d = \dim_x \Delta - 1 .$$

However, this contradicts the fact that $F' \cap \Delta$ is of codimension $\geqq 2$ in Δ. Thus V must contain the diagonal Δ as claimed.

It remains to show that this fact implies $U = Z$. Due to 2.5/5 it is enough to show that there exists a faithfully flat S-morphism $f: Z' \longrightarrow Z$ from a smooth S-scheme Z' of finite type to Z such that $u \circ f$ is defined everywhere. So, let Z' be the intersection of V with $Z \times_S U$ in $Z \times_S Z$. Then the first projection from $Z \times_S Z$ to Z gives rise to a faithfully flat morphism $f: Z' \longrightarrow Z$. Namely, since smooth morphisms are flat, it only remains to show that f is surjective. So, let $z: T \longrightarrow Z$ be a geometric point of Z; i.e., T is the spectrum of a field. Viewing V as a Z-scheme with respect to the first projection, the scheme $T \times_Z V$ is non-empty since V contains the diagonal Δ of $Z \times_S Z$. Furthermore, the domain of definition U of u is S-dense open in Z. Hence the intersection of $T \times_Z V$ with $T \times_S U$ in $T \times_S Z$ is not empty. Thus we see that the morphism f is surjective and, hence, faithfully flat. Now look at the morphism

$$V \cap (Z \times_S U) \longrightarrow G , \qquad (z_1, z_2) \longmapsto v(z_1, z_2) u(z_2).$$

It is clear that this map coincides with $u \circ f$, in terms of S-rational maps. Thus, the S-rational map u is defined everywhere, and we have finished the proof of Theorem 1. □

Remark 3. The method of reduction to the diagonal which was used in the proof of Theorem 1 works also within the context of formal schemes or rigid analytic spaces. So, if R is a complete discrete valuation ring, the assertion of Theorem 1 remains true if Z is of type $\operatorname{Spec} R[[t]]$ or $\operatorname{Spec} R\{t\}$ (formal or strictly convergent power series in a finite number of variables $t_1, \ldots, t_n$).

Now it is easy to show that the R-group scheme $\bar{X}$ we have introduced in Section 4.3 satisfies the Néron mapping property and thereby to end the proof of the existence theorem 1.3/1 for Néron models over a discrete valuation ring R (modulo the proof of Theorem 4.3/6). Recall the situation of 4.3. Starting with a smooth K-group scheme of finite type X_K such that the set of its K^{sh}-valued points is bounded in X_K, we have constructed in 4.3/4 a smooth and separated R-model of finite type X such that the group law on X_K extends to an R-birational group law on X; cf. 4.3/5. In 4.3/6 we have claimed that there is a unique extension of X to a smooth and separated R-group scheme of finite type $\bar{X}$ containing X as an R-dense open subscheme.

Corollary 4. *Let X be the R-model of X_K as considered in* 4.3/4 *and* 4.3/5, *and let $\bar{X}$ be the associated R-group scheme in the sense of* 4.3/6. *Then $\bar{X}$ is a Néron model of X_K over the ring R.*

Furthermore, for each ω-minimal R-model X' of X_K, the identity on X_K extends to an open immersion $X' \hookrightarrow \bar{X}$ over R.

Proof. In order to show that $\bar{X}$ satisfies the Néron mapping property let Z be a smooth R-scheme and let $u_K: Z_K \longrightarrow X_K$ be a K-morphism. We have to show that u_K extends to an R-morphism $u: Z \longrightarrow \bar{X}$.

It is enough to consider the case where Z has an irreducible special fibre. Let ζ be the generic point of the special fibre of Z, and let $R' = \mathcal{O}_{Z,\zeta}$ be the local ring of Z at ζ.

Look first at the rational map

$$Z \times_R X \dashrightarrow Z \times_R X , \qquad (z, x) \longrightarrow (z, u_K(z)x) ,$$

which is defined on the generic fibre. Applying the base change $\operatorname{Spec} R' \longrightarrow Z$, this map is turned into an R'-rational map; cf. 4.3/4. Then it follows from 2.5/5 that the map

$$\tau: Z \times_R \bar{X} \dashrightarrow \bar{X} , \qquad (z, x) \longrightarrow u_K(z)x ,$$

is defined at all generic points of the special fibre of $Z \times_R \bar{X}$ which project to ζ under the first projection. So τ is an R-rational map. Since it is defined at the generic fibre, it is defined everywhere by Theorem 1. Therefore, if we denote by p the structural morphism of Z, and by ε the unit section of $\bar{X}$, the composition of the morphism

$$(\mathrm{id}_Z, \varepsilon \circ p): Z \longrightarrow Z \times_R \bar{X}$$

with τ yields an R-morphism $u: Z \longrightarrow \bar{X}$ extending u_K. The uniqueness of u follows from the separatedness of $\bar{X}$.

If X' is an ω-minimal R-model of X_K, the identity on X_K extends to an R-rational map from X' to X by 4.3/4. Hence it extends to an R-morphism from X' to $\bar{X}$ by Theorem 1. Then it is an open immersion, due to 4.3/1. □

Chapter 5. From Birational Group Laws to Group Schemes

For the construction of Néron models, we need the fact that an S-birational group law on a smooth S-scheme with non-empty fibres can be birationally enlarged to a smooth S-group scheme; see 4.3/6. The purpose of the present section is to prove this result in the case where S is strictly henselian. In Chapter 6, the result will be extended to a more general base.

The technique of constructing group schemes from birational group laws is originally due to A. Weil [2], § V, n°33, Thm. 15; he considered birational group laws over fields. More general situations were dealt with by M. Artin in [SGA 3_{II}], Exp. XVIII, among them birational group laws over strictly henselian rings. The proof we give in this chapter, essentially follows the exposition of M. Artin [9]. Finally, in Chapter 6, descent techniques can be used to handle the case where the base is of a more general type.

5.1 Statement of the Theorem

In the following, let S be a scheme, and let X be a smooth separated S-scheme of finite type. Furthermore, we will assume that X has non-empty fibres over S or, which amounts to the same, that X is faithfully flat over S.

Definition 1. *An S-birational group law on X is an S-rational map*

$$m : X \times_S X \dashrightarrow X , \qquad (x, y) \longmapsto xy ,$$

such that

(a) *the S-rational maps*

$$\Phi : X \times_S X \dashrightarrow X \times_S X , \qquad (x, y) \longmapsto (x, xy) ,$$

$$\Psi : X \times_S X \dashrightarrow X \times_S X , \qquad (x, y) \longmapsto (xy, y) ,$$

are S-birational, and

(b) *m is associative; i.e., $(xy)z = x(yz)$ whenever both sides are defined.*

Just as in the case of group schemes, the maps Φ and Ψ will be referred to as universal left or right translations.

Note that, in place of (a), it is enough to require Φ and Ψ to be open immersions on an S-dense open subscheme U of $X \times_S X$. To see this, one has only to verify that

the images $V = \Phi(U)$ and $W = \Psi(U)$ are S-dense in $X \times_S X$. Since each fibre of U over S has the same number of components as the corresponding fibre of $X \times_S X$ over S, the same is true for the fibres of V and W over S. Hence V and W are S-dense in $X \times_S X$ if Φ and Ψ are open immersions on U.

