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à la mémoire d'André Néron

Preface

Néron models were invented by A. Néron in the early 1960's with the intention to study the integral structure of abelian varieties over number fields. Since then, arithmeticians and algebraic geometers have applied the theory of Néron models with great success, usually without going into the details of Néron's construction process. In fact, even for experts the existence proof given by Néron was not easy to follow. Quite recently, in connection with new developments in arithmetic algebraic geometry, the desire to understand more about Néron models, and even to go back to the basics of their construction, was reactivated. We have taken this as an incentive to present a treatment of Néron models in the form of a book.

The three of us have approached Néron models from different angles. The senior author has been involved in the developments from the beginning on. Immediately after the discovery of Néron models, it was one of his first assignments from A. Grothendieck to translate Néron's construction to the language of schemes. The other two authors worked in the early 1980's on the uniformization of abelian varieties, thereby finding a rigid analytic approach to Néron models. It was at this time that we realized that we had a common interest in the field and decided to write a book on Néron models and related topics.

At first we had the idea of covering a much wider variety of subjects than we actually do here. We wanted to start with a presentation of the construction of Néron models, on an elementary level and understandable by beginners, and then to continue with a general structure theory for rigid analytic groups, with the intention of applying it to the discussion of uniformizations and polarizations of abelian varieties. However, it did not take long to realize that an appropriate treatment of Néron models would require a book of its own. So we changed our plans; colleagues watching the project encouraged us in doing so. Now, having finished the manuscript, we hope that the "elementary" part of the book, which consists of Chapters 1 to 7, is, indeed, understandable by beginners.

We are, of course, indebted to Néron for the original ideas leading to the construction of Néron models, and to the work of Grothendieck which provides language and methods of expressing these ideas in an adequate context. There are other sources from which we have borrowed, most noteworthy the work of A. Weil as well as various contributions of M. Artin.

In preparing this book we received help from many sides. We thank the Deutsche Forschungsgemeinschaft for its constant support during the entire project. Similarly we wish to thank the Centre National de la Recherche Scientifique, as well as the Institute des Hautes Etudes Scientifiques for its hospitality. Finally, we are indebted to our home universities and Mathematics departments in Münster and

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During the project Dr. W. Heinen from Münster was of invaluable help to us; he proofread the manuscripts and set up the index. We thank him heartily for his work. Last but not least, our thanks go to the publishers for their cooperation.

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Introduction

Let K be a number field, S the spectrum of its ring of integers, and A_K an abelian variety over K. Standard arguments show that A_K extends to an abelian scheme A' over a non-empty open part S' of S. Thus A_K has good reduction at all points s of S' in the sense that A_K extends to an abelian scheme or, what amounts to the same, to a smooth and proper scheme over the local ring at s. In general, one cannot expect that A_K also has good reduction at the finitely many points in $S - S'$. However, one can ask if, even at these points, there is a notion of "good" models which generalizes the notion of good reduction. It came as a surprise for arithmeticians and algebraic geometers when A. Néron, relaxing the condition of properness and concentrating on the group structure and the smoothness, discovered in the years 1961–1963 that such models exist in a canonical way; see Néron [2], see also his lecture at the Séminaire Bourbaki [1]. Gluing these models with the abelian scheme A', one obtains a smooth S-group scheme A of finite type which may be viewed as a best possible integral group structure over S on A_K. It is called a Néron model of A_K and is characterized by the universal property that, for any smooth S-scheme Z and any K-morphism $u_K: Z_K \longrightarrow A_K$, there is a unique S-morphism $u: Z \longrightarrow A$ extending u_K. In particular, rational points of A_K can be interpreted as integral points of A.

Néron himself used his models to study rational points of abelian varieties over global fields, especially their heights. In his paper [3], he shows that the local height contribution at a non-archimedean place can be calculated on the local Néron model in terms of intersection multiplicities between divisors and integral points.

Before Néron's discovery, in 1955, Shimura systematically studied the reduction of algebraic varieties over a discrete valuation ring R, in the affine, projective, as well as in the "abstract" case; see Shimura [1]. In particular, he defined the specialization of subvarieties as well as the reduction of algebraic cycles. In the years 1955 to 1960, several other authors became interested in the reduction of abelian varieties, either in the abstract form or in the form of Albanese and Picard varieties. Koizumi [1] proved that if an abelian variety A_K over K extends to a proper and smooth R-scheme A, then the group structure of A_K also extends. Furthermore, it follows from Koizumi and Shimura [1] that A is essentially uniquely determined by A_K. The latter corresponds to the fact that A is a Néron model of A_K and therefore satisfies the universal mapping property characterizing Néron models. Igusa [1] showed that the Jacobian of a curve with good reduction has good reduction. He also considered the case where the reduction of the curve has an ordinary double point as singularity.

Concerning the reduction of elliptic curves, a systematic investigation of degenerate fibres was carried out by Kodaira [1] for the special case of holomorphic fibrations of smooth surfaces by elliptic curves. Among other things, he classified the possible diagrams of the fibres for minimal fibrations by using the intersection form.

On the other hand, starting with an elliptic curve over the field of fractions of an arbitrary Dedekind ring R, equations of Weierstraß type can provide natural R-models, even at bad places. It seems certain that, at least in characteristic different from 2 and 3, the minimal Weierstraß model was known to arithmeticians at the time Néron worked on his article [2]. However, it was Néron's idea to consider minimal models which are regular and proper, but not necessarily planar. In [2], after constructing Néron models for general abelian varieties, he turns to elliptic curves, shows the existence of regular and proper minimal models, and works out their different types. The classification of special fibres which he obtains is the same as Kodaira's. In order to pass to the "Néron model" as considered in the case of general abelian varieties, one has to restrict to the smooth locus of the corresponding regular and proper minimal model. Furthermore, the identity component coincides with the smooth part of the minimal Weierstraß model.

In his paper [2], Néron uses a terminology which is derived from that in Weil's Foundations of Algebraic Geometry [1]. The terminology has earned its merits when working with varieties over fields. However, applying it to a relative situation, even if the base is as simple as a discrete valuation ring, one cannot avoid a number of unpleasant technical problems. For example, since there are two fibres, namely the generic and the special fibre, it is necessary to work with two universal domains, one for each fibre. Both domains have to behave well with respect to specialization, and so on. Clearly, Weil's terminology was not adapted to handle problems of this kind.

Néron's paper appeared at a time when Grothendieck had just started a revolution in algebraic geometry. With his theory of schemes, he had developed a new machinery, specially designed for treating problems in relative algebraic geometry. Néron knew of this fact, but he did not want to abandon the framework in which he was used working. In the introduction to his article [2], he says that the notion of a scheme over a commutative ring will frequently intervene in his text, in a more or less explicit way. However—and now we quote—"faute d'être suffisamment accoutumé à ce langage, nous avons estimé plus prudent de renoncer à son emploi systématique, et d'utiliser le plus souvent un langage dérivé de celui des *Foundations* de Weil ... ou de celui de Shimura ..., laissant les spécialistes se charger de la traduction."

Certainly, a few specialists did the translation, but mainly for themselves and without publishing proofs. It was only about 20 years later, in 1984, at the occasion of a conference on Arithmetic Algebraic Geometry, that M. Artin wrote a Proceedings article [9] explaining the construction of Néron models from a scheme viewpoint. So, at Néron's time, the situation remained somewhat mysterious. On the one hand, it was very hard to follow Néron's arguments concerning the construction of his models. On the other, arithmeticians were able to use the notion of Néron models with great success, for example, in the investigation of Galois cohomology of abelian varieties. Since Néron models are characterized by a simple universal

property, it is possible to work with them without knowing about the actual construction process.

After Néron's work, substantial progress on the structure of Néron models was achieved with the so-called semi-abelian reduction theorem. It states that, up to finite extension of the ground field, Néron models of abelian varieties are semi-abelian. A first proof of this result was carried out by Grothendieck during the fall of 1964; he explained it in a series of letters to Serre, using regular models for curves and l-adic monodromy. The proof was published later in [SGA 7_I]. Independently, Mumford was able to obtain the semi-abelian reduction theorem via his theory of algebraic theta-functions, at least for the case where the residue characteristic is different from 2; for this proof see the Appendix II to Chai [1]. The behavior of a Néron model with respect to base change can be difficult to follow; however, in the semi-abelian case it is particularly simple because the identity component is preserved.

In the late sixties, Raynaud [6] further developed the relative Picard functor over discrete valuation rings R in such a way that, in quite general situations, the Néron model of the Jacobian of a curve could be described in terms of the relative Picard functor of a regular R-model of this curve. Using Abhyankar's desingularization of surfaces, one thereby obtains, at least in the case of Jacobians, a second method of constructing Néron models which is largely independent of the original construction given by Néron.

Today, using the relative Picard functor, the semi-abelian reduction theorem is viewed as a consequence of the corresponding semi-stable reduction theorem on curves; see, for example, Artin and Winters [1], or see Bosch and Lütkebohmert [3] for an approach through rigid analytic uniformization theory. To a certain extent, the semi-abelian reduction theorem has changed the view on the reduction of abelian varieties. Namely, it is sometimes enough to work with semi-abelian models and to consider the corresponding monodromy at torsion points. As an example, we refer to Faltings' proof [1] of the Mordell conjecture.

On the other hand, there are questions where, in contrast to the above, Néron models are involved with all their beautiful structure, with their Lie algebra, and with their group of connected components. An example is given by the precise form of the Taniyama-Weil conjecture on modular elliptic curves over $\mathbb{Q}$; cf. Mazur and Swinnerton-Dyer [1].

For further applications of Néron models, we refer to the work of Ogg [1] and Shafarevich [1] concerning moderately ramified torsors over function fields. This was extended by Grothendieck to arbitrary torsors; cf. Raynaud [1].

It should also be noted that the Néron model is of interest when studying the Shafarevich-Tate group Ш. Namely, let A be the Néron model over a Dedekind scheme S of an abelian variety A_K where K is the field of fractions of S. Then Ш is the group of "locally trivial" torsors under A_K, a group which is closely related to the group $H^1(S, A)$. In this way the Néron model is involved in questions concerning the group Ш. For example, concerning its conjectural finiteness in the global arithmetic case.

Finally, to give another application involving torsors under abelian varieties, we mention that Tate studied in [1] the group $H^1(K, A_K)$, where A_K is an abelian variety over a local field K of characteristic 0 having a finite residue field. He used the compact group $\hat{A}_K(K)$ (where $\hat{A}_K$ is the dual abelian variety of A_K) as well as its Pontryagin dual. Later, when the theory of Néron models was available, there appeared some variants of this work for algebraically closed residue field; cf. Bégueri [1] and Milne [2]. Here the Néron model of A_K, in particular, its proalgebraic structure plays an important role.

The aim of the present book is to provide an exposition of the theory of Néron models and of related methods in algebraic geometry. Using the language and techniques of Grothendieck, we describe Néron's construction, discuss the basic properties of Néron models, and explain the relationship between these models and the relative Picard functor in the case of Jacobians. Finally, using generalized Néron models which are just locally of finite type, we study Néron models of not necessarily proper algebraic groups.

We now describe the contents in more detail. Chapter 1 is meant as a first orientation on Néron models. The actual construction of Néron models in the local case takes place in Chapters 3 to 6. Instead of just using Grothendieck's [EGA] as a general reference, we have chosen to explain in Chapter 2 some of the basic notions we need. So, for the convenience of the reader, we give a self-contained exposition of the notion of smoothness relating it closely to the Jacobi criterion. A discussion of henselian rings, an overview on flatness, as well as a presentation of the basics on relative rational maps follows. Also, at the beginning of Chapter 6, we have included an introduction to descent theory.

In Chapter 3, we start the construction of Néron models with the smoothening process. Working over a discrete valuation ring R with field of fractions K, this process modifies any R-model X (of finite type and with a smooth generic fibre X_K) by means of a sequence of blowing-ups with centers in special fibres to an R-model X' such that each integral point of X lifts to an integral point of the smooth locus of X'. This leads to the construction of so-called weak Néron models. Since there is a strong analogy between the smoothening process and the technique of Artin approximation, we have included the latter, although it is not actually needed for the construction of Néron models.

Next, in Chapter 4, we look at group schemes. We consider a smooth K-group scheme of finite type X_K admitting a weak Néron model X and show that the group law on X_K extends to an R-birational group law on X if we remove all non-minimal components from the special fibre of X; the minimality is measured with respect to a non-trivial left-invariant differential form of maximal degree on X_K. In Chapter 5, working over a strictly henselian base and following ideas of M. Artin, we associate to the R-birational group law on X an R-group scheme. The latter is, by a generalization of a theorem of Weil for rational maps from smooth schemes into group schemes, already the Néron model of X_K. The generalization to an arbitrary discrete valuation ring is done in Chapter 6 by means of descent. After we have finished the construction of Néron models in Chapter 6, we discuss their properties in Chapter 7.