The notion of S-birational group law is compatible with base change. Furthermore, an S-birational group law on X induces an S-birational group law on each S-dense open subscheme of X. In particular, if $\bar{X}$ is an S-group scheme and if X is an S-dense open subscheme of $\bar{X}$, the group law of $\bar{X}$ induces an S-birational group law on X. But there are S-birational group laws which do not occur in this way. Namely, even if the base consists of a field, one can blow up a subscheme of a group scheme $\bar{X}$ and consider the induced birational group law on the blowing-up. So it is natural to shrink X in order to expect that an S-birational group law on X extends to a group law on an S-scheme $\bar{X}$ containing X.

Definition 2. *Let m be an S-birational group law on a separated and smooth S-scheme X which is faithfully flat and of finite type over S. A solution of m is a separated and smooth S-group scheme $\bar{X}$ of finite type over S with group law $\bar{m}$, together with an S-dense open subscheme $X' \subset X$ and an open immersion $X' \hookrightarrow \bar{X}$ such that*

(a) *the image of X' is S-dense in $\bar{X}$, and*

(b) *$\bar{m}$ restricts to m on X'.*

First we want to show that solutions of S-birational group laws are unique.

Proposition 3. *Let m be an S-birational group law on a separated and smooth S-scheme X which is faithfully flat and of finite type over S. If there exists a solution of m, it is uniquely determined up to canonical isomorphism.*

For the proof we need the following well-known lemma.

Lemma 4. *Let G be a smooth S-group scheme, and let U be an S-dense open subscheme of G. Then the morphism*

$$U \times_S U \longrightarrow G , \qquad (x, y) \longmapsto xy$$

is smooth and surjective.

Proof of Proposition 3. Let

$$\sigma_1 : X_1' \hookrightarrow \bar{X}_1 \qquad \text{and} \qquad \sigma_2 : X_2' \hookrightarrow \bar{X}_2$$

be solutions of the S-birational group law m on X, and set $Y := X_1' \cap X_2'$. Then Y is an S-dense open subscheme of X, and each $\sigma_i(Y)$ is S-dense open in $\bar{X}_i$, $i = 1, 2$. The group laws $\bar{m}_i$ of $\bar{X}_i$ give rise to morphisms

$$\bar{m}_i \circ (\sigma_i \times \sigma_i) : Y \times_S Y \longrightarrow \bar{X}_i , \qquad i = 1, 2,$$

which are faithfully flat by Lemma 4. Furthermore, the morphisms σ_1 and σ_2 yield an S-birational map

$$\alpha = \sigma_2 \circ \sigma_1^{-1} : \bar{X}_1 \dashrightarrow \bar{X}_2$$

which is compatible with the group laws; i.e.,

$$\bar{m}_2 \circ (\sigma_2 \times \sigma_2) = \alpha \circ \bar{m}_1 \circ (\sigma_1 \times \sigma_1).$$

So, due to 2.5/5, the map α is defined everywhere. Since α is compatible with the group laws, it is clear that α is a group homomorphism. Similarly, $\beta = \sigma_1 \circ \sigma_2^{-1}$ is a group homomorphism which is defined everywhere. Since α and β are inverse to each other, they yield S-isomorphisms between $\bar{X}_1$ and $\bar{X}_2$. □

Next we want to look at the existence of solutions of S-birational group laws. In the present chapter we will consider the case where the base consists of a discrete valuation ring; see 6.6/1 for the case where the base is more general.

Theorem 5. *Let S be the spectrum of a field or of a discrete valuation ring, and let m be an S-birational group law on a smooth separated S-scheme X which is faithfully flat and of finite type over S. Then there exists a solution of m, i.e., a smooth separated S-group scheme $\bar{X}$ of finite type with a group law $\bar{m}$, together with an S-dense open subscheme $X' \subset X$ and an S-dense open immersion $X' \hookrightarrow \bar{X}$ such that $\bar{m}$ restricts to m on X'.*

The group scheme $\bar{X}$ is unique, up to canonical isomorphism. If (in the case where the base S consists of a valuation ring) the generic fibre X_K of X is a group scheme under the law m_K, the above assertion is true for $X' = X$. So, in this case, it is not necessary to shrink X.

The proof of the existence will follow in the subsequent sections (cf. 5.2/2, 5.2/3, and 6.5/2), whereas the uniqueness has already been proved. So, accepting the existence of $\bar{X}$, let us concentrate on the additional assertion on the domain X' where the group laws on X and $\bar{X}$ coincide. Assume that the base S consists of a discrete valuation ring and that the generic fibre X_K is a group scheme. By the uniqueness assertion, the S-rational map

$$\imath : X \dashrightarrow \bar{X}$$

induced by $X' \hookrightarrow \bar{X}$ restricts to a K-isomorphism

$$\imath_K : X_K \longrightarrow \bar{X}_K .$$

Hence $\imath$ is defined in codimension ≤ 1 so that, by 4.4/1, the rational map $\imath$ is defined everywhere. Now let ω be a differential form generating $\Omega^d_{\bar{X}/S}$, where d is the relative dimension of $\bar{X}$ over S; cf. 4.2/3. Pulling back ω, we get a differential form $\imath^*\omega$ on X which generates $\Omega^d_{X/S}$ over $X' \cup X_K$; hence $\imath^*\omega$ generates $\Omega^d_{X/S}$ in codimension ≤ 1. Since on a normal scheme, the zero set of a non-vanishing section of a line bundle is empty or of pure codimension ≤ 1, we see that $\imath^*\omega$ has no zeros. Thus $\imath$ is étale by 2.2/10. Since $\imath$ is birational, Zariski's Main Theorem 2.3/2′ implies that $\imath$ is an open immersion. □

5.2 Strict Birational Group Laws

In the following, let S be a scheme, and let X be a smooth separated S-scheme of finite type. Furthermore, we assume that X is faithfully flat over S.

If X is an S-dense open subscheme of an S-group scheme $\bar{X}$, then, for each T-valued point $x : T \longrightarrow X$, the set of points $y \in T \times_S X$ which is characterized symbolically by the conditions

$$xy \in T \times_S X , \quad x^{-1}y \in T \times_S X , \quad \text{and} \quad xy^{-1} \in T \times_S X$$

is T-dense and open in $T \times_S X$. Thus, we see that the group law of $\bar{X}$ induces an S-birational group law on X which is of a special type. Namely, there is an open subscheme U of $X \times_S X$ which is X-dense in $X \times_S X$ (with respect to both projections $p_i : X \times_S X \longrightarrow X$, $i = 1, 2$; i.e., X-dense when $X \times_S X$ is viewed as an X-scheme via each p_i), such that the universal translations

$$\Phi : X \times_S X \dashrightarrow X \times_S X , \qquad (x, y) \longmapsto (x, xy),$$

$$\Psi : X \times_S X \dashrightarrow X \times_S X , \qquad (x, y) \longmapsto (xy, y) ,$$

are defined and open immersions on U, and their images $V := \Phi(U)$ and $W := \Psi(U)$ are X-dense in $X \times_S X$. Just take for U the intersection of $X \times_S X$ with the inverse images of $X \times_S X$ under the group law and both universal translations on $\bar{X}$. So it is natural to introduce the following terminology:

Definition 1. *An S-birational group law on X is called a strict (S-birational) group law if it satisfies the following condition: There is an X-dense open subscheme U of $X \times_S X$, on which m is defined, such that the universal translations*

$$\Phi : X \times_S X \dashrightarrow X \times_S X , \qquad (x, y) \longmapsto (x, xy) ,$$

$$\Psi : X \times_S X \dashrightarrow X \times_S X , \qquad (x, y) \longmapsto (xy, y) ,$$

are isomorphisms from U onto X-dense open subschemes $V := \Phi(U)$ and $W := \Psi(U)$ in $X \times_S X$. (As before, X-density is meant with respect to both projections from $X \times_S X$ onto its factors.)