The next topic to be dealt with is the relative Picard functor and, in particular, its relationship to Néron models in the case of Jacobians of curves. Since there seems to be no systematic exposition of the relative Picard functor $\mathrm{Pic}_{X/S}$ available which takes into account developments after Grothendieck's lectures [FGA], we thought it necessary to include a chapter on this topic. In Chapter 8 we explain the various representability results for $\mathrm{Pic}_{X/S}$ in terms of schemes or algebraic spaces, mainly due to Grothendieck [FGA] and Artin [5]. From this point on, due to lack of space, it was impossible to give detailed proofs for all the results we mention. It is our strategy to list the important results, to prove them whenever possible without too much effort, or to sketch proofs otherwise. In any case, we attempt to give precise references and to point out improvements which have appeared in the subsequent literature.

The same can be said for the first half of Chapter 9 where we deal with relative Jacobians of curves. Among other things, modulo some considerations contained in Chapter 7, we show here how to derive the semi-abelian reduction theorem for Néron models from the semi-stable reduction theorem for curves. A proof of the latter theorem has not been included in the book since a detailed discussion of models for curves and of related methods would be a topic of its own, too large to be dealt with in the present book. Instead, for a proof using Abhyankar's desingularization, we refer to Artin and Winters [1] or, for a proof using rigid geometry, to Bosch and Lütkebohmert [1]. Finally, in Sections 5 to 7 of Chapter 9, we compare the Néron model with the relative Picard functor in the case of Jacobians. As an application, we show how to compute the group of connected components of a Néron model.

The book ends with a chapter on Néron models of commutative, but not necessarily proper algebraic groups. In the local case, we prove a criterion for a smooth commutative K-group scheme X_K of finite type to admit a Néron model which, over an excellent strictly henselian base, amounts to the condition that X_K does not contain subgroups of type $\mathbb{G}_a$ or $\mathbb{G}_m$. We also indicate how to globalize this result. In doing so, it is natural to admit Néron models which are locally of finite type (lft), but not necessarily of finite type. This way we can construct Néron models for tori as well as study the same problem for K-wound unipotent groups. Since our investigations seem to have few applications at the moment and, since some of the statements are still at a conjectural stage, we have chosen only to give short indications of proofs.

Bibliographical references are given by mentioning the author, with a number in square brackets to indicate the particular work we are referring to. An exception is made for Grothendieck, where we also use the familiar abbreviations [FGA], [EGA], and [SGA], as listed at the beginning of the bibliography. Cross references to theorems, propositions, etc., like Theorem 1.3/1, usually contain the number of the chapter, the section number, and the number of the particular result. For references within the same section, the chapter and the section numbers will not be repeated.

Chapter 1. What Is a Néron Model?

This chapter is meant to provide a first orientation to the basics of Néron models. Among other things, it contains an explanation of the context in which Néron models are considered, as well as a discussion of the main results on the construction and existence, including some examples.

We start by looking at models over Dedekind schemes. In particular, the notion of étale integral points is introduced, and models of finite type satisfying the extension property for étale integral points are considered. For a local base, the existence of such models is characterized in terms of a boundedness condition. Then, in Section 1.2, we define Néron models and prove some elementary properties which follow immediately from the definition. We also discuss the relationship between global and local Néron models as well as a criterion for a smooth group scheme of finite type to be a Néron model. Next, in Section 1.3, we state the main existence theorem for Néron models in the local case and explain the skeleton of its proof, anticipating some key results which are obtained in later chapters.

In Section 1.4, we discuss the case of abelian varieties. More precisely, we study the notion of good reduction and show how the existence of local Néron models leads to the existence of global Néron models. In Section 1.5, in order to provide some explicit examples, we consider elliptic curves. In particular, we compare the Néron model with the minimal proper and regular model and with the minimal Weierstraß model. The chapter ends with a look at Néron's article [2] which serves as a basis for the construction of Néron models. For this section, a certain familiarity with the contents of later Chapters 3 to 6 is advisable.

1.1 Integral Points

When dealing with Néron models, one usually works over a base scheme S which is a *Dedekind scheme*, i.e., a noetherian normal scheme of dimension ≤ 1. The local rings of S are either fields or discrete valuation rings. For example, S can be the spectrum of a Dedekind domain. We will talk about the *local case* if S consists of a local scheme and, thus, is the spectrum of a discrete valuation ring or even of a field; the general case will be referred to as the *global case*. Any Dedekind scheme S decomposes into a disjoint sum of finitely many irreducible components S_i with a generic point η_i each. We set $K := \bigoplus k(\eta_i)$, so K is the *ring of rational functions on* S. Furthermore, the affine scheme Spec K is referred to as the *scheme of generic points of* S. If S is connected—and this is the case to keep in mind—there is a unique

generic point $\eta \in S$. Its residue field is K and we can identify η with the associated geometric point Spec $K \longrightarrow S$. It is only for technical reasons that we do not require Dedekind schemes to be connected.

There are three examples of Dedekind schemes, which are of special interest. To describe the first one, let K be a number field, i.e., a finite extension of $\mathbb{Q}$, and let R be the ring of integers of K. Then set $S = \operatorname{Spec} R$. Similarly, we can consider an algebraic function field K of dimension 1 over a constant field k and define S to be the normal proper k-curve associated to K. In both cases, S is a Dedekind scheme. On the other hand, we can start with a normal noetherian local scheme of dimension 2 and remove the closed point from it. Also this way we obtain a Dedekind scheme.

Now let S be an arbitrary Dedekind scheme with ring of rational functions K and consider an S-scheme X. We define its *generic fibre* (or, more precisely, its *scheme of generic fibres*) by $X_K := X \otimes_S K$, viewed as a scheme over K. Conversely, if we start with a K-scheme X_K, any S-scheme Y extending X_K, i.e., with generic fibre $Y_K = X_K$, will be called an *S-model of* X_K. There is an abundance of such models. For example, any change of Y (such as blowing up or removing a closed subscheme) which takes place in fibres disjoint from X_K, will produce a new S-model of the same K-scheme X_K. On the other hand, X_K can be viewed as an S-model of itself. In the local case, the latter is even of finite type over S if X_K is of finite type over K.

The main problem we will be concerned with when studying the existence of Néron models is to construct S-models X of X_K which satisfy certain natural properties. One of them is the extension property concerning étale integral points, or just étale points, as we will say; for the notion of étale see Section 2.2.

Definition 1. *Let X be a scheme over a Dedekind scheme S. Then we say that X satisfies the extension property for étale points at a closed point $s \in S$ if, for each étale local $\mathcal{O}_{S,s}$-algebra R' with field of fractions K', the canonical map $X(R') \longrightarrow X_K(K')$ is surjective.*

Each étale local $\mathcal{O}_{S,s}$-algebra is a discrete valuation ring again. In fact, it can be seen from Chapter 2, in particular, from 2.4/8 and 2.3/9, that the étale local $\mathcal{O}_{S,s}$-algebras R' correspond bijectively to the (faithfully flat) extensions of discrete valuation rings $\mathcal{O}_{S,s} \subset R'$ with the properties that a uniformizing element of $\mathcal{O}_{S,s}$ is also uniformizing for R', that the extension of fraction fields of $\mathcal{O}_{S,s} \subset R'$ is finite and separable, and that the residue extension of $\mathcal{O}_{S,s} \subset R'$ is finite and separable. So we conclude from the valuative criterion of separatedness [EGA II], 7.2.3, that the map $X(R') \longrightarrow X_K(K')$ is injective if X is separated over S. Furthermore, the extension property for étale points as formulated in Definition 1 is similar to the one occurring in the valuative criterion of properness [EGA II], 7.3.8; the only difference is that we restrict ourselves to valuation rings R' which are étale over $\mathcal{O}_{S,s}$.

Instead of considering all étale local $\mathcal{O}_{S,s}$-algebras R' one can just as well apply limit arguments and work with a strict henselization R^{sh} of $\mathcal{O}_{S,s}$. The latter is the inductive limit over all pairs (R', α) where R' is an étale local $\mathcal{O}_{S,s}$-algebra and where α is an R-homomorphism from R' into a fixed separable algebraic closure of the residue field $k(s)$; see Section 2.3. Then, if K^{sh} is the field of fractions of R^{sh}, it follows that X satisfies the extension property for étale points at $s \in S$ if and only if the map

$X(R^{sh}) \longrightarrow X_K(K^{sh})$ is surjective. Furthermore, let us mention that X satisfies the extension property for étale points at $s \in S$ if and only if $X \otimes_S \mathcal{O}_{S,s}$, viewed as a scheme over $\mathcal{O}_{S,s}$, does.

A simple method for constructing S-models of finite type is the method of *chasing denominators*. It applies to the case where S is affine, say $S = \operatorname{Spec} R$, and where X_K is affine of finite type over K (resp. projective over K). The resulting models are affine of finite type over R (resp. projective over R). To explain the affine case, let X_K be the spectrum of a ring

$$A_K = K[t_1, \ldots, t_n]/I_K ;$$

i.e., of a quotient of a free polynomial ring by an ideal I_K. Then I_K is generated by finitely many polynomials $f_1, \ldots, f_r$, which we may assume to have coefficients in R. So set

$$A := R[t_1, \ldots, t_n]/I ,$$

where I is the ideal generated by $f_1, \ldots, f_r$. Then $X := \operatorname{Spec} R$ is an R-model of finite type of X_K. Furthermore, since a module over a valuation ring is flat as soon as there is no torsion, we see that X will be flat over R if we saturate I; i.e., if we set

$$I := I_K \cap R[t_1, \ldots, t_n].$$

Then, by its definition, X is just the schematic closure of X_K in the affine n-space over R; for the notion of schematic closure see Section 2.5. Finally, the projective case is completely analogous; here one works with the Proj of homogeneous coordinate rings.

If X_K is projective, any R-model X obtained by chasing denominators is projective and, thus, satisfies the extension property for étale points by the valuative criterion of properness. If X_K is just of finite type, but not projective, the construction of an S-model of finite type satisfying the extension property for étale points can be quite complicated or even impossible as the example of the affine n-space $\mathbb{A}_K^n$ shows. As a necessary condition in the local case, we will introduce the notion of boundedness.

So assume that S consists of a discrete valuation ring R with field of fractions K. Furthermore, consider a faithfully flat extension of discrete valuation rings $R \subset R'$ and let K' be the field of fractions of R'. Then R and R' give rise to absolute values on K and on K'; we denote them by $|\ |$ assuming that both coincide on K. For us the case where R' is a strict henselization R^{sh} of R will be of interest. Now, for any K-scheme X_K, for any point $x \in X_K(K')$, and for any section g of $\mathcal{O}_{X_K}$ being defined at x, we may view $g(x)$ as an element of K' so that its absolute value $|g(x)|$ is well-defined. In particular, it makes sense to say that g is bounded on a subset of $X_K(K')$. Applying this procedure to the coordinate functions of the affine n-space $\mathbb{A}_K^n$, we arrive at the notion of a *bounded subset of* $\mathbb{A}_K^n(K')$.

Definition 2. *As before, let $R \subset R'$ be a faithfully flat extension of discrete valuation rings with fields of fractions K and K'. Furthermore, let X_K be a K-scheme of finite type and consider a subset $E \subset X_K(K')$.*

(a) *If X_K is affine, E is called bounded in X_K if there exists a closed immersion $X_K \hookrightarrow \mathbb{A}_K^n$ mapping E onto a bounded subset of $\mathbb{A}_K^n(K')$.*

(b) *In the general case, E is called bounded in X_K if there exists a covering of X_K by finitely many affine open subschemes $U_1, \ldots, U_s \subset X_K$ as well as a decomposition $E = \bigcup E_i$ into subsets $E_i \subset U_i(K')$ such that, for each i, the set E_i is bounded in U_i in the sense of* (a).

It should be kept in mind that the definition of boundedness takes into account the choice of valuation rings $R \subset R'$ and, thereby, the choice of particular valuations on K and K', although the latter is not expressed explicitly when we say that a subset $E \subset X_K(K')$ is bounded in X_K.

If X_K is affine, say if $X_K = \operatorname{Spec} A_K$, condition (a) of the definition means that there are elements $g_1, \ldots, g_n \in A_K$ generating A_K as a K-algebra which, as maps $X_K(K') \longrightarrow K'$, are bounded on E. The latter is equivalent to the fact that each $g \in A_K$ is bounded on E and it is easily seen that, in the affine case, conditions (a) and (b) of the definition are equivalent. Moreover, if there is one closed immersion $X_K \hookrightarrow \mathbb{A}_K^n$ mapping E onto a bounded subset of $\mathbb{A}_K^n(K')$, it follows that the latter property is enjoyed by all closed immersions of type $X_K \hookrightarrow \mathbb{A}_K^m$.