Note that X-density implies S-density. So the subschemes U, V, and W above are S-dense in $X \times_S X$. The first step in the existence proof of 5.1/5 consists in showing that each S-birational group law on X induces a strict group law on an S-dense open subscheme of X if S consists of a field or of a discrete valuation ring.

Proposition 2. *Let S consist of a field or of a discrete valuation ring. Let X be a smooth separated S-scheme of finite type, and consider an S-birational group law m on X. Then there exists an S-dense open subscheme X' of X such that m restricts to a strict group law on X'.*

Proof. Let U be the S-dense open subscheme of $X \times_S X$ such that m is defined at U and such that the universal translations Φ and Ψ are open immersions on U. Set $V = \Phi(U)$ and $W = \Psi(U)$. Since U, V, and W are S-dense in $X \times_S X$, the set

$$Z = U \cap V \cap W$$

is again S-dense open in $X \times_S X$. We want to show that there exists an S-dense open subscheme Ω_1 of X such that $Z \cap (\Omega_1 \times_S X)$ is Ω_1-dense in $\Omega_1 \times_S X$ with respect to the first projection p_1. Due to 2.5/1, the set

$$T_1 = \{x \in X; Z \cap (x \times_S X) \text{ is not dense in } x \times_S X\}$$

is constructible in X. Since Z is S-dense in $X \times_S X$, the generic points of the fibres of X over S do not belong to T_1. Hence the closure $\overline{T}_1$ of T_1 in X cannot be dense in any fibre of X if S consists of a discrete valuation ring. So the open subscheme $\Omega_1 = X - \overline{T}_1$ is S-dense in X and has the required property. Similarly, one defines a subscheme Ω_2 of X by considering the second projection. Then the subscheme

$$X' = \Omega_1 \cap \Omega_2$$

is S-dense open in X, and $Z \cap (X' \times_S X')$ is X'-dense in $X' \times_S X'$ (with respect to both projections).

Setting

$$U' := U \cap (X' \times_S X') \cap (m|_U)^{-1}(X'),$$
$$V' := \Phi(U'),$$
$$W' := \Psi(U'),$$

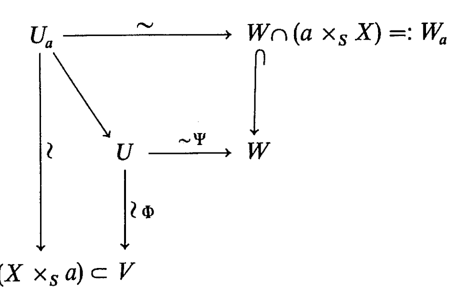
it remains to show that these open subschemes are X'-dense in $X' \times_S X'$. As a general argument, we will use the fact that U, V, and W give rise to X'-dense open subschemes in $X' \times_S X'$, because $Z = U \cap V \cap W$. Now consider a point $a \in X'$. We may assume that the base S is a field and that a is an S-valued point of X'. First we will show that U' is X'-dense in $X' \times_S X'$ with respect to the first projection p_1. If we view $X \times_S X$ as an X-scheme via p_1, the base change $a \longrightarrow X$ transforms Φ into

$$\Phi(a, \cdot) : U \cap (a \times_S X) \xrightarrow{\sim} V \cap (a \times_S X) \subset a \times_S X,$$

which is an open immersion with dense image. Then the open subscheme

$$\Phi(a, \cdot)^{-1}(V \cap (a \times_S X')) = (m|_U)^{-1}(X') \cap (a \times_S X)$$

is also dense in $a \times_S X$. This shows that U' is p_1-dense, i.e., X'-dense with respect to p_1. In a similar way, using Ψ, one shows U' is p_2-dense. Next, it is clear that V' is p_1-dense, since $V' \cap (a \times_S X')$ is the image of the dense open subscheme $U' \cap (a \times_S X')$ of $a \times_S X$ under the open immersion $\Phi(a, \cdot)$; the latter has a dense image in $a \times_S X$. By the same argument, using $\Psi(\cdot, a)$, we see that W' is p_2-dense. In order to show that W' is p_1-dense, set $U_a := m^{-1}(a)$, and consider the diagram of isomorphisms



Since a belongs to X', the set V_a is dense in $X \times_S a$, and W_a is dense in $a \times_S X$. The same is true if we replace V_a by its restriction to $X' \times_S a$ and W_a by its restriction to $a \times_S X'$. Taking inverse images with respect to Φ and Ψ, the set

$$U_a \cap U' = \Phi^{-1}(V_a \cap (X' \times_S a)) \cap \Psi^{-1}(W_a \cap (a \times_S X'))$$

is open and dense in U_a. Hence its image under Ψ, which is $W' \cap (a \times_S X)$, is open and dense in $a \times_S X$. Thereby we see that W' is p_1-dense. Similarly, one shows that V' is p_2-dense. □

The proposition reduces the proof of Theorem 5.1/5 to the problem of enlarging a strict group law on X to a group law on a group scheme $\overline{X}$. If the base scheme S is normal and strictly henselian (of any dimension), we will construct the group scheme $\overline{X}$ in a direct way. The case where S consists of a field or of a discrete valuation ring, without assuming that the latter is strictly henselian, will be reduced to the preceding one by means of descent theory, cf. 6.5/2. For further generalizations see Section 6.6.

Theorem 3. *Let S be the spectrum of a strictly henselian local ring which is noetherian and normal, and let m be a strict group law on a separated smooth S-scheme X which is faithfully flat and of finite type over S. Then there exists an open immersion $X \hookrightarrow \overline{X}$ with S-dense image into a smooth separated S-group scheme $\overline{X}$ of finite type such that the group law $\overline{m}$ of $\overline{X}$ restricts to m on X.*

The S-group scheme $\overline{X}$ is unique, up to canonical isomorphism.

The uniqueness assertion of Theorem 5.1/5, which has already been proved in Section 5.1, yields the uniqueness assertion of the present theorem. A proof of the existence part will be given in Section 5.3, assuming that the base S is strictly henselian. The idea is easy to describe, although a rigorous proof requires the consideration of quite a lot of unpleasant technical details. Namely, a smooth scheme X over a strictly henselian base S admits many sections in the sense that the points of the special fibre X_k which lift to S-valued points of X are schematically dense in X_k; cf. 2.3/5. So the idea is to construct $\overline{X}$ by gluing "translates" of X. More precisely, consider an S-valued point a of X and a copy $X(a)$ of X, thought of as a left translate of X by a. Then one can glue X and $X(a)$ along the correspondence given by the left translation by a

$$\Phi(a, \cdot) : X \dashrightarrow X.$$

The result is a new S-scheme $X' = X \cup X(a)$, and it has to be verified that the strict group law m on X extends to a strict group law m' on X'. The left translation by a

$$\Phi'(a, \cdot) : X' \dashrightarrow X'$$

is now defined on the open subscheme X of X'. Repeating such a step finitely many times with suitable S-valued points $a_1, \ldots, a_n \in X(S)$, and applying a noetherian argument, one ends up with an S-scheme $\overline{X} = X^{(n)}$ such that the strict group law m on X extends to a strict group law $\overline{m}$ on $\overline{X}$, such that the S-rational map

$$\overline{m} : \overline{X} \times_S \overline{X} \dashrightarrow \overline{X}$$

is defined on the open subscheme $X \times_S X \subset \bar{X} \times_S \bar{X}$. Then it is not difficult to show that $\bar{m}$ defines a group law on $\bar{X}$, and that $\bar{X}$ is the S-group scheme we are looking for.