We want to show that condition (b) of Definition 2 is independent of the particular affine open covering $\{U_i\}$ of X.

Lemma 3. *Let $R \subset R'$ be a faithfully flat extension of discrete valuation rings with fields of fractions K and K'. Furthermore, let X_K be a K-scheme of finite type and consider a subset $E \subset X_K(K')$. If there exists a finite affine open covering $\mathfrak{U} = \{U_i\}$ of X_K such that condition* (b) *of Definition 2 is satisfied, then the latter condition is satisfied independently of the particular covering $\mathfrak{U}$. More precisely, given any finite affine open covering $\mathfrak{V} = \{V_j\}$ of X_K, there is a partition $E = \bigcup F_j$ into subsets $F_j \subset V_j(K')$ such that F_j is bounded in V_j for each j.*

Proof. Since conditions (a) and (b) of Definition 2 are equivalent in the affine case, we may assume that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. Now pick an element $U_i \in \mathfrak{U}$, say $U_i = \operatorname{Spec} A$, and let it be covered by the elements $V_1, \ldots, V_r \in \mathfrak{V}$. Then we may assume that V_ρ is of type $\operatorname{Spec} A_{f_\rho}$, $\rho = 1, \ldots r$, where $f_1, \ldots, f_r$ generate the unit ideal in A. So there is an equation $\Sigma a_\rho f_\rho = 1$ with coefficients $a_\rho \in A$. Let E_i be a bounded subset of $U_i(K')$. Then it follows from the equation representing the unit 1 that

$$\varepsilon := \inf\{\max\{|f_\rho(x)| ; \rho = 1, \ldots, r\} ; x \in E_i\}$$

is positive. Therefore, setting

$$F_\rho = \{x \in E_i ; |f_\rho(x)| \geq \varepsilon\} ,$$

we have $E_i = F_1 \cup \ldots \cup F_r$, and each F_ρ is bounded in $V_\rho = \operatorname{Spec} A_{f_\rho}$. Proceeding in the same way with all $U_i \in \mathfrak{U}$, we see that $\mathfrak{V}$ satisfies condition (b) of Definition 2 if $\mathfrak{U}$ does. □

We want to give two immediate applications of the above lemma, the first one saying that the image of a bounded set is bounded again and the second one that the notion of boundedness, in some sense, is compatible with extensions of the field K.

Proposition 4. *Let $R \subset R'$ be a faithfully flat extension of discrete valuation rings with fields of fractions K and K' and consider a K-morphism $f: X_K \longrightarrow Y_K$ between K-schemes of finite type. Then, for any bounded subset $E \subset X_K(K')$, its image under $X_K(K') \longrightarrow Y_K(K')$ is bounded in Y_K.*

Proposition 5. *Let $R \subset R'$ be a faithfully flat extension of discrete valuation rings with fields of fractions K and K'. Furthermore, let X_K be a K-scheme of finite type. Then a subset $E \subset X_K(K')$ is bounded in X_K if and only if the corresponding subset $E' \subset X_{K'}(K')$ is bounded in $X_{K'}$.*

Both assertions are obvious in the affine case; the reduction to this case is done using Lemma 3. Next we want to show that properness always implies boundedness.

Proposition 6. *Let $R \subset R'$ be a faithfully flat extension of discrete valuation rings with fields of fractions K and K', and consider a proper K-scheme X_K. Then any subset $E \subset X_K(K')$ is bounded in X_K.*

Proof. Let us begin with the remark that the notion of boundedness as introduced in Definition 2 works just as well without the discreteness assumption if we restrict to faithfully flat extensions of valuation rings $R \subset R'$ corresponding to valuations of height 1 on K and K'. The above mentioned properties of boundedness remain true. So, for the purposes of the present proposition, we may extend the valuation of K' to an algebraic closure of K' and thereby assume that K' is algebraically closed.

Due to Chow's lemma [EGA II], 5.6.1, there is a surjective K-morphism $Y_K \longrightarrow X_K$, where Y_K is projective. Then, using Proposition 4, we see that it is enough to look at the case where X_K is projective or, more specifically, where $X_K = \mathbb{P}^n_K$ and where $E = \mathbb{P}^n_K(K')$. To do this, fix a set of homogeneous coordinates on $\mathbb{P}^n_K$ and consider the associated standard covering of $\mathbb{P}^n_K$. For $i = 0, \dots n$, let $U_i \simeq \mathbb{A}^n_K$ be the affine open part of $\mathbb{P}^n_K$ where the i-th coordinate does not vanish. Writing points $x \in \mathbb{P}^n_K(K')$ in homogeneous coordinates in the form $x = (x_0, \dots, x_n)$ with $x_0, \dots, x_n \in K'$, we can set

$$E_i := \{x = (x_0, \dots, x_n) \in \mathbb{P}^n_K(K') \,;\, |x_i| = \max(|x_0|, \dots, |x_n|)\}\,.$$

Then $\mathbb{P}^n_K(K') = \bigcup E_i$ with $E_i \subset U_i(K')$ being bounded in U_i. So it follows that $\mathbb{P}^n_K(K')$ is bounded in $\mathbb{P}^n_K$. □

If X_K is a closed subscheme of $\mathbb{A}^n_K$, and if X is its schematic closure in $\mathbb{A}^n_R$, the image of the canonical map

$$X(R') \longrightarrow X_K(K') \subset \mathbb{A}^n_K(K')$$

consists of those points in $X_K(K')$ whose coordinates are bounded by 1. In particular, multiplying coordinate functions on $\mathbb{A}^n_K$ by suitable constants, we can always assume that the image of $X(R') \longrightarrow X_K(K')$ contains a given subset $E \subset X_K(K')$

provided E is bounded in X_K. So, for affine schemes, we see that the following characterization of boundedness is valid:

Proposition 7. *Let $R \subset R'$ be a faithfully flat extension of discrete valuation rings with fields of fractions K and K'. Furthermore, let X_K be a K-scheme (resp. an affine K-scheme) of finite type. Then a subset $E \subset X_K(K')$ is bounded in X_K if and only if there is an R-model (resp. an affine R-model) X of X_K of finite type such that the image of the canonical map $X(R') \longrightarrow X_K(K')$ contains E.*

In particular, taking for R' a strict henselization R^{sh} of R and for K' the field K^{sh} of fractions of R^{sh}, there is an R-model (resp. an affine R-model) X of X_K of finite type satisfying the extension property for étale points if and only if $X_K(K^{sh})$ is bounded in X_K.

Proof. If, in the general case, $E \subset X_K(K')$ is bounded in X_K, one considers an affine open covering $\{U_{i,K}\}$ of X_K and a decomposition $E = \bigcup E_i$ into subsets $E_i \subset U_{i,K}(K')$ which are bounded in $U_{i,K}$. Then one can find an affine R-model U_i of each $U_{i,K}$ such that E_i belongs to the image of $U_i(R') \longrightarrow U_{i,K}(K')$. Gluing the U_i along the generic fibre, one ends up with an R-model X of X_K such that the image of $X(R') \longrightarrow X_K(K')$ contains E.

Remark 8. If X_K is a separated K-scheme, the R-model X we obtain in Proposition 7 will not, in general, be separated. It requires substantial extra work to modify X in such a way that it becomes separated; see 3.5/6.

Using the approximation theorem of Greenberg [2], we want to add here a non-trivial criterion for boundedness.

Proposition 9. *Let R be an excellent henselian discrete valuation ring with field of fractions K and let X_K be an open subscheme of a K-scheme $\bar{X}_K$ of finite type. Furthermore, consider a subset $E \subset X_K(K)$ which is bounded in $\bar{X}_K$. Then, if $(\bar{X}_K - X_K)(K) = \emptyset$, the set E is bounded in X_K, too.*

Proof. We may assume that $\bar{X}_K$ is affine. Let $\bar{X} = \operatorname{Spec} \bar{A}$ be an affine R-model of $\bar{X}_K$ such that each point of E extends to an R-valued point of $\bar{X}$. Furthermore, let Z be the schematic closure of $\bar{X}_K - X_K$ in $\bar{X}$ so that $X_K = \bar{X}_K - Z_K$. Therefore $Z(K)$ and, thus, also $Z(R)$ are empty. Now fix a uniformizing element π of R and set $R_n = R/(\pi^n)$. It follows then from Greenberg [2], Cor. 2, that $Z(R_n)$ is empty if n is large enough. Therefore, if Z is defined in $\bar{X}$ by the elements $f_1, \dots, f_r \in \bar{A}$, we must have

$$\max\{|f_1(x)|, \dots, |f_r(x)|\} > |\pi^n|$$

for all $x \in X_K(K)$.

Using the latter fact, it is easy to show that $E \subset X_K(K)$ is bounded in X_K. Namely set

$$E_i = \{x \in E; |f_i(x)| > |\pi^n|\}.$$

Then E is the union of the E_i and X_K is the union of the affine open subschemes $\operatorname{Spec} \bar{A}_K[f_i^{-1}]$. Furthermore, since E_i is bounded in $\bar{X}_K$, it is obvious that E_i is bounded in $\operatorname{Spec} \bar{A}_K[f_i^{-1}]$. Thus E is bounded in X_K. □

Each separated K-scheme of finite type X_K admits a compactification; i.e., there is a proper K-scheme $\bar{X}_K$ containing X_K as a dense open subscheme; cf. Nagata [1], [2]. If there exists a compactification with $(\bar{X}_K - X_K)(K) = \varnothing$, we say that X_K has no rational point at infinity. Using this terminology, we can conclude from Propositions 6 and 9:

Corollary 10. *Let R be an excellent discrete valuation ring with field of fractions K and let X_K be a separated K-scheme of finite type with no rational point at infinity. Then $X_K(K)$ is bounded in X_K.*

1.2 Néron Models

In the following, let S be a Dedekind scheme with ring of rational functions K. Considering a smooth and separated K-scheme X_K of finite type, we are interested in constructing S-models X of X_K which are smooth, separated, and of finite type over S. Furthermore, we may ask if among all such models X one can select a minimal one; i.e., an S-model X such that for any other S-model Y of this type there is a unique morphism $Y \longrightarrow X$ restricting to the identity on the generic fibre. Requiring this mapping property for arbitrary smooth S-schemes Y, we arrive at the notion of Néron models.

Definition 1. *Let X_K be a smooth and separated K-scheme of finite type. A Néron model of X_K is an S-model X which is smooth, separated, and of finite type, and which satisfies the following universal property, called Néron mapping property:*

For each smooth S-scheme Y and each K-morphism $u_K : Y_K \longrightarrow X_K$ there is a unique S-morphism $u : Y \longrightarrow X$ extending u_K.

The restriction to schemes of finite type is not really necessary. In Chapter 10 we will consider Néron models, so-called Néron lft-models, which are locally of finite type (by the smoothness condition), but not necessarily of finite type. However, adding the finiteness condition simplifies things to a certain extent. In many important cases, Néron models are automatically of finite type; see, for example, the case of abelian varieties.

As a first step towards Néron models, we will have to consider a weaker form, so-called *weak Néron models* of X_K. Thereby we understand smooth S-models X of finite type which satisfy the extension property for étale points 1.1/1; see also 3.5/1 for the definition we will work with in later chapters.

We want to list some elementary properties of Néron models which follow immediately from the definition.

Proposition 2. *Let X be a smooth and separated S-scheme which is a Néron model of its generic fibre X_K.*

(a) *X is uniquely determined by X_K, up to canonical isomorphism.*

(b) *X is a weak Néron model of its generic fibre; in particular, it satisfies the extension property for étale points.*

(c) *The formation of Néron models commutes with étale base change; i.e., if $S' \longrightarrow S$ is an étale morphism and if K' is the ring of rational functions on S', then $X_{S'} = X \times_S S'$ is a Néron model over S' of the K-scheme $X_{K'} = X_K \times_K K'$.*

Proof. Assertion (a) follows immediately from the Néron mapping property. The same is true for assertion (b) (modulo a limit argument as provided by Lemma 5 below); one has to apply the Néron mapping property to schemes Y which are étale over S. To verify assertion (c), we only have to show the Néron mapping property for $X_{S'}$. So consider a smooth S'-scheme Y' and a K'-morphism $Y'_{K'} \longrightarrow X_{K'}$. Composing the latter morphism with the projection $X_{K'} \longrightarrow X_K$, we obtain a K-morphism $Y'_{K'} \longrightarrow X_K$ which uniquely extends to an S-morphism $Y' \longrightarrow X$ since X is a Néron model of X_K; namely, Y' is smooth over S since the composition of the structural morphism $Y' \longrightarrow S'$, which is smooth, with the étale morphism $S' \longrightarrow S$ is smooth again. Now $Y' \longrightarrow X$ yields an S'-morphism $Y' \longrightarrow X_{S'}$ and the latter is a unique extension of the K'-morphism $Y'_{K'} \longrightarrow X_{K'}$. □

Next, we mention that the notion of Néron models is local on the base:

Proposition 3. *Let S be a Dedekind scheme and let (S_i) be an open covering of S. Furthermore, let X be an S-scheme. Then X is a Néron model of its generic fibre if and only if, for each i, the same is true for the S_i-scheme $X \times_S S_i$.*

In the above assertion, one can replace the open subschemes $S_i \subset S$ by the localizations of S at closed points. However, then it is necessary to require the scheme we start with to be of finite type.

Proposition 4. *Let S be a Dedekind scheme and let X be an S-scheme of finite type. Then the following assertions are equivalent:*

(a) *X is a Néron model of its generic fibre.*

(b) *For each closed point $s \in S$, the $\mathcal{O}_{S,s}$-scheme $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ is a Néron model of its generic fibre.*

If we want to verify the implication (a) $\Longrightarrow$ (b), we cannot just apply an argument of base change as provided by Proposition 2 (c). The reason is that $\operatorname{Spec} \mathcal{O}_{S,s}$ is a limit of open subschemes of S but not, in general, an étale extension of S. So we will have to combine limit arguments with arguments of base change. Let us mention the necessary facts on limits.

Lemma 5 ([EGA IV_3], 8.8.2). *Let S be a base scheme and let s be a point of S.*

(a) *Let X and Y be S-schemes which are of finite presentation. Then the canonical map*

$$\varinjlim \mathrm{Hom}_{S'}(X \times_S S', Y \times_S S') \longrightarrow \mathrm{Hom}_{\mathcal{O}_{S,s}}(X \otimes_S \mathcal{O}_{S,s}, Y \otimes_S \mathcal{O}_{S,s})$$

is bijective, the direct limit being taken over all open neighborhoods S' of s in S.

(b) *Let $X_{(s)}$ be an $\mathcal{O}_{S,s}$-scheme of finite presentation. Then there are an open neighborhood S' of s in S and an S'-scheme X' of finite presentation such that $X' \otimes_{S'} \mathcal{O}_{S,s}$ is isomorphic to $X_{(s)}$.*

Proof of Proposition 4. To verify the implication (a) $\Longrightarrow$ (b), pick a point $s \in S$ and write $X_{(s)} = X \otimes_S \mathcal{O}_{S,s}$. Let K be the field of fractions of $\mathcal{O}_{S,s}$. It is only to show that $X_{(s)}$ satisfies the Néron mapping property. So consider a K-morphism $u_K : Y_{(s),K} \longrightarrow X_{(s),K}$ where $Y_{(s)}$ is a smooth $\mathcal{O}_{S,s}$-scheme; we may assume that $Y_{(s)}$ is of finite type and, thus, of finite presentation over $\mathcal{O}_{S,s}$. Then we can extend $Y_{(s)}$ to a scheme Y' over a connected open neighborhood $S' \subset S$ of s and, taking S' small enough, we may even suppose that Y' is smooth just as $Y_{(s)}$ is; cf. the definition of smoothness in 2.2/3. Using the fact that $X' := X \times_S S'$ is a Néron model of its generic fibre, it follows that u_K extends uniquely to an S'-morphism $u' : Y' \longrightarrow X'$. Then $u' \otimes_{S'} \mathcal{O}_{S,s} : Y_{(s)} \longrightarrow X_{(s)}$ is a unique $\mathcal{O}_{S,s}$-morphism extending u_K. So $X_{(s)}$ is a Néron model of its generic fibre.

The opposite implication (b) $\Longrightarrow$ (a) is obtained similarly. Let K be the ring of rational functions on S and consider a K-morphism $u_K : Y_K \longrightarrow X_K$ where Y is a smooth S-scheme. Again we may assume that Y is of finite type and, thus, of finite presentation over S. Then condition (b) implies that, over a neighborhood $S(s)$ of each closed point $s \in S$, the morphism u_K extends uniquely to an $S(s)$-morphism $u(s) : Y \times_S S(s) \longrightarrow X \times_S S(s)$. Gluing all $u(s)$ yields a unique S-morphism $u : Y \longrightarrow X$ extending u_K. Since the smoothness and the separatedness of the $\mathcal{O}_{S,s}$-scheme $X \otimes_S \mathcal{O}_{S,s}$ imply the smoothness and separatedness of X over a neighborhood of s, we see that X is a Néron model of X_K. □

In the situation of condition (a) of Proposition 4 we will say that X is a *global* Néron model of the generic fibre X_K whereas in the situation of condition (b) the schemes $X \times_S \mathrm{Spec}\, \mathcal{O}_{S,s}$ will be called the *local* Néron models of X_K. Thus we see that if X_K admits a global Néron model, all its local Néron models exist. The converse of this assertion is not true as we will see in 10.1/11.

A further consequence of the Néron mapping property is the fact that Néron models respect group schemes.

Proposition 6. *Let X be a smooth and separated S-scheme which is a Néron model of its generic fibre X_K. Assume that X_K is a K-group scheme. Then the group scheme structure of X_K extends uniquely to an S-group scheme structure on X.*

Remark 7. When dealing with group schemes, the separatedness occurring as a condition in Definition 1 is superfluous. Indeed, a group scheme is separated over its base as soon as the unit section is a closed immersion; cf. 7.1/2. So group schemes over fields are automatically separated. Furthermore, let X be a smooth S-group scheme of finite type which satisfies the Néron mapping property. In order to show that X is separated over S, we may apply Proposition 4 and thereby assume that S is local. Then, due to the Néron mapping property, the unit section $\mathrm{Spec}\, K \longrightarrow X_K$ of the generic fibre X_K extends uniquely to a section $S \longrightarrow X$, namely to the unit section of X. It follows that the latter is a closed immersion, as can be seen from 7.1/1 and its proof. Thus X is separated as claimed.

Although Néron models have been defined within the setting of schemes, their importance seems to be restricted to group schemes or, more generally, to torsors under group schemes as we will see in Chapter 6. For example, $\mathbb{P}^1_K$ admits $\mathbb{P}^1_S$ as a smooth and separated S-model which, due to the properness, satisfies the extension property for étale points. But $\mathbb{P}^1_S$ is not a Néron model of its generic fibre since not all K-automorphisms of $\mathbb{P}^1_K$ extend to S-automorphisms of $\mathbb{P}^1_S$; cf. 3.5/5. The situation is much better in the group scheme case as can be seen from an extension theorem of Weil for rational maps into group schemes; cf. 4.4/1:

Let $u : Y \dashrightarrow X$ be a rational map between S-schemes where Y is smooth and where X is a smooth and separated S-group scheme. Then, if u is defined in codimension ≤ 1, it is defined everywhere.

Using this result, one can show without difficulties that abelian schemes over S, i.e., proper and smooth S-group schemes with connected fibres, provide examples of Néron models.

Proposition 8. *Let X be an abelian scheme over S. Then X is a Néron model of its generic fibre X_K.*

Proof. Let Y be a smooth S-scheme and let $u_K : Y_K \longrightarrow X_K$ be a K-morphism. We claim that u_K extends to a rational map $u : Y \dashrightarrow X$ with a domain of definition $V \subset Y$ which is S-dense; i.e., which is dense in each fibre of Y over S. Namely, consider a closed point $s \in S$ and a generic point ζ of the fibre over s in Y. Then the local ring $\mathcal{O}_{Y,\zeta}$ is a discrete valuation ring; cf. 2.3/9. So the valuative criterion of properness implies that u_K extends to a morphism $\mathrm{Spec}\, \mathcal{O}_{Y,\zeta} \longrightarrow X$ or, using Lemma 5, to a rational map $Y \dashrightarrow X$ which is defined in a neighborhood of ζ. Therefore u is defined in codimension ≤ 1 and, thus, by Weil's extension theorem, it is defined everywhere. The uniqueness of the extension follows from the separatedness of X. □

We have seen that Néron models satisfy the extension property for étale points. On the other hand, using a similar argument as the one given in the above proof, one can show that a smooth and separated group scheme satisfying the extension property for étale points is already a Néron model; see also 7.1/1.

Criterion 9. *Let X be a smooth and separated S-group scheme of finite type. Then X is a Néron model of its generic fibre if and only if X satisfies the extension property for étale points.*

Describing the necessary steps of the *proof*, we mention first of all that, due to Proposition 4, the criterion has only to be verified in the local case. So assume that

S is a local scheme. Then one has to use the fact that X, as a weak Néron model of its generic fibre, satisfies the so-called weak Néron mapping property; cf. 3.5/3. The latter means that each K-morphism $u_K : Y_K \longrightarrow X_K$ extends to an S-rational map $u : Y \dashrightarrow X$; i.e., to a rational map which is defined on an S-dense open subscheme of Y. So, just as in the case of abelian schemes, the if-part of the assertion is reduced to Weil's extension theorem for morphisms into group schemes. □

1.3 The Local Case: Main Existence Theorem

As we have seen in 1.2/4, the existence of a Néron model over a global Dedekind scheme S implies the existence of the local Néron models at closed points of S. In fact, if global Néron models are to be constructed, the first step is to obtain all local ones. Then one can try to glue them in order to build a global model; see Section 1.4 for the case of abelian varieties. The purpose of the present section is to present the existence theorem for Néron models in the local case.

Theorem 1. *Let R be a discrete valuation ring with field of fractions K, with a strict henselization R^{sh}, and with field of fractions K^{sh} of R^{sh}. Let X_K be a smooth K-group scheme of finite type. Then X_K admits a Néron model X over R if and only if $X_K(K^{sh})$ is bounded in X_K.*

In particular, since properness implies boundedness, abelian varieties admit Néron models in the local case:

Corollary 2. *Let A_K be an abelian variety over the field of fractions K of a discrete valuation ring R. Then A_K admits a Néron model over R.*

The only-if-part of Theorem 1 is a trivial consequence of 1.1/7 since Néron models are of finite type. The proof of the if-part, however, is more complicated and will be carried out in Chapters 3 to 6, each one of them dealing with a certain aspect of the construction of local Néron models. At this place we have to content ourselves with a simplified description of the necessary steps.

We start the construction by choosing a separated R-model X of X_K of finite type which satisfies the extension property for étale points. If X_K is projective, we can take for X the schematic closure of X_K in a projective n-space over R. Similarly, if X_K is affine, we may use the boundedness condition and take for X the schematic closure of X_K in a suitable affine n-space over R. In the general case we use 1.1/7. Since the model X obtained from 1.1/7 might not be separated and since we want to avoid the result 3.5/7 saying that a separated R-model can be found, we will generalize the situation slightly in Chapters 3 and 4 by working with a finite family (X_i) of *separated* R-models of X_K such that the canonical map

$$\coprod X_i(R^{sh}) \longrightarrow X_K(K^{sh})$$

is surjective.

For simplicity, let us consider a separated R-model $X^{(1)}$ of finite type of X_K satisfying the extension property for étale points. Then we apply the so-called smoothening process to $X^{(1)}$, which will be explained in Chapter 3. Thereby we obtain a proper R-morphism $X^{(2)} \longrightarrow X^{(1)}$ consisting of a sequence of blowing-ups with centers in special fibres. It has the property that each R^{sh}-valued point of $X^{(1)}$ lifts to an R^{sh}-valued point of $X^{(2)}$ which factors through the smooth locus $X^{(2)}_{\text{smooth}}$ of $X^{(2)}$; cf. 3.1/3. Thus $X^{(3)} := X^{(2)}_{\text{smooth}}$ is a *smooth* R-model of finite type of X_K which satisfies the extension property for étale points. In other words, $X^{(3)}$ is a weak Néron model of X_K. It satisfies the so-called weak Néron mapping property which means that, for each smooth R-scheme Y and each K-morphism $u_K : Y_K \longrightarrow X^{(3)}_K$, there is an R-rational extension $u : Y \dashrightarrow X^{(3)}$; i.e., a rational extension which is defined on an R-dense open part of Y; cf. 3.5/3. Hence $X^{(3)}$ satisfies certain aspects of a Néron model. However, weak Néron models are not unique and it might be that the group structure of X_K does not extend to a group scheme structure on $X^{(3)}$. Thus, one cannot expect that $X^{(3)}$ is already a Néron model of X_K.