The technical problems in the proof of Theorem 3 are due to the fact that, for a point $a \in X$, the product ax is only defined for "generic" $x \in X$. This drawback disappears, when we look at the situation from the point of view of group functors. Let m be a strict group law on X, as in Theorem 3, and consider the group functor

$$\mathscr{R}_{X/S} : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Sets})$$

which associates to each S-scheme T the set of T-birational maps from $X_T = X \times_S T$ onto itself. Identifying X with its functor of points $h_X = \mathrm{Hom}(\cdot, X)$, cf. 4.1, we claim that there is a monomorphism $X \hookrightarrow \mathscr{R}_{X/S}$ respecting the laws of composition on X and $\mathscr{R}_{X/S}$. Namely, due to the definition of strict group laws, one knows that the universal left translation

$$\Phi : X \times_S X \dashrightarrow X \times_S X , \qquad (x, y) \longmapsto (x, m(x, y))$$

is X-birational if $X \times_S X$ is viewed as an X-scheme via the first projection. So, for any S-scheme T and any T-valued point $a \in X(T)$, the map

$$\tau_a : T \times_S X \dashrightarrow T \times_S X ,$$

the "left translation" by a obtained from Φ by means of the base change $a : T \longrightarrow X$, is T-birational and thus belongs to $\mathscr{R}_{X/S}(T)$. It is clear that the maps

$$X(T) \longrightarrow \mathscr{R}_{X/S}(T) , \qquad a \longmapsto \tau_a ,$$

constitute a morphism of functors $X \longrightarrow \mathscr{R}_{X/S}$.

Lemma 4. *The morphism $X \longrightarrow \mathscr{R}_{X/S}$ is a monomorphism which respects the laws of composition on X and on $\mathscr{R}_{X/S}$; i.e., for any S-scheme T and all T-valued points a, b, $c \in X(T)$ satisfying $m(a, b) = c$, one has $\tau_a \circ \tau_b = \tau_c$.*

Proof. We have to show that all maps $X(T) \longrightarrow \mathscr{R}_{X/S}(T)$ are injective. So consider $a, b \in X(T)$ with $\tau_a = \tau_b$. Applying the base change $T \longrightarrow S$ to our situation, we may consider T as the new base, writing S instead of T. Let U be the X-dense open subscheme of $X \times_S X$ required by Definition 1 (on which the universal translations are open immersions). Using the X-density of U with respect to the first projection, we see that the compositions

$$\Psi_a : S \times_S X \xrightarrow{a \times \mathrm{id}_X} X \times_S X \overset{\Psi}{\dashrightarrow} X \times_S X ,$$
$$\Psi_b : S \times_S X \xrightarrow{b \times \mathrm{id}_X} X \times_S X \overset{\Psi}{\dashrightarrow} X \times_S X$$

are defined as S-rational maps. Since $\Psi_a = (\tau_a, \mathrm{id}_X)$ and $\Psi_b = (\tau_b, \mathrm{id}_X)$ when $S \times_S X$ is identified with X, we see that $\tau_a = \tau_b$ yields $\Psi_a = \Psi_b$. Now Ψ is an open immersion on U, so $a \times \mathrm{id}_X$ and $b \times \mathrm{id}_X$ must coincide on the S-dense open subscheme

$$X' := (a \times \mathrm{id}_X)^{-1}(U) \cap (b \times \mathrm{id}_X)^{-1}(U)$$

of $S \times_S X$, hence on all of $S \times_S X$. In particular, their first components agree, i.e., $a = b$. Thus we see that $X \longrightarrow \mathscr{R}_{X/S}$ is a monomorphism. That this transformation

respects the laws of composition follows immediately from the associativity of m. □

If X has been expanded into an S-group scheme $\bar{X}$ such that X is S-dense and open in $\bar{X}$ and such that the group law on $\bar{X}$ restricts to the strict group law m on X, then there is a canonical commutative diagram of natural transformations

$$\begin{array}{ccc} X & \longrightarrow & \mathscr{R}_{X/S} \\ \downarrow & & \downarrow \wr \\ \bar{X} & \longrightarrow & \mathscr{R}_{\bar{X}/S} \end{array}$$

where the vertical arrow on the right-hand side is an isomorphism, since X is S-dense in $\bar{X}$. Although it is not in general true that the group functor $\bar{X}$ is generated by X, i.e., that $X(T)$ generates the group $\bar{X}(T)$ for all S-schemes T, the latter is nevertheless correct if T is a strictly henselian local S-scheme. Namely the group law on $\bar{X}$ induces a surjective and smooth S-morphism

$$X \times_S X \longrightarrow \bar{X} ,$$

c.f. 5.1/4, so that, by 2.3/5, each T-valued point of $\bar{X}$ lifts to a T-valued point of $X \times_S X$.

5.3 Proof of the Theorem for a Strictly Henselian Base

We have already seen in 5.2/2 that Theorem 5.2/3 implies Theorem 5.1/5 if the base is strictly henselian. So we may restrict ourselves to strict group laws and give only a proof of 5.2/3. In this section we assume that *the base S consists of a strictly henselian local ring* which is *noetherian* and *normal*. Furthermore, let X be a smooth and separated S-scheme which is faithfully flat and of finite type over S, and let m be a strict group law on X; the symbols Φ, Ψ, and U, V, W will be used in the sense of 5.2/1.

Introducing further notational conventions, let X^n be the n-fold fibred product of X over S, and, for integers $1 \leqq i_1 < \ldots < i_r \leqq n$, let

$$p_{i_1 \ldots i_r} : X^n \longrightarrow X^r$$

be the projection of X^n onto the product of the factors with indices $i_1, \ldots, i_r$. In such a situation, we can view X^n as an X^r-scheme with respect to the morphism $p_{i_1 \ldots i_r}$. So we have the notion of X^r-density in X^n; to be more precise, we will speak of $p_{i_1 \ldots i_r}$-density. Sometimes, we will write $x = (x_1, \ldots, x_n)$ for points in X^n and $(x_{i_1}, \ldots, x_{i_r})$ instead of $p_{i_1 \ldots i_r}(x)$ for their projections onto X^r. As usual, the S-rational map $m : X^2 \dashrightarrow X$ will be characterized by $(x_1, x_2) \longmapsto x_1 x_2$.

Lemma 1. *Let Ω be the set of points $(x, y, z, w) \in X^4$ such that*

$$(z, w) \in U , \qquad (y, w) \in U , \qquad \text{and} \qquad (x, yw) \in U .$$

Then Ω is p_{123}-dense in X^4.

Proof. Recall that the intersection of finitely many p_{123}-dense open subschemes of X^4 is p_{123}-dense and open again. Since U is p_1-dense in X^2, the first two conditions pose no problem. So it remains to show that the set Ω' of all points $(x, y, w) \in X^3$, satisfying $(y, w) \in U$ and $(x, yw) \in U$, is p_{12}-dense and open in X^3. We can describe Ω' as the inverse image of U with respect to the following morphism:

$$X \times_S U \xrightarrow{\mathrm{id}_X \times \Phi} X^3 \xrightarrow{p_{13}} X^2 ,$$
$$(x, y, w) \longmapsto (x, y, yw) \longmapsto (x, yw) .$$

Since U is p_1-dense in X^2, and since Φ leaves the first component fixed and is an open immersion on U with a p_1-dense image in X^2, we see that Ω' is p_{12}-dense and open in X^3. □

The assertion of Lemma 1 is only an example for similar assertions of this type. Roughly speaking, it says that, fixing x, y, and z, the stated conditions form open conditions on w; these are satisfied if w is generic.