In general, it is necessary to modify $X^{(3)}$. This can be done by using the group structure on X_K; cf. Section 4.3. To simplify the notation, write X instead of $X^{(3)}$. Furthermore, let π be a uniformizing element of R, and let $k = R/\pi R$ be the residue field of R. Fixing a non-trivial left-invariant differential form ω on X_K of degree $d = \dim X_K$, we define its π-order over each component Y_k of the special fibre X_k of X. Namely, let η be the generic point of Y_k. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with uniformizing element π. Since the sheaf of relative differential forms $\Omega^d_{X/R}$ is a line bundle, there is an integer n such that $\pi^{-n}\omega$ extends to a generator of $\Omega^d_{X/R}$ at η, and we can set $\operatorname{ord}_{Y_k} \omega := n$. Then the ω-minimal components of X_k, i.e., those components for which the π-order of ω is minimal, are uniquely determined by X_K up to R-birational isomorphism. They occur in each weak Néron model of X_K and have to be interpreted as the components which have largest volume. More precisely, any isomorphism $u_K : X_K \longrightarrow X_K$, which leaves ω invariant, extends to an R-rational map $X \dashrightarrow X$ which maps the ω-minimal components of X_k birationally onto each other; cf. 4.3/2. So if X' is the open subscheme obtained from X by removing all non-minimal components of the special fibre X_k, the isomorphism u_K gives rise to an R-birational map $X' \dashrightarrow X'$ which even is an open immersion on its domain of definition; see 4.3/1 (ii). Applying this argument to general translations on X_K, one can realize that the group multiplication $m_K : X_K \times X_K \longrightarrow X_K$ extends to an R-birational map $m : X' \times X' \dashrightarrow X'$. In fact, m defines a so-called R-birational group law on X'; cf. 4.3/5. The R-scheme X' is, as we will see in the end (cf. 4.4/4), already an R-dense open subscheme of the Néron model we are going to construct, although X' will not, in general, satisfy the extension property for étale points any more.

Now a Néron model of X_K can be derived from X' by considering its "saturation" under the birational group law. There is a standard procedure, first invented by Weil for the case where the base consists of a field and then generalized by A. Néron and M. Artin, which associates group schemes to R-birational group laws. We will explain it in Chapter 5 for the case where the base ring R is strictly henselian; the generalization to an arbitrary discrete valuation ring is done in Chapter 6 by means of descent. Thereby we will see, cf. 5.1/5, that X' can be enlarged to an R-group scheme X'' which is an R-model of X_K of finite type and which has the property that the group multiplication on X'' restricts to the R-birational group law m on X'. Then one uses a translation argument to show that X'' satisfies the extension property for étale points so that X'' is a Néron model of X_K by Criterion 1.2/9.

1.4 The Global Case: Abelian Varieties

In the preceding section we have discussed the existence of Néron models in the local case. If a global Néron model is to be constructed, one has to find a way to glue the local Néron models. The problem is that the resulting global model might not be of finite type again, a property which is necessary for Néron models. However, as we want to show in the present section, when dealing with abelian varieties the gluing works well and we do obtain global Néron models this way. To start with, let us state Proposition 1.2/4, which describes the relationship between local and global Néron models, in a form which is more useful for applications.

Proposition 1. *Let S be a Dedekind scheme with ring of rational functions K and let X_K be a smooth and separated K-scheme of finite type. Then the following assertions are equivalent:*

(a) *There exists a global Néron model X of X_K over S.*

(b) *There exists a dense open subscheme $S' \subset S$ such that X_K admits a Néron model over S' as well as local Néron models at the finitely many closed points of $S - S'$.*

Proof. The implication (a) $\Longrightarrow$ (b) is trivial, due to 1.2/3 and 1.2/4. To obtain the opposite, we may assume that S is connected. Let $s_1, \ldots, s_r$ be the closed points which form the complement of S' in S and let X' be a Néron model of X_K over S'. Furthermore, let $X_{(s_i)}$ be a local Néron model of X_K over the ring $\mathcal{O}_{S,s_i}$. Then, using 1.2/5, $X_{(s_i)}$ extends to a smooth and separated scheme of finite type X_i over a suitable open neighborhood S_i of s_i. Since X_i and X' coincide at the generic point of S, both must coincide over a non-empty open part of S'. Removing finitely many closed points from S_i, we may assume that $S_i \cap (S - S') = \{s_i\}$ and that X_i coincides with X' over $S' \cap S_i$. But then we can glue each X_i with X' over $S' \cap S_i$ to obtain a smooth and separated S-model X of finite type satisfying $X \times_S S' = X'$ and $X \otimes_S \mathcal{O}_{S,s_i} = X_{(s_i)}$. Thus X is a global Néron model of X_K by 1.2/4. □

Now consider a connected Dedekind scheme S with field of rational functions K and an abelian variety A_K over K. One says that A_K has *good reduction at a closed point $s \in S$* if A_K extends to a smooth and proper scheme $A_{(s)}$ over $\mathcal{O}_{S,s}$. We want to show that $A_{(s)}$ is automatically an abelian scheme in this case and, thus, a Néron model of A_K.

Proposition 2. *Let S be a connected Dedekind scheme with field of fractions K and let A_K be an abelian variety over K. Assume that A_K extends to an S-scheme A which is smooth and proper. Then A is an abelian scheme under a group structure which extends the given group structure on A_K. In particular, A is a Néron model of A_K.*

Proof. Using 1.2/4 we may assume that we are in the local case where S consists of a discrete valuation ring. Since A is proper, the valuative criterion of properness shows that A is already a weak Néron model of A_K. Furthermore, the special fibre A_k of A is connected by [EGA III$_1$], 5.5.1. Therefore A_k has to be viewed as an ω-minimal component, with ω being a generating differential form of degree $\dim A_K$ on A_K; use the weak Néron mapping property 3.5/3 and the result 4.3/1. On the other hand, we know from 1.3/2 that A_K admits a Néron model X. Thus, by the Néron mapping property, there is a canonical S-morphism $A \longrightarrow X$ which is an open immersion by 4.3/1 (ii) or 4.4/1. Because A is proper, its image is closed in X. However, X is connected due to the fact that X is flat over S, with the generic fibre $X_K = A_K$ being connected. So $A \longrightarrow X$ is an isomorphism and A is a Néron model of A_K. Thus, applying the Néron mapping property, the group structure of A_K extends to a group scheme structure on A and A is seen to be an abelian scheme. □

In order to apply Proposition 1 in the case of abelian varieties A_K, we have to show that A_K has good reduction at almost all closed points of S and even more: that A_K extends to an abelian scheme A' over a dense open subscheme S' of S. Looking at a simple example, assume that the characteristic of K is different from 2 and consider the case where A_K is an elliptic curve in $\mathbb{P}^2_K$ given by an equation in Weierstraß form

$$y^2z = x^3 + \beta xz^2 + \gamma z^3$$

with a non-zero discriminant $\Delta = 4\beta^3 + 27\gamma^2$. Then the elements β, γ, Δ, and Δ^{-1} belong to almost all local rings $\mathcal{O}_{S,s}$ at closed points $s \in S$. So there exists a non-empty open subscheme $S' \subset S$ such that β, γ, and Δ extend to sections in $\mathcal{O}_S(S')$ and such that Δ and 2 are invertible in $\mathcal{O}(S')$. Consequently, A_K extends to a smooth projective family A' of elliptic curves in $\mathbb{P}^2_{S'}$. Then A' is an abelian scheme extending A_K as we have shown in Proposition 2. Alternatively, we can apply limit arguments of type 1.2/5 and see directly that, after a possible shrinking of S', the scheme A' gives rise to an abelian scheme over S'. In principle, the same reasoning applies to any abelian variety A_K over K.

Theorem 3. *Let S be a connected Dedekind scheme with field of fractions K and let A_K be an abelian variety over K. Then A_K admits a global Néron model A over S. Furthermore, let S' be the subset of S consisting of the generic point and of all closed points in S where A_K has good reduction. Then S' is a dense open subscheme of S and $A \times_S S'$ is an abelian scheme over S'.*

Proof. We have to show that A_K extends to a smooth and proper scheme over a neighborbood of the generic point of S as well as over a neighborhood of each closed point of S where A_K has good reduction. Then all such schemes are abelian schemes by Proposition 2 and, using the Néron mapping property, they can be glued to give an abelian scheme over S'. Furthermore, due to the existence of local Néron models 1.3/2, we conclude from Proposition 1 that A_K admits a global Néron model A.

In order to show that A_K extends to a smooth proper scheme over a non-empty open part of S, choose a closed embedding $A_K \longrightarrow \mathbb{P}^n_K$ into some projective n-space and consider the schematic closure A of A_K in $\mathbb{P}^n_S$. Then A is smooth over the generic point of S and, thus, smooth over an open neighborhood S'' of this point. So $A'' = A \times_S S''$ is a smooth projective S''-model of A_K. Alternatively, we can use 1.2/5 to extend A_K to a scheme A'' of finite type over an open neighborhood S'' of the generic point in S. If S'' is small enough, A'' will be smooth and, by [EGA IV_3], 8.10.5, also proper. The same argument applies if we consider a closed point $s \in S$ where A_K has good reduction. Namely, then A_K extends to a smooth and proper scheme $A_{(s)}$ over $\mathcal{O}_{S,s}$ and we can extend the latter over an open neighborhood of s. □

It follows from the valuative criterion of properness that any K-rational map $u_K : Y_K \dashrightarrow A_K$ from a smooth K-scheme Y_K into an abelian variety A_K is defined in codimension 1 and, thus, is defined everywhere by Weil's extension theorem 4.4/1. Thereby it is seen that, in the case of abelian varieties, the Néron mapping property can be strengthened.

Proposition 4. *Let S be a connected Dedekind scheme with field of fractions K and let A_K be an abelian variety over K with Néron model A over S. Then, for each smooth S-scheme Y, and for each K-rational map $u_K : Y_K \dashrightarrow A_K$, there is a unique S-morphism $u : Y \longrightarrow A$ extending u_K.*

For further generalizations of this result see 8.4/6 and 10.3/1.

1.5 Elliptic Curves

In order to illustrate the construction of Néron models, we want to look at Néron models of elliptic curves. In this particular case, the procedure of construction can be made quite explicit. The reader who is interested in a more profound discussion of models of elliptic curves is referred to Kodaira [1], Néron [2], and Tate [2]. In our terminology, an elliptic curve will always be understood to have a rational point.

We will work over a base scheme S consisting of a strictly henselian discrete valuation ring R with field of fractions K and with an *algebraically closed* residue field k. First we want to clarify the interdependence between Néron models and regular and proper minimal models of elliptic curves over K. So consider an elliptic curve E_K over K. Then E_K admits a Néron model, as we have stated in 1.3/2. It also admits a proper minimal model. By the latter we mean a proper flat R-model E which is a regular scheme and which is minimal among all models E' of this type in the sense that each R-morphism $E \longrightarrow E'$ which is an isomorphism on generic

fibres is an isomorphism itself. So there are no irreducible components of the special fibre of E which can be contracted without loosing the regularity of E. Regular and proper minimal models of curves are unique; see Abhyankar [1] and Lipman [1] for the existence of regular and proper models and Lichtenbaum [1], Shafarevich [1], or Néron [2] for the existence of regular and proper minimal models.

Proposition 1. *Assume that R is a strictly henselian discrete valuation ring. Let E be a regular and proper minimal model over R of the elliptic curve E_K. Then the smooth locus of E is a Néron model of E_K.*

Proof. Write E' for the smooth locus of E. It follows from 3.1/2 that each R-valued point of E factors through E'. So, by the valuative criterion of properness, we see that E' satisfies the extension property for étale points and, thus, is a weak Néron model of E_K. Furthermore, it follows from 2.3/5 that all k-valued points of the special fibre E'_k lift to R-valued points of E'.

Fix an invariant differential form ω of degree 1 on E_K. We claim that all components of the special fibre E'_k are ω-minimal. To see this, consider two components X_1 and X_2 of E'_k and two k-valued points $y_k \in X_1$ and $z_k \in X_2$. Lift them to R-valued points y, z of E' and restrict them to K-valued points $y_K, z_K \in E_K$. Then the translation by $z_K y_K^{-1}$ is a K-isomorphism of E_K mapping y_K to z_K. Due to the uniqueness of regular and proper minimal models, this isomorphism extends to an R-isomorphism of E and, thus, of E', mapping y onto z. So there are R-isomorphisms of E' which operate transitively on the components of the special fibre E'_k and which leave ω invariant. Consequently, all components of E'_k must be ω-minimal; cf. 4.3/1.

Now, as explained in Section 1.2 or, in more detail, in Section 4.3 and Chapter 5, the group structure on E_K extends to an R-birational group law on E' and, then, to a group scheme structure on a bigger R-scheme E'' containing E' as an R-dense open subscheme; cf. 5.1/5. However, using the fact that all translations by K-valued points on E_K extend to isomorphisms on E', and to the translations by the corresponding R-valued points on E'', it follows that E' and E'' coincide. So E' is a Néron model of E_K. □

If E is a proper and flat R-model of an elliptic curve E_K over K, then E is smooth over R at all points of the generic fibre. Furthermore, E is smooth at a point x of the special fibre E_k if and only if this fibre is smooth over k at x, or equivalently since k is algebraically closed, if and only if E_k is regular at x. So, in order to pass to the smooth locus of E, one removes all irreducible components with multiplicities > 1 from E_k as well as from the remaining part of E_k all singular points; the latter form a finite set. For algebraically closed residue field k, special fibres of regular and proper minimal models of elliptic curves have been classified by Néron [2], see also Kodaira [1]; there is only a finite list of possible types. An algorithm to compute the type of the special fibre from a given equation for E_K has been given in Tate [2].

If one is interested in a Néron model E of an elliptic curve E_K and not so much in its regular and proper minimal model, one can construct E directly without too much effort starting out from an equation describing E_K in $\mathbb{P}^2_K$, at least when the residue characteristic of K is different from 2 and 3. To do this, one classifies Weierstraß equations into a finite list of types, according to certain conditions involving the values of their coefficients, discriminants, and j-invariants. Then one can construct the Néron model E by direct computation in each of these cases. To demonstrate this, assume that R is a strictly henselian discrete valuation ring with residue characteristic char k different from 2 and 3 and consider an elliptic curve E_K over K, defined in $\mathbb{P}^2_K$ by an equation in Weierstraß form

$$(*) \qquad y^2z = x^3 + \beta xz^2 + \gamma z^3 .$$

Then discriminant Δ and j-invariant j are given by

$$\Delta = 4\beta^3 + 27\gamma^2 , \qquad j = 2^6 \cdot 3^3 \cdot 4\beta^3/\Delta .$$

Viewing E_K as a group scheme, we assume that the point $(0, 1, 0)$ defines the unit section of E_K. Let π be a uniformizing element of R, and let $v: K \longrightarrow \mathbb{Z}$ be the additive valuation given by R which satisfies $v(\pi) = 1$. We need some elementary properties of the equation $(*)$.

Lemma 2. *For $n \in \mathbb{Z}$, the change of homogeneous coordinates in $\mathbb{P}^2_K$*

$$(x, y, z) \longmapsto (\pi^{-2n}x, \pi^{-3n}y, z)$$

induces on the equation of E_K the change

$$\beta \longmapsto \pi^{4n}\beta , \qquad \gamma \longmapsto \pi^{6n}\gamma , \qquad \Delta \longmapsto \pi^{12n}\Delta .$$

Lemma 3. (a) *If $v(j) \geq 0$, then $v(\Delta) = \min(v(\beta^3), v(\gamma^2))$. In particular, $v(\Delta) \equiv 0\ (2)$ or $v(\Delta) \equiv 0\ (3)$.*

(b) *If $v(j) < 0$, then $v(\Delta) > v(\beta^3) = v(\gamma^2)$. In particular, $v(\beta) \equiv 0\ (2)$ and $v(\gamma) \equiv 0\ (3)$.*

Making a change of coordinates as described in Lemma 2, we can assume that the coefficients β and γ of $(*)$ belong to R and, furthermore, that $\min(v(\beta^3), v(\gamma^2))$ is minimal. Thereby we arrive at a so-called minimal Weierstraß equation of E_K; i.e., at a Weierstraß equation with coefficients in R such that $v(\Delta)$ is minimal. We list the possible cases which remain.

Lemma 4. *Let the equation $(*)$ be a minimal Weierstraß equation for E_K. Then, if $v(j) \geq 0$, we have $v(\Delta) \in \{0, 2, 3, 4, 6, 8, 9, 10\}$. Furthermore, if $v(j) < 0$, either $v(\beta) = v(\gamma) = 0$, or $v(\beta) = 2$ and $v(\gamma) = 3$.*

Using Néron's symbols as introduced in his table [2], p. 124/125, the possibilities for a minimal Weierstraß equation for E_K as mentioned in the above lemma split into the following subcases; see also the table in Tate [2], p. 46.

(a) $\quad v(j) \geq 0 , \quad v(\Delta) = 0$

(b_m) $\quad v(j) = -m < 0 , \quad v(\beta) = v(\gamma) = 0$

(c1) $\quad v(j) \geq 0 , \quad v(\Delta) = 2$

(c2) $\quad v(j) \geq 0 , \quad v(\Delta) = 3$

(c3) $\quad v(j) \geq 0 , \quad v(\Delta) = 4$

(c4) $\quad v(j) \geq 0 , \quad v(\Delta) = 6$

($c5_m$) $\quad v(j) = -m < 0 , \quad v(\beta) = 2 , \quad v(\gamma) = 3$

(c6) $\quad v(j) \geq 0 , \quad v(\Delta) = 8$

(c7) $\quad v(j) \geq 0 , \quad v(\Delta) = 9$

(c8) $\quad v(j) \geq 0 , \quad v(\Delta) = 10$

Now, to construct a Néron model of E_K, one proceeds as follows. One chooses a minimal Weierstraß equation for E_K and uses it for the definition of an R-model $\bar{E}$ of E_K in $\mathbb{P}^2_R$. Let E^0 be the smooth part of $\bar{E}$. Then one verifies by direct computation, or by using general properties of planar cubics, that E^0 is a smooth R-group scheme extending E_K. In fact, we will see that it is the so-called identity component of the Néron model of E_K. There are three possibilities which we characterize by the first letters of Néron's symbols:

(a) $v(\Delta) = 0$. Then $\bar{E}$ is smooth, so $E^0 = \bar{E}$ is an abelian scheme extending E_K. It follows that E_K is an elliptic curve with good reduction and that $\bar{E}$ is its Néron model.

(b) $v(\Delta) > 0$ and $\min(v(\beta), v(\gamma)) = 0$. Then $\bar{E}$ is not smooth; the special fibre of E^0 is the multiplicative group $\mathbb{G}_{m,k}$.

(c) $v(\Delta) > 0$ and $\min(v(\beta), v(\gamma)) > 0$. Also in this case, $\bar{E}$ is not smooth; the special fibre of E^0 is the additive group $\mathbb{G}_{a,k}$.

Consider the invariant differential $\omega = \dfrac{dx}{y}$ on E_K. Then ω has π-order 0 over E^0. We claim that, for the construction of the Néron model of E_K, it is enough to extend E^0 into a weak Néron model E of E_K with the property that the special fibre of E consists of ω-minimal components, all of them being isomorphic to E^0_k.

Lemma 5. *Let $E^1, \ldots, E^r$ be smooth and separated R-models of E_K. Assume that, for all ρ, the special fibre E^ρ_k, as a k-scheme, is isomorphic to E^0_k, that ω has π-order 0 over E^ρ_k, and that the canonical map $\coprod_{\rho=0}^{r} E^\rho(R) \longrightarrow E_K(K)$ is bijective. Then, gluing the E^ρ along the generic fibre E_K, we obtain a Néron model E of E_K. Furthermore, E^0 is the identity component of E.*

Proof. It is clear that E is a smooth R-model of finite type of E_K which satisfies the extension property for étale points 1.1/1. So E is a weak Néron model of E_K. Furthermore, E is separated since, for $\rho \neq \tau$, the intersection of $E^\rho \times_R E^\tau$ with the diagonal in $E \times_R E$ is just E_K. By the assumption on the π-order of ω, all components of the special fibre E_k are ω-minimal. So, denoting by N the Néron model of E_K, we have an open immersion $E \hookrightarrow N$ by 4.3/1 or 4.4/4. Then E^0 must coincide with the identity component N^0 of the Néron model N. Thereby we see that the special fibre N_k consists of $r + 1$ copies of E^0_k which, in case (c) is the affine

1-space $\mathbb{A}^1_k$, and in case (b) is $\mathbb{A}^1_k$ minus the origin. Since the same is true for E, we conclude from the special type of E^0_k that $E \hookrightarrow N$ is bijective. So E is a Néron model of E_K. □

In each of Néron's cases, a Néron model E of E_K can be constructed via the above lemma. To show how to proceed, we will look at the cases (c1) and (c2) which are quite simple, as well as at case (b_m) which is more complicated. First note that $e_k := (0, 1, 0) \in \bar{E}(k)$ is a non-singular point of the special fibre of $\bar{E}$; in fact, it is the unit section of E^0_k. So the singularities of $\bar{E}_k$ belong to the affine part $\bar{E}_z$ of $\bar{E}$ which is described in $\mathbb{A}^2_R$ by the equation

$$(**) \qquad y^2 = x^3 + \beta x + \gamma \, .$$

There is precisely one singularity p_k in $\bar{E}_{z,k}$ in the cases (b) or (c); it corresponds to a multiple zero of the right-hand side of (**). So, in order to apply Lemma 5, we have to concentrate on R-models $E^\dagger$ of E_K such that the image of $E^\dagger(R) \longrightarrow E_K(K)$ consists of K-valued points which, in $\bar{E}$, specialize into the singular point p_k.

Case (c1). Then $v(\beta) \geq 1$ and $v(\gamma) = 1$ by Lemma 3; hence $p_k = (0, 0)$, using affine coordinates of $\bar{E}_{z,k}$. Since

$$\{(x, y) \in \bar{E}_z(K); v(x) > 0, v(y) > 0\} = \varnothing \, ,$$

it follows from Lemma 5 that $E^0 = \bar{E} - \{p_k\}$ is the Néron model of E_k. Also it is easily checked that the minimal Weierstraß model is regular and, thus, coincides with the regular and proper minimal model. □

Case (c2). We have $v(\beta) = 1$ and $v(\gamma) \geq 2$ by Lemma 3. Again, $p_k = (0, 0)$ is the singular point of $\bar{E}_{z,k}$. Thus all points $(x, y) \in \bar{E}_z(K)$ which do not extend to R-valued points of E^0 must satisfy $v(x) \geq 1$ and $v(y) \geq 1$. Use $\hat{x} := \pi^{-1}x$ and $\hat{y} := \pi^{-1}y$ as new coordinates and let E^1 be the R-model of E_K obtained by gluing

$$\operatorname{Spec} R[\hat{x}, \hat{y}]/(\hat{y}^2 - \pi\hat{x}^3 - \pi^{-1}\beta\hat{x} - \pi^{-2}\gamma)$$

along its generic fibre to E_K. Then all points $(x, y) \in \bar{E}_z(K)$, which satisfy $v(x) \geq 1$ and $v(y) \geq 1$, extend to R-valued points of E^1. In addition, E^1 is smooth and separated and has special fibre $E^1_k \simeq \mathbb{A}^1_k \simeq E^0_k$ as required. Furthermore, since $\hat{x}$ and $\hat{y}$ do not vanish at the generic point of E^1_k, we see that $\omega = dx/y = d\hat{x}/\hat{y}$ is of π-order 0 over E^1_k. Thus Lemma 5 can be applied. The Néron model of E_K is obtained by gluing E^0 and E^1 along the generic fibre E_K; its special fibre consists of two components. □

We mention here that the process of replacing a variable x by $\hat{x} = \pi^{-1}x$ is a special case of a dilatation, a technique to be applied systematically when we work out the smoothening process in Chapter 3. In fact, the method we have used above for the construction of E is a very explicit form of the smoothening process. It has to be applied in a similar way for treating the remaining cases.

Case (b_m). This case is of special interest if R is complete; then E_K is a so-called Tate elliptic curve. We have $v(j) = -m < 0$, $v(\beta) = v(\gamma) = 0$, and, hence, $v(\Delta) = m > 0$. Furthermore, $E^0_k \simeq \mathbb{G}_{m,k}$. Let us write

$$P(x) = x^3 + \beta x + \gamma$$

for the right-hand side of (**) and $\bar{P}(x)$ for the polynomial obtained from $P(x)$ by reducing coefficients from R to k. Then $\bar{P}(x)$ has a single root $\bar{a} \in k$ and a double root $\bar{b} \in k$. So $p_k = (\bar{b}, 0)$ is the singular point of $\bar{E}_{z,k}$ and all points $(x, y) \in \bar{E}_z(K)$ which do not extend to R-valued points of E^0 must reduce to p_k.

The root $\bar{a}$ lifts to a root $a \in R$ of $P(x)$ since R is strictly henselian. Set $Q(x) := P(x)/(x - a)$. Then $Q(x)$ has coefficients in R and $\bar{Q}(x) = (x - \bar{b})^2$ is the polynomial obtained from it by reducing coefficients from R to k. Extending the valuation v from K to the algebraic closure K^{alg}, the root $\bar{b}$ lifts to two roots $b_1, b_2 \in K^{\mathrm{alg}}$ of $Q(x)$, where $v(a - b_i) = 0$ for $i = 1, 2$. Thus, the discriminant of $P(x)$, which is Δ, coincides with the discriminant of $Q(x)$, up to a unit in R. Since $v(\Delta) = m$, we have

$$v(b_1 - b_2) = m/2 \, .$$

Furthermore, since R is strictly henselian, the extension of v from K to $K(b_1, b_2)$ is unique, just as for complete fields. So $v(b_1) = v(b_2)$. Using an inductive argument on m, interpreted as the value of the discriminant of $Q(x)$, we want to construct R-models $E^1, \ldots, E^{m-1}$ which, together with E^0, will satisfy the conditions of Lemma 5.