Lemma 2. *Let Γ be the schematic closure in X^3 of the graph of $m : U \longrightarrow X$. Let T be an S-scheme. If (a, b, c) is a T-valued point in $\Gamma(T) \subset X^3(T)$, then, using the functor $\mathcal{R}_{X/S}$ of 5.2, the T-birational maps τ_a, τ_b, and τ_c of X_T satisfy $\tau_a \circ \tau_b = \tau_c$ in $\mathcal{R}_{X/S}(T)$.*

Proof. Let Ω be the p_{123}-dense open subscheme of X^4 which was considered in Lemma 1. Then the S-rational maps

$$\lambda : X^4 \dashrightarrow X^4 , \qquad (x, y, z, w) \longmapsto (x, y, x(yw), w),$$
$$\mu : X^4 \dashrightarrow X^4 , \qquad (x, y, z, w) \longmapsto (x, y, zw, w) ,$$

are defined on Ω. Next, let $\Omega' := \Omega \cap p_{12}{}^{-1}(U)$. We claim that $\Omega' \cap (\Gamma \times_S X)$ is schematically dense in $\Omega \cap (\Gamma \times_S X)$. Namely, $p_{12}{}^{-1}(U) \cap (\Gamma \times_S X)$ is schematically dense in $\Gamma \times_S X$ by the definition of Γ (since X is flat over S), and this density is not destroyed when we intersect both sets with an open subscheme of X^4 such as Ω. Since the law m is associative, the morphism $\mu|_{(\Gamma \times_S X) \cap \Omega'}$ factors through Λ, the schematic image of $\lambda|_\Omega$. By continuity, also $\mu|_{(\Gamma \times_S X) \cap \Omega}$ factors through Λ, and thus yields a morphism

$$\mu : (\Gamma \times_S X) \cap \Omega \longrightarrow \Lambda .$$

Now set

$$\varphi = (a, b, c) \times \mathrm{id}_X : T \times_S X \longrightarrow X^4 ,$$

and $\Omega_{a,b,c} := \varphi^{-1}(\Omega)$. Then $\Omega_{a,b,c}$ is T-dense and open in X_T. Let $\varphi_T : X_T \longrightarrow X_T^4$ be the T-morphism derived from φ, and let μ_T be the T-morphism obtained from μ by means of the base change $T \longrightarrow S$. Then $p_3 \circ \mu_T \circ \varphi_T$ coincides with τ_c on $\Omega_{a,b,c}$, but

also with $\tau_a \circ \tau_b$ since $\mu \circ \varphi$ factors through Λ. Hence, we have $\tau_a \circ \tau_b = \tau_c$ in $\mathcal{R}_{X/S}(T)$. □

We state an important consequence of Lemma 2.

Lemma 3. *Let Γ be the schematic closure in X^3 of the graph of $m : U \longrightarrow X$, and let $q_{ij} : \Gamma \longrightarrow X^2$ be the morphisms induced from the projections $p_{ij} : X^3 \longrightarrow X^2$. Then each q_{ij} is an open immersion and has an image which is p_1-dense and p_2-dense in X^2*

Proof. First we want to show that each q_{ij} is injective as a map of sets. If (a, b, c) is a T-valued point in $\Gamma(T)$ for some S-scheme T, then $\tau_a \circ \tau_b = \tau_c$ by Lemma 2. Since this is an identity in the group $\mathcal{R}_{X/S}(T)$, any two of the maps τ_a, τ_b, τ_c determine the third one. As stated in 5.2/4, the natural transformation $X \longrightarrow \mathcal{R}_{X/S}$ is a monomorphism. Hence a point of Γ is known if two of its components are given. This implies that q_{ij} is injective as a map of sets and, hence, that q_{ij} is quasi-finite. We claim that the maps q_{ij} are, in fact, S-birational. Namely, using the notation of 5.2/1, the projection q_{12} gives rise to an isomorphism $q_{12}^{-1}(U) \xrightarrow{\sim} U$ because m is defined on U. Furthermore, q_{13} defines an isomorphism $q_{13}^{-1}(V) \xrightarrow{\sim} V$ because q_{13} is injective and because $\Phi_{|U}$ is an isomorphism $U \xrightarrow{\sim} V$. Likewise, q_{23} defines an isomorphism $q_{23}^{-1}(W) \xrightarrow{\sim} W$ because q_{23} is injective and because $\Psi_{|U}$ is an isomorphism $U \xrightarrow{\sim} W$. Thus, by Zariski's Main Theorem 2.3/2′ (recall that S is normal), each q_{ij} is an open immersion and, due to the X-density of U, V, and W in X^2, the image of each q_{ij} is X-dense in X^2 (with respect to p_1 and p_2). □

Fixing points a, b, $c \in X(T)$ for some S-scheme T, we see from the preceding lemma that there exists at most one point $x \in X(T)$ such that $ax = c$ and at most one point $y \in X(T)$ such that $yb = c$. Suggestively, we will write $a^{-1}c$ for x and cb^{-1} for y. With this notation the assertion of Lemma 3 can be interpreted as follows: The maps

$$q_{13} \circ q_{12}^{-1} : X^2 \dashrightarrow X^2 , \qquad (a, b) \longmapsto (a, ab) ,$$
$$q_{23} \circ q_{12}^{-1} : X^2 \dashrightarrow X^2 , \qquad (a, b) \longmapsto (b, ab) ,$$
$$q_{23} \circ q_{13}^{-1} : X^2 \dashrightarrow X^2 , \qquad (a, c) \longmapsto (a^{-1}c, c) ,$$
$$q_{12} \circ q_{13}^{-1} : X^2 \dashrightarrow X^2 , \qquad (a, c) \longmapsto (a, a^{-1}c) ,$$
$$q_{13} \circ q_{23}^{-1} : X^2 \dashrightarrow X^2 , \qquad (b, c) \longmapsto (cb^{-1}, c) ,$$
$$q_{12} \circ q_{23}^{-1} : X^2 \dashrightarrow X^2 , \qquad (b, c) \longmapsto (cb^{-1}, b) ,$$

are S-birational. They are open immersions on their domains of definition; the latter as well as the corresponding images are X-dense in X^2 (with respect to both projections). In addition, the lemma shows that the law $m : X^2 \dashrightarrow X$ is defined at a point $(x, y) \in X^2$ as soon as the fibre $q_{12}^{-1}((x, y))$ is non-empty. This fact will be needed in the next lemma.