To do this, choose an arbitrary lifting $b \in R$ of $\bar{b}$ and use $x - b$ as a new variable instead of x; denote it by x again. The effect is that the singular point $p_k = (\bar{b}, 0)$ is transformed into the origin $(0, 0)$ this way. We will denote transformed polynomials and roots by $P(x)$, $Q(x)$, a, b_i, etc., again, so that

$$P(x) = (x - a)Q(x) \, , \qquad Q(x) = (x - b_1)(x - b_2)$$

where now

$$v(a) = 0 \, , \qquad v(b_1) = v(b_2) \geq 1/2 \, .$$

For $m = 1$ we obtain $v(b_1 - b_2) = 1/2$ and, hence, $v(b_i) = 1/2$. Then each $x \in R\pi$ satisfies $v(P(x)) = 1$ and we see that $P(x)$ cannot have a square root in R. So there are no points $(x, y) \in \bar{E}_z(K)$ satisfying $v(x) \geq 1$ and $v(y) \geq 1$, and we can conclude from Lemma 5 that, in this case, E^0 is already the Néron model of E_K. Furthermore, the minimal Weierstraß model is regular in this case.

If $m > 1$, we use $\pi^{-1}x$ and $\pi^{-1}y$ as new variables, writing x and y for them again. Then, looking for points $(x, y) \in \bar{E}_z(K)$ satisfying $v(x) \geq 1$ and $v(y) \geq 1$, we have to look for integral solutions of the equation

$$y^2 = (a - \pi x) \cdot Q(x) \, ,$$

where we have written $Q(x)$ instead of $\pi^{-2}Q(\pi x)$ again. This way the discriminant of $Q(x)$ has been divided by π^2 so that its value is now $m - 2$. Assume $m = 2$. Then

$$\operatorname{Spec} R[x, y]/(y^2 - (a - \pi x) \cdot Q(x))$$

is smooth over R. Gluing it along its generic fibre to E_K, we obtain an R-model E^1 as required in Lemma 5. Namely, the special fibre of E^1 is

$$\operatorname{Spec} k[x, y]/(y^2 - \bar{a}\bar{Q}(x))$$

with $\bar{Q}(x)$ having two distinct roots in k. So it is $\mathbb{P}^1_k$ minus two closed points and, thus, isomorphic to E^0_k. That the differential ω has π-order 0 over E^1_k, is easily checked. So, for $m = 2$, the Néron model is obtained by gluing E^0 and E^1 along the generic fibre E_K; its special fibre consists of two components.

If $m > 2$, the polynomial $\bar{Q}(x)$ has a root of multiplicity 2 and the scheme

$$\operatorname{Spec} R[x, y]/(y^2 - (a + \pi x) \cdot Q(x))$$

is not smooth over R; its special fibre consists of two affine lines intersecting each other. Removing the intersection point, we can construct two R-models E^1 and E^2 of E_K with special fibre isomorphic to E^0_k each. If $m = 3$, one is reduced to the case considered above where the discriminant of $Q(x)$ has value 1. Thereby it is seen that E^0, E^1, E^2 satisfy the conditions of Lemma 5. If $m > 3$, the value of the discriminant of $Q(x)$ is > 1 and can be reduced by 2 again as shown above. One continues this way until the value of the discriminant of $Q(x)$ is 1 or 0. Thereby one constructs R-models $E^1, \ldots, E^{m-1}$ of E_K which, together with E^0 satisfy the conditions of Lemma 5. So the special fibre of the Néron model E of E_K consists of m components. With a little bit of extra work one can show that the group E_k/E^0_k is cyclic of order m. Also, by means of the arguments we have given, one can determine the regular and proper minimal model of E_K. Its special fibre consists of a chain of m projective lines forming a loop (if $m > 1$) or of a rational curve with a double point (if $m = 1$). In particular, we can thereby see that the regular and proper minimal model of E_K will not be planar if $m > 3$, because a planar cubic cannot have more than 3 components. □

It is useful to look at Tate elliptic curves also from the rigid analytic viewpoint. So let R be a complete discrete valuation ring. We do not need that R is strictly henselian or that the residue field k is perfect. An elliptic curve E_K over K is called a Tate curve if, in the sense of rigid analytic geometry, it can be represented as a quotient $\mathbb{G}_{m,\mathrm{rig}}/q^{\mathbb{Z}}$ where $\mathbb{G}_{m,\mathrm{rig}}$ is the analytification of the multiplicative group $\mathbb{G}_{m,K}$ and where $q \in K^*$ satisfies $m := v(q) > 0$. The quotient $\mathbb{G}_{m,\mathrm{rig}}/q^{\mathbb{Z}}$ can be thought of as being constructed by gluing m annuli of type $\{x \in \mathbb{G}_{m,\mathrm{rig}}; |\pi| \leq |x| \leq 1\}$ in a cyclical way. Using this covering, we can extend $\mathbb{G}_{m,\mathrm{rig}}/q^{\mathbb{Z}}$ into a formal scheme X whose special fibre X_k is a projective line with a double point if $m = 1$ and a chain of m projective lines forming a loop if $m > 1$.

Choosing an effective Cartier divisor D on X whose support is contained in the smooth locus of X and which is very ample on all components of X_k and on the generic fibre X_{rig}, one constructs a projective embedding of X and, thus, an R-model E' of E_K whose formal completion is X. Then it turns out that the smooth locus E of E' is a Néron model of E_K. The special fibre E_k coincides with the smooth locus of X_k and, thus, is an extension of $\mathbb{G}_{m,k}$ by $\mathbb{Z}/m\mathbb{Z}$. See Bosch and Lütkebohmert [3] for a generalization of the construction to abelian varieties.

1.6 Néron's Original Article

We want to give here some analysis of Néron's article "Modèles minimaux des variétés abéliennes sur les corps locaux et globaux"[2] which appeared in 1964 and which serves as a basis for the construction of Néron models as done in this book; see also the lecture [1] given by Néron in 1961 at the Séminaire Bourbaki. Consider an abelian variety A_K over a local field K and think of it as being embedded into a projective space $\mathbb{P}^N_K$. Let X be the schematic closure of A_K in $\mathbb{P}^N_R$ where R is the discrete valuation ring of integers of K. Then X is an R-model of A_K on which integral points might not be read as nicely as possible. Moreover, it will be likely that the group structure of A_K does not extend to the smooth part of X. To obtain R-models of A_K which do not have these disadvantages, Néron had to apply a series of substantial modifications to X and, in doing so, he had to overcome a lot of technical difficulties.

His article is divided into three chapters. The first one develops a language of varieties over discrete valuation rings, taking Weil's "Foundations" [1] as point of departure. The main results are "Théorème 3" on p. 57, which corresponds to our smoothening process (see 3.1/3), and, as a corollary, "Théorème 6" on p. 61, which yields the existence of weak Néron models (see 3.5/2). In the second chapter, one finds the construction of Néron models for abelian varieties or, more generally, for torsors under abelian varieties; Néron uses the terminology "modèle faiblement minimal". The existence of Néron models is asserted in "Théorème 2" on p. 79 for the local case and in "Théorème 4" on p. 87 for the global case. Finally, the third chapter, which is fairly independent of the others, contains the construction of regular proper minimal models for elliptic curves.

Néron's article has to be viewed as a contribution to relative algebraic geometry over a discrete valuation ring; the applications he gives in the global case are easily deduced from the local case. Concerning the construction of Néron models, Chapters 1 and 2 of his article are quite difficult to read. To a substantial extent, this is due to the fact that they are very technical and also to the fact that the terminology Néron applies is not commonly used; it has been abandoned since.

To give some impression of his terminology, let us explain the basic setting considered by Néron. We start with a discrete valuation ring R with maximal ideal $\mathfrak{p}$. Denote by K the field of fractions as well as by k the residue field of R. The latter is assumed to be perfect. Néron, familiar with the notion of generic points in the sense of Weil's "Foundations" [1], works with universal domains on two levels. First he chooses a universal domain $\mathfrak{k}$ for the residue field k and then a universal domain $\mathfrak{K}$ for the field of fractions K. The latter is done in such a way that $\mathfrak{K}$ is a universal domain of the field of fractions of a ring $\mathfrak{R}$ which serves as an "integral" universal domain. To define $\mathfrak{R}$ in the equal characteristic case, he considers a lifting of k to the completion of R as well as a uniformizing element T of R and takes for $\mathfrak{R}$ the formal power series ring $\mathfrak{k}[[T]]$. In the unequal characteristic case, he sets $\mathfrak{R} = \hat{R} \otimes_{W(k)} W(\mathfrak{k})$ where $\hat{R}$ is the completion of R and where W indicates rings of Witt vectors. The interference of Witt vectors is the main reason why the residue field k is assumed to be perfect. Then he works with relative schemes over R, so-called $\mathfrak{p}$-varieties. To be precise, a $\mathfrak{p}$-variety corresponds to a flat R-scheme of finite type; its points have values in the universal domains $\mathfrak{K}$ or $\mathfrak{k}$ or, when considering integral points, in the subring $\mathfrak{R}$ of $\mathfrak{K}$. Such a $\mathfrak{p}$-variety is called $\mathfrak{p}$-simple if it is regular; it is called simple modulo $\mathfrak{p}$ at a point of the special fibre if it is smooth over R at this point. For both notions, Néron discusses the Jacobi criterion.

In the following, we want to examine Néron's approach to the smoothening process as presented in his Chapter 1, without pursuing his terminology any further; we will use the language of schemes, as generally applied in this book. Let X be a flat R-scheme of finite type with a smooth generic fibre X_K and consider R'-valued points of X where R' is a discrete valuation ring over R having same uniformizing element as R. (So R' is of ramification index 1 over R, since the residue field k of R is perfect.) For such points $x \in X(R')$, Néron defines the integer $l(x, X)$ which measures the defect of smoothness of X along x; see his section n°17 starting on p. 35 or our section 3.3. He shows that $l(x, X)$ is bounded as a function of x. Then he works out the smoothening process by relying on two techniques: the first one is a generic smoothening and the second is the theory of pro-varieties.

The generic smoothening can be formulated as follows:

Let $u : \operatorname{Spec} R' \longrightarrow X$ be an R'-valued point of X where R' is as above. Reducing modulo the maximal ideal $\mathfrak{p}$ of R, one obtains a morphism $\bar{u} : \operatorname{Spec} k' \longrightarrow X_k$. Let $\bar{Y}$ be the closure of its image and let $f : \tilde{X} \longrightarrow X$ be the blowing-up of $\bar{Y}$ on X. Then, if $\tilde{u} : \operatorname{Spec} R' \longrightarrow \tilde{X}$ is the lifting of u to $\tilde{X}$, one has

$$l(\tilde{u}, \tilde{X}) < \max(l(u, X), 1)\ .$$

In particular, after a finite repetition, one ends up with a factorization of u through the smooth locus of a blowing-up of X.

The statement may be viewed as an individual smoothening for R'-valued points x of X. In order to obtain some form of smoothening which works simultaneously for several x and R', Néron relies on the technique of pro-varieties; this is one of the most delicate points in his construction. To give a sketch of his approach, consider an affine open part of X and thereby suppose that X is embedded into an affine space $\mathbb{A}^N_R$. Using the coefficients of formal series in $\mathfrak{k}[[T]]$ in the equal characteristic case and Witt coordinates in the unequal characteristic case, Néron introduces on the set of $R/\mathfrak{p}^n$-valued points of $\mathbb{A}^N_R$ a structure of k-variety ${}^n\mathbb{A}^N_k$. Since X has a smooth generic fibre, the image of $X(R)$ in ${}^n\mathbb{A}^N_k$ gives rise to a constructible subset ${}^nX$ and one obtains a projective system of morphisms

$$\cdots \longrightarrow {}^{n+1}X \longrightarrow {}^nX \longrightarrow \cdots$$

defining a k-pro-variety.

The possibility of parametrizing solutions of X modulo $\mathfrak{p}^n$ by a k-variety or, more specifically, of points of X with values in the completion $\hat{R}$ of R by a k-pro-variety, had been systematically studied by M. Greenberg [1] within the context of schemes and representable functors; see also Serre [3]. The technique is referred to as the Greenberg functor. However, since Néron did not use the language of functors, he gave proofs of his own for the facts he needed.