Lemma 4. *Let a be an S-valued point of X, and consider another point $b \in X$. Then $a \times_S b$ can be viewed as a point in X^2, and the law $m: X^2 \dashrightarrow X$ is defined at $a \times_S b$ if and only if the birational map $\tau_a: X \dashrightarrow X$ is defined at b.*

Proof. It is only necessary to verify the if-part of the assertion. Considering the S-dense open subscheme $U_a := U \cap (a \times_S X)$ of $a \times_S X \cong X$, we know that τ_a is at least defined on U_a. Let Γ_a be the schematic closure in X^2 of the graph of $\tau_{a|U_a}$. Then we have

$$(a \times_S \Gamma_a) \cap (a \times_S U_a \times_S X) \subset \Gamma$$

and, by continuity, also $a \times_S \Gamma_a \subset \Gamma$. Since the image of the morphism

$$a \times_S \Gamma_a \hookrightarrow \Gamma \xrightarrow{q_{12}} X^2$$

contains the point $a \times_S b$, the fibre over it with respect to q_{12} is non-empty. Thus, the assertion follows from Lemma 3. □

The preceding lemma is very useful if one wants to expand the domain of definition of $m: X^2 \dashrightarrow X$ by means of enlarging X. Namely, one has only to enlarge the domain of definition of $\tau_a: X \dashrightarrow X$ for suitable sections $a \in X(S)$. This can be done by introducing sort of a translate of X by a and by gluing it to X.

Therefore, fix a section $a \in X(S)$ and, as in the proof of Lemma 4, consider the schematic closure Γ_a in X^2 of the graph of the S-birational map τ_a. Then $a \times_S \Gamma_a \subset \Gamma$ and, by Lemma 3, both projections $p_i: \Gamma_a \longrightarrow X$ are injective as maps of sets. Since τ_a is S-birational, Zariski's Main Theorem implies that p_1 and p_2 are open immersions; furthermore, p_1 and p_2 have S-dense images in X. So these projections define gluing data, and we obtain an S-scheme

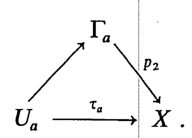
$$X' = X \cup_{\Gamma_a} X ,$$

which is smooth and of finite type over S, and which is covered by two S-dense open subschemes isomorphic to X. Due to its definition, Γ_a is closed in X^2, hence X' is separated over S.

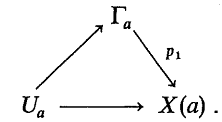
We need to distinguish between the two copies of X which cover X'. So let us write more precisely

$$p_1: \Gamma_a \longrightarrow X(a) ,$$
$$p_2: \Gamma_a \longrightarrow X$$

for the gluing data, where $X(a)$ is another copy of X. This way we have fixed one of the two canonical embeddings of our original S-scheme X into X'. We want to show that $X(a)$ can be interpreted as a "left translate" (in X') of X by a. Namely, consider the S-birational map $\tau_a: X \dashrightarrow X$. It is defined at least on U_a so that we have the following factorization:



Working in X', we can write this diagram also in the form



Since the horizontal map is the restriction to U_a of the canonical isomorphism $X \xrightarrow{\sim} X(a)$, we see that $\tau_a: X \dashrightarrow X$ extends to an isomorphism $\tau_a: X \xrightarrow{\sim} X(a)$, namely the canonical one. In particular, τ_a extends to an S-birational map $X' \dashrightarrow X'$ which is defined on X.

Lemma 5. *As before, let X' be the S-scheme obtained by gluing a left translate $X(a) = \tau_a(X)$ for some point $a \in X(S)$ to X. Then X' contains X as an S-dense open subscheme, and the strict group law m on X extends to a strict group law m' on X'.*

Proof. We have already seen that X is S-dense in X'. So it is clear that m extends to an S-birational group law m' on X', and we have only to show that m' is strict, i.e., that there exists an X'-dense (with respect to both projections) open subscheme $U' \subset X' \times_S X'$ satisfying the following conditions:

(a) m' is defined on U',

(b) the universal translations

$$\Phi': X' \times_S X' \dashrightarrow X' \times_S X' , \qquad (x, y) \longmapsto (x, xy) ,$$
$$\Psi': X' \times_S X' \dashrightarrow X' \times_S X' , \qquad (x, y) \longmapsto (xy, y) ,$$

are open immersions on U', and the images $V' := \Phi(U')$ and $W' := \Psi(U')$ are X'-dense in $X' \times_S X'$ (with respect to both projections).

The product $X' \times_S X'$ is the union of the open subschemes

$$X \times_S X , \quad X(a) \times_S X , \quad X \times_S X(a) , \quad \text{and} \quad X(a) \times_S X(a) .$$

In order to define U', let U, as before, be the open subscheme of $X \times_S X$ whose existence is required in Definition 5.2/1 for the strict group law m on X. Furthermore, let U_1 be the image of U under the isomorphism

$$\tau_a \times \mathrm{id}_X: X \times_S X \xrightarrow{\sim} X(a) \times_S X .$$

Then m' is defined on U since m is defined on U, and the isomorphism $\tau_a: X \xrightarrow{\sim} X(a)$ can be used in order to obtain the morphism

$$U_1 \longrightarrow X(a) , \qquad (\tau_a(x), y) \longmapsto \tau_a(xy) ,$$

from $m: U \longrightarrow X$. Both morphisms coincide on an S-dense open part of U, due to the associativity of m. Thus m' is defined on the open subscheme $U \cup U_1$ of $X' \times_S X'$; the latter is X'-dense with respect to the first projection.

Next consider the open subscheme

$$\{(x, y, z) \in X^3 ; (x, y) \in U , (xy, z) \in U\}$$

of X^3. Similarly as in the proof of Lemma 1, one shows that it is p_{23}-dense in X^3. Hence, intersecting it with $X \times_S a \times_S X$ and applying the isomorphism

$$X \times_S a \times_S X \xrightarrow{p_{13}} X^2 \xrightarrow{\mathrm{id}_X \times \tau_a} X \times_S X(a)\ ,$$

we obtain an open subscheme U_2 of $X \times_S X(a)$ which is $X(a)$-dense with respect to the second projection. Then the morphism

$$(*) \qquad U_2 \longrightarrow X\ , \qquad (x, \tau_a(y)) \longmapsto (xa)y$$

is defined and, using the associativity of m, it coincides with the multiplication $m: U \longrightarrow X$ on an S-dense open part of U. Thus, writing U' for the X'-dense (with respect to both projections) open subscheme $U \cup U_1 \cup U_2$ of $X' \times_S X'$, we see that m' is defined on U' and, hence, that U' satisfies condition (a).

In order to verify condition (b), notice that the universal translations Φ' and Ψ' corresponding to m' extend the universal translations Φ and Ψ corresponding to m. Thus, since Φ and Ψ are open immersions on U, we see that Φ' and Ψ' are open immersions on each one of the schemes U, U_1, and U_2. In particular, Φ' and Ψ' are quasi-finite on U'. Since these are S-birational maps on $X' \times_S X'$, Zariski's Main Theorem 2.3/2′ implies that they are open immersions on U'.

As in 5.2/1, set $V := \Phi(U)$. Furthermore, let V_1 be the image of V under the isomorphism

$$\tau_a \times \tau_a : X \times_S X \xrightarrow{\sim} X(a) \times_S X(a)\ .$$

Then $V' := \Phi'(U')$ contains $V \cup V_1$, and the latter is X'-dense in $X' \times_S X'$ (with respect to both projections); in particular, V' is X'-dense in $X' \times_S X'$.

Similarly, one shows that $W' := \Psi'(U')$ is X'-dense in $X' \times_S X'$ with respect to the first projection. In order to see that the same is true for the second projection, notice that $W_1 := \Psi'(U')$ is X-dense in $X' \times_S X$ with respect to the second projection. Furthermore, consider the open subscheme

$$W_2 := \Psi'(U_2) \subset X \times_S X(a)$$

and look at the description $(*)$ of m' on U_2 which was discussed above. Then W_2 is seen to be $X(a)$-dense in $X' \times_S X(a)$ with respect to the second projection since, for any T-valued point z of X, the right translation

$$X_T \dashrightarrow X_T\ , \qquad x \longmapsto xz\ ,$$

is T-birational. Hence $W' = \Psi'(U')$ is X'-dense in $X' \times_S X'$ with respect to both projections. The latter finishes the verification of condition (b). □

Now consider a sequence $a_1, a_2, \ldots$ of S-valued points of X. Iterating the construction of X' by using these points, we obtain a sequence of S-schemes

$$X = X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \ldots,$$

where $X^{(i)} = X^{(i-1)} \cup X^{(i-1)}(a_i)$. Each $X^{(i)}$ contains X as an S-dense open subscheme, and $X^{(i)}$ is separated, smooth, and of finite type over S. Furthermore, Lemma 5 shows that the strict group law m on X extends to a strict group law $m^{(i)}$ on each $X^{(i)}$. Using a noetherian argument, we want to show that the sequence $X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \ldots$ becomes stationary at a certain $X^{(n)}$. Then, for a suitable choice of the a_i, we will see that $X^{(n)}$ is the S-group scheme we are looking for.

Lemma 6. *There exist finitely many S-valued points $a_1, \ldots, a_n \in X(S)$ such that, for $X^{(n)}$ as above, the S-rational map $m: X \times_S X \dashrightarrow X$ extends to an S-morphism $X \times_S X \longrightarrow X^{(n)}$.*

Proof. First we show that we can find $a_1, \ldots, a_n \in X(S)$ in such a way that, for each $a \in X(S)$, the S-birational map $\tau_a: X \dashrightarrow X$ extends to an S-morphism $X \longrightarrow X^{(n)}$. Proceeding indirectly, consider a sequence $a_1, a_2, \ldots$ in $X(S)$ such that

$$\tau_{a_{i+1}}: X \dashrightarrow X^{(i)}\ , \qquad i = 1, 2, \ldots,$$

is not defined everywhere on X. Let $\Gamma^{(i)}$ be the schematic closure in $(X^{(i)})^3$ of the graph of $m: U \longrightarrow X$. It coincides with the schematic closure of the graph of the induced strict group law $m^{(i)}$ on $X^{(i)}$; so we know from Lemma 3 that

$$p_{12}: \Gamma^{(i)} \longrightarrow X^{(i)} \times_S X^{(i)}$$

is an open immersion. Setting

$$Q^{(i)} := p_{12}(\Gamma^{(i)}) \cap (X \times_S X)\ ,$$

the $Q^{(i)}$ form an increasing sequence of open subschemes of $X \times_S X$, since the $\Gamma^{(i)}$ form an increasing sequence. However, the base S consists of a noetherian ring, which implies that the topological space $X \times_S X$ is noetherian. Thus the $Q^{(i)}$ must become stationary at a certain index $n \in \mathbb{N}$, and we claim that, for $a = a_{n+1}$, the map $\tau_a: X \dashrightarrow X^{(n)}$ is defined everywhere. Namely, consider a point $b \in X$. By the definition of $X^{(n+1)}$, the birational map $\tau_a: X \dashrightarrow X^{(n+1)}$ is defined everywhere. So we see from Lemma 4 that the law $m^{(n+1)}$ on $X^{(n+1)}$ is defined at $a \times_S b$. Hence the fibre over $a \times_S b$ of

$$p_{12}: \Gamma^{(n+1)} \longrightarrow X^{(n+1)} \times_S X^{(n+1)}$$

is non-empty, and $a \times_S b \in Q^{(n+1)}$. But, since $Q^{(n+1)} = Q^{(n)}$, the fibre over $a \times_S b$ of

$$p_{12}: \Gamma^{(n)} \longrightarrow X^{(n)} \times_S X^{(n)}$$

cannot be empty, and we see from Lemma 3 that the law $m^{(n)}$ on $X^{(n)}$ is defined at $a \times_S b$. In particular, $\tau_{a_{n+1}} = \tau_a: X \dashrightarrow X^{(n)}$ is defined at b. This contradicts our assumption on the sequence $a_1, a_2, \ldots$; so there must exist $a_1, \ldots, a_n \in X(S)$ such that $\tau_a: X \dashrightarrow X^{(n)}$ is defined everywhere for each $a \in X(S)$.

It remains to show that, in this situation, the S-rational map $m: X \times_S X \dashrightarrow X^{(n)}$ is defined everywhere. We know already from Lemma 4 that m is defined on $a \times_S X$ for each S-valued point a of X. However, this is not enough, and we now have to use the fact that our assumption on X to be a faithfully flat and smooth scheme over a *strictly henselian base* S yields the following property:

Let t be a point of S, and let C_t be the reduced subscheme of $X \times_S t$ whose underlying topological space is the closure in $X \times_S t$ of the set of points $\{a(t); a \in X(S)\}$. Then there exists a component X_t^0 of X_t contained in C_t; cf. Lemma 7 below.

Moreover, let k' be an extension field of $k(t)$, and let t' be the scheme of k'. Then $C_t \times_t t'$ coincides with the reduced subscheme of $X \times_S t'$ whose underlying topological space is the closure of the points $\{a(t'); a \in X(S)\}$; cf. [EGA IV_3], 11.10.7.

In particular, if $Z_{t'}$ is a dense open subscheme of $X \times_S t'$, there exists a point $a \in X(S)$ such that $a \times_S t'$ gives rise to a point of $Z_{t'}$.

Now let us continue the proof of Lemma 6. Using the notation of Lemma 3, we know that

$$q_{23} \circ q_{13}^{-1} : X \times_S X \dashrightarrow X \times_S X , \qquad (w, x) \longmapsto (w^{-1}x, x) ,$$

is an S-birational map. It is an open immersion on its domain of definition D, and this domain as well as its image are X-dense in X^2 with respect to both projections. Now consider a point $t \in X^2$. It follows that the set

$$Z := \{(w, x, y) \in X^3 \; ; (w, x) \in D \text{ and } (w^{-1}x, y) \in U\} ,$$

where U is as in 5.2/1, is open and p_{23}-dense in X^3 and, hence, open and dense in $X \times_S t$. So, applying the base change $t \longrightarrow X^2$ to $X \times_S X^2$, the assumption on X as explained above implies the existence of a point $a \in X(S)$ such that $a \times_S t \in Z$. Then the S-rational map

$$X \times_S X \dashrightarrow X , \qquad (x, y) \longmapsto (a^{-1}x)y ,$$

is defined at t. Furthermore, since the left translation

$$\tau_a : X \dashrightarrow X^{(n)}$$

is defined everywhere, we see that

$$X \times_S X \dashrightarrow X^{(n)} , \qquad (x, y) \longmapsto a((a^{-1}x)y) ,$$

is defined at t. However, this map coincides on $X \times_S X$ with the strict group law m, since m is associative. So we see that m extends to an S-rational map

$$X \times_S X \dashrightarrow X^{(n)}$$

which is defined at all points of X^2. □

Lemma 7. *Let T be a noetherian scheme, let $Y \longrightarrow T$ be a morphism of finite type, and let $\{a_i, i \in I\}$ be a family of sections of Y. Let t_1 and t_0 be points of T such that t_0 is a specialization of t_1. Let C_j be the closure of the set of points $\{a_i(t_j), i \in I\}$ in the fibre $Y_{t_j}, j = 0, 1$. Then $\dim C_1 \geqq \dim C_0$.*

In particular, if T is strictly henselian and noetherian, and if $Y \longrightarrow T$ is smooth and surjective, then, for each point $t \in T$, there exists a connected component Y_t^0 of the fibre Y_t such that the set of the points $\{a(t), a \in Y(T)\}$ is dense in Y_t^0.

Proof. It suffices to show the first assertion after a base change $\varphi : T' \longrightarrow T$ such that the points t_0, t_1 belong to the image of φ. So, due to [EGA II], 7.1.4, we may assume that T consists of a discrete valuation ring with generic point t_1 and closed point t_0. Denote by V the schematic closure of C_1 in Y; so V is flat over T, since T consists of a discrete valuation ring. Then it is clear that

$$\dim V_{t_1} \geqq \dim V_{t_0} ;$$

cf. [EGA IV_3], 14.3.10. Since $C_0 \subset V$, the first assertion is clear.

For the second, we may assume that the relative dimension of Y over T is constant on Y. Due to 2.3/5 the closure of the set of points $\{a(t_0), a \in Y(T)\}$ is Y_{t_0} for the closed point t_0 of T. Hence the second assertion follows from the first one. □

Now the proof of Theorem 5.2/3 is quite easy. Namely, let $\bar{X}$ be the S-scheme $X^{(n)}$ constructed in Lemma 6. Then $\bar{X}$ is separated, smooth, of finite type, and contains X as an S-dense open subscheme. Furthermore, by Lemmata 5 and 6, the strict group law m on X extends to a strict group law $\bar{m}$ on $\bar{X}$, and the S-rational map $\bar{m} : \bar{X}^2 \dashrightarrow \bar{X}$ is defined on X^2. It is a general fact that $\bar{X}$ is an S-group scheme in this situation; so we can end the proof of 5.2/3 by establishing the following result:

Lemma 8. *Let $\bar{X}$ be a smooth and separated S-scheme of finite type which is equipped with a strict group law $\bar{m}$. Assume that $\bar{X}(S)$ is non-empty and that there exists an S-dense open subscheme X of $\bar{X}$ such that $\bar{m}$ is defined on the open subscheme X^2 of $\bar{X}^2$. Then $\bar{X}$ is an S-group scheme with respect to the law $\bar{m}$.*

Proof. First we want to show that

$$\bar{m} : \bar{X} \times_S \bar{X} \dashrightarrow \bar{X} , \qquad (x, y) \longmapsto xy ,$$

is defined everywhere. Since the domain of definition is compatible with faithfully flat base change (2.5/6), it suffices to show that, for each point $(b, c) \in \bar{X}^2$, the map

$$\bar{m}_X = \mathrm{id}_X \times \bar{m} : X \times_S \bar{X}^2 \dashrightarrow X \times_S \bar{X}$$

is defined at some point $(a, b, c) \in X \times_S \bar{X}^2$ above (b, c). For example, let (a, b, c) be a generic point of the fibre over (b, c). Then $(a, b) \in X \times_S \bar{X}$ is a generic point in the fibre over b and the map

$$X \times_S \bar{X} \dashrightarrow X , \qquad (w, x) \longmapsto xw ,$$

is defined at (a, b), since $\bar{m}$ is a strict group law on $\bar{X}$. Likewise, using Lemma 3, the map

$$X \times_S \bar{X} \dashrightarrow X , \qquad (w, y) \longmapsto w^{-1}y ,$$

is defined at (a, c) which is a generic point in the fibre over c. Since $\bar{m}$ is defined on X^2, the map

$$m' : X \times_S \bar{X} \times_S \bar{X} \dashrightarrow X \times_S \bar{X} , \qquad (w, x, y) \longmapsto (w, (xw)(w^{-1}y)) ,$$

is defined at (a, b, c), and the associativity of $\bar{m}$ shows that m' coincides with $\bar{m}_X$. Thus $\bar{m}$ is defined on all of $\bar{X}^2$.

Similar arguments show that the map

$$\bar{X} \times_S \bar{X} \longrightarrow \bar{X} , \qquad (x, y) \longmapsto x^{-1}y ,$$

is defined everywhere. But then $\bar{m}$ defines on $\bar{X}$ the structure of an S-group scheme. Namely, returning to the functorial point of view, consider the monomorphism

$$\bar{X} \hookrightarrow \mathscr{R}_{\bar{X}/S}$$

of 5.2/4. The group law on $\mathscr{R}_{\bar{X}/S}$ restricts to the law $\bar{m}$ on $\bar{X}$, and $\bar{X}(T) \neq \varnothing$ for $T = S$ and, hence, for all S-schemes T. Thus, since the map $(x, y) \longmapsto (x^{-1}y)$ is defined on $\bar{X} \times_S \bar{X}$, we see that each $\bar{X}(T)$ is a subgroup of $\mathscr{R}_{\bar{X}/S}(T)$. So $\bar{X}$ is a subgroup functor of $\mathscr{R}_{\bar{X}/S}$ and in fact, the representability being granted, an S-group scheme with group law $\bar{m}$. □

So we have finished the proof of Lemma 8 and thereby also the proofs of 5.2/3 and of 5.1/5 for the case where the base S consists of a strictly henselian valuation ring or of a separably closed field.

Chapter 6. Descent

During the years 1959 to 1962, Grothendieck gave a series of six lectures at the Séminaire Bourbaki, entitled "Technique de descente et théorèmes d'existence en géométrie algébrique". In the first lecture [FGA], n°190, the general technique of faithfully flat descent is introduced. It is an invaluable tool in algebraic geometry. Quite often it happens that a certain construction can be carried out only after faithfully flat base change. Then one can try to use descent theory in order to go back to the original situation one started with. Before Grothendieck, descent was certainly known in the form of Galois descent.

We begin by describing the basic facts of Grothendieck's formalism and by discussing some general criteria for effective descent, including several examples. Then, working over a Dedekind scheme, our main objective is to study the descent of torsors under smooth group schemes; see Raynaud [4]. As a preparation, we discuss the theorem of the square and use it to show the quasi-projectivity of torsors. Relying on the latter fact, effective descent of torsors can be described in a very convenient form; we do this in Section 6.5. As an application, we look at existence and descent of Néron models for torsors. Also, working over a more general base, we are able to extend the technique of associating group schemes to birational group laws as discussed in Chapter 5. The chapter ends with an example of non-effective descent.

6.1 The General Problem

Let $p: S' \longrightarrow S$ be a morphism of schemes and consider the functor $\mathscr{F} \longrightarrow p^*\mathscr{F}$, which associates to each quasi-coherent S-module $\mathscr{F}$ its pull-back under p. Then, in its simplest form, the problem of descent relative to $p: S' \longrightarrow S$ is to characterize the image of this functor. The procedure of solution is as follows. Set $S'' := S' \times_S S'$, and let $p_i: S'' \longrightarrow S'$ be the projection onto the i-th factor ($i = 1, 2$). For any quasi-coherent S'-module $\mathscr{F}'$, call an S''-isomorphism $\varphi: p_1^*\mathscr{F}' \longrightarrow p_2^*\mathscr{F}'$ a *covering datum* of $\mathscr{F}'$. Then the pairs $(\mathscr{F}', \varphi)$ of quasi-coherent S'-modules with covering data form a category in a natural way. A morphism between two such objects $(\mathscr{F}', \varphi)$ and $(\mathscr{G}', \psi)$ consists of an S'-morphism $f: \mathscr{F}' \longrightarrow \mathscr{G}'$ which is compatible with the covering data φ and ψ; thereby we mean that the diagram