Let us return to the situation of a generic smoothening as above where we consider a blowing-up $f : \tilde{X} \longrightarrow X$ with center $\bar{Y}$. Then there is an induced morphism ${}^nf : {}^n\tilde{X} \longrightarrow {}^nX$ for each n and, taking limits over n, a bijection $\tilde{X}(\hat{R}) \simeq X(\hat{R})$. To obtain a simultaneous smoothening, Néron has to consider partial inverses of the maps ${}^nf$. More precisely, for each n, there is a constructible subset ${}^nY$ of ${}^nX$ given by the points in $X(\hat{R})$ which reduce to points of $\bar{Y}$ and he shows that there is a constructible map ${}^{n+1}Y \longrightarrow {}^n\tilde{X}$ such that the diagram

$$\begin{array}{ccc} {}^{n+1}Y & \longrightarrow & {}^n\tilde{X} \\ \downarrow & & \downarrow {}^nf \\ {}^nY & \hookrightarrow & {}^nX \end{array}$$

commutes. (In the case of Witt coordinates, a map of type ${}^{n+1}Y \longrightarrow {}^n\tilde{X}$ involves radicial morphisms of extracting p-th roots. Later, to overcome this kind of difficulties, Serre [2] worked with quasi-algebraic varieties.)

Now set $l = \max l(x, X)$ where the maximum is taken over all R'-valued points of X and let Z be an irreducible component of ${}^lX$. Combining blowing-ups and shiftings as above, Néron shows the following assertion: there exists a non-empty open part U of Z such that there is a simultaneous smoothening of X with respect to all points of $X(R')$ whose image in ${}^lX$ is already contained in U. Using this assertion, he can finish the smoothening process by a constructibility argument; cf. his "Théorème 3" on p. 57.

The proof we will give for the existence of the smoothening process is basically the same as Néron's, except for the fact that we can avoid using pro-varieties and the Greenberg functor. We do this by establishing a more precise form of the generic smoothening; cf. 3.3/5. Namely, as we will see, considering the blowing-up $f : \tilde{X} \longrightarrow X$, there exists a non-empty open subscheme $V \subset \bar{Y}$, described in terms of differential calculus, such that, for each R'-valued point v of X whose special fibre factors through V, and for the lifting $\tilde{v}$ of v to $\tilde{X}$, we have

$$l(\tilde{v}, \tilde{X}) < \max(l(v, X), 1)\ .$$

Then it is possible to end the smoothening process directly by a constructibility argument without looking at solutions of X modulo higher powers of $\mathfrak{p}$.

At the end of Néron's Chapter 1, there is the discussion of what we call weak Néron models and the measuring of the size of their components. The latter is done with respect to a non-zero differential form ω of maximal degree of X_K. The smoothening process implies that, up to birational equivalence, there are only finitely many components of "maximal volume" with respect to ω. The arguments are the same as we will present them later at the corresponding places in our Chapters 3 and 4.

Let us dicuss now Néron's Chapter 2. It starts with the definition of torsors, or principally homogeneous spaces in his terminology. The definition is given in terms of ternary laws of composition in such a way that the underlying group of the torsor is hidden. Presumably this is done in order not to separate the construction of Néron models into the group case and the case of a torsor under a group scheme. So consider a torsor X_K under an abelian variety A_K over K and a projective R-model X' of X_K. Néron applies the smoothening process to X', restricts to the smooth locus, and removes from the special fibre all irreducible components which do not have maximal volume. The volume is measured with respect to a non-zero invariant differential form of maximal degree on X_K; write X'' for the resulting R-model of X_K. Then he shows that the structure of torsor on X_K extends to a birational law of torsor on X''.

The next step is to show that finitely many "translates" of X'' (defined over certain unramified extensions of R) cover all points of X' with values in unramified extensions R' of R. The same problem occurs in our presentation at the end of the construction of Néron models, where we want to prove their universal mapping property; cf. 4.4/4.

To construct the Néron model X of X_K, it is, of course, necessary to really glue translates of X''; the latter is not a standard procedure since the translates are only defined over certain unramified extensions of R. Starting with an ample invertible sheaf on X'', Néron shows that it extends to an ample invertible sheaf on the translates of X'' and, finally, on the Néron model X. So this part contains in one step the construction of X in terms of gluing translates under the birational law on X'' as well as the descent and the quasi-projectivity of the resulting model. It presents a tremendous accumulation of difficulties. In addition, explanations which are given are not very detailed and in most cases quite complicated to follow. In order to simplify things, it is possible to separate the construction into two steps. First one constructs the Néron model over an étale extension R' of R, where one has enough integral points to perform translations and where it is enough to consider the group scheme case. Then, as a second step, one goes back from R' to R by means of descent, using ample invertible sheaves and thereby proving the quasi-projectivity of the model. This is how M. Artin proceeds in [9]; the same strategy will be applied in the present book.

Finally, the universal mapping property of Néron models is established (in a rudimentary form) quite early in Néron's article, see n°4, pp. 71–73, even before Néron models are constructed. It is based on Weil's arguments [2], concerning rational maps from smooth varieties into algebraic groups.

It remains to say a few words about Néron's Chapter 3 where he constructs proper and regular minimal R-models for elliptic curves with a rational point over K. Except for a few examples, already mentioned in Section 1.5, the subject will not be touched in this book. Néron studies minimal Weierstraß equations and classifies them according to the values of their coefficients, discriminants, and j-invariants. Then he obtains the regular and proper minimal model as a successive joint of new components. His construction leads to the same diagrams as the ones obtained by Kodaira [1]. But Néron's approach of discussing minimal Weierstraß equations case by case is quite different, it does not use the existence of regular models nor does it use the intersection form. An improved version of his method was later published by Tate [2] in algorithmical form; it is known as the Tate algorithm.

Chapter 2. Some Background Material from Algebraic Geometry

In this chapter we give a review of some basic tools which are needed in later chapters for the construction of Néron models. Assuming that the reader is familiar with Grothendieck's definition of schemes and morphisms, we treat the concept of smooth and étale morphisms, of henselian rings, and of S-rational maps; moreover, we have included some facts on differential calculus and on flatness. Concerning the smoothness, we give a self-contained exposition of this notion, relating it closely to the Jacobi criterion. For the other topics we simply state results, sometimes without giving proofs. Most of the material presented in this chapter is contained in Grothendieck's treatments [EGA IV_4] and [SGA 1].

2.1 Differential Forms

In this section we define the sheaf of relative differential forms of one scheme over another. We introduce it by a purely algebraic method using derivations. So let us first review the basic facts on derivations; detailed explanations and proofs can be found in [EGA 0_{IV}], 20.5.

In the following let R be a ring, and let A be an R-algebra. An R-derivation of A into an A-module M is an R-linear map $d : A \longrightarrow M$ such that

$$d(fg) = fd(g) + gd(f) \quad \text{for all} \quad f, g \in A .$$

In particular, $d(r \cdot 1) = 0$ for all $r \in R$. The set $\mathrm{Der}_R(A, M)$ of all R-derivations of A into an A-module M is canonically an A-module. One defines the *module of relative differential forms (of degree* 1) *of A over R* as an A-module $\Omega^1_{A/R}$, together with an R-derivation $d_{A/R} : A \longrightarrow \Omega^1_{A/R}$, which is universal in the following sense: For each A-module M, the canonical map

$$\mathrm{Hom}_A(\Omega^1_{A/R}, M) \xrightarrow{\sim} \mathrm{Der}_R(A, M) , \qquad \varphi \longmapsto \varphi \circ d_{A/R} ,$$

is bijective. The map $d_{A/R}$ is called the *exterior differential.* Such a couple $(\Omega^1_{A/R}, d_{A/R})$ is uniquely determined up to canonical isomorphism. The existence can easily be verified in the following way. If A is a free R-algebra $R[T_i]_{i \in I}$ of polynomials in the variables T_i, $i \in I$, then let Ω^1 be the free A-module generated by the symbols dT_i, $i \in I$, and define $d : A \longrightarrow \Omega^1$ by the formula

$$d(P) := \sum_{i \in I} \frac{\partial P}{\partial T_i} \cdot dT_i ,$$

where $\partial P/\partial T_i$ is the usual partial derivative of P with respect to T_i. It is easy to see that (Ω^1, d) is the A-module of relative differential forms of A over R. In general, an R-algebra B is a residue ring $B = A/\mathfrak{a}$ of a free R-algebra of polynomials A. Then the B-module of relative differential forms of B over R is given by the B-module

$$\Omega^1_{A/R}/(\mathfrak{a}\Omega^1_{A/R} + A d_{A/R}\mathfrak{a}) ,$$

and the exterior differential is canonically induced by $d_{A/R}$.

We give an alternate method for proving the existence of modules of differentials. Let $m : A \otimes_R A \longrightarrow A$ be the map induced by the multiplication on A, set $I = \ker(m)$ and consider the map

$$d : A \longrightarrow I/I^2 , \quad f \longmapsto 1 \otimes f - f \otimes 1 \mod I^2 .$$

The $(A \otimes_R A)$-module I/I^2 is actually an $((A \otimes_R A)/I)$-module. Using the canonical isomorphism

$$(A \otimes_R A)/I \xrightarrow{\sim} A$$

one can view I/I^2 as an A-module, and one verifies that $(I/I^2, d)$ is the A-module of relative differential forms of A over R.

The universal property of $\Omega^1_{A/R}$ implies certain functorial properties. For example, each morphism $\varphi : A \longrightarrow B$ of R-algebras induces a unique A-linear map

$$\Omega^1_{A/R} \longrightarrow \Omega^1_{B/R} , \quad \sum_i f_i d_{A/R}(g_i) \longmapsto \sum_i \varphi(f_i) d_{B/R}(\varphi(g_i)) ,$$

and hence a B-linear map

$$\Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} .$$

Moreover, since each A-derivation of B is also an R-derivation, one obtains a map

$$\Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} , \quad \sum_i f_i d_{B/R}(g_i) \longmapsto \sum_i f_i d_{B/A}(g_i) .$$

Thus we have a canonical sequence

$$\Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow \Omega^1_{B/A} \longrightarrow 0$$

which can be shown to be exact. If B is a residue algebra of A, say $B = A/\mathfrak{a}$, the R-derivation $d_{A/R}$ induces a canonical B-linear map

$$\delta : \mathfrak{a}/\mathfrak{a}^2 \longrightarrow \Omega^1_{A/R} \otimes_A B , \quad \bar{a} \longmapsto d_{A/R}(a) \otimes 1$$

where $\bar{a}$ denotes the residue class of $a \in \mathfrak{a}$ modulo $\mathfrak{a}^2$. As a second important fact on the behavior of differentials, one shows that the sequence

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\delta} \Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow 0$$

is exact.

Next we want to globalize the notion of modules of differentials in terms of sheaves over schemes. One can either show that the formation of modules of differentials is compatible with localization or, what is more elegant, use the alternate description we have given above. Proceeding the latter way, consider a base scheme S and an S-scheme X. The diagonal morphism

$$\Delta : X \longrightarrow X \times_S X$$

yields an isomorphism of X onto its image $\Delta(X)$ which is a locally closed subscheme of $X \times_S X$; i.e., $\Delta(X)$ is a closed subscheme of an open subscheme W of $X \times_S X$. Let $\mathscr{I}$ be the sheaf of ideals defining $\Delta(X)$ as a closed subscheme of W. Then we define the *sheaf of relative differential forms* (*of degree* 1) *of* X *over* S as the sheaf

$$\Omega^1_{X/S} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$$

on X. Note that $\mathscr{I}/\mathscr{I}^2$ has a natural structure of an $\mathcal{O}_{\Delta(X)}$-module; hence $\Delta^*(\mathscr{I}/\mathscr{I}^2)$ is canonically an $\mathcal{O}_X$-module. It is clear that $\Omega^1_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module, which is of finite type if X is locally of finite type over S. The canonical map

$$d_{X/S} : \mathcal{O}_X \longrightarrow \Omega^1_{X/S} ,$$

induced by the map sending a section f of $\mathcal{O}_X$ to the section $p_2^* f - p_1^* f$ of $\mathscr{I}$ (where $p_i : X \times_S X \longrightarrow X$ is the projection onto the i-th factor), is called the *exterior differential.*

Since $\Omega^1_{X/S}$ is quasi-coherent, $(\Omega^1_{X/S}, d_{X/S})$ can be described in local terms: for each open affine subset $V = \operatorname{Spec} R$ of S and for each open affine subset $U = \operatorname{Spec} A$ of X lying over V, the sheaf $\Omega^1_{X/S}|_U$ is the quasi-coherent $\mathcal{O}_X|_U$-module associated to the A-module $\Omega^1_{A/R}$, and the map $d_{X/S}|_U$ is associated to the canonical map $d_{A/R} : A \longrightarrow \Omega^1_{A/R}$.

The sheaf of relative differential forms has similar functorial properties as the module of relative differential forms. Given an S-morphism $f : X \longrightarrow Y$, one can pull back differential forms on Y to X. So one obtains a canonical $\mathcal{O}_X$-morphism

$$f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} .$$

Each section ω of $\Omega^1_{Y/S}$ gives rise to a section ω' of $f^*\Omega^1_{Y/S}$ and hence to a section ω'' of $\Omega^1_{X/S}$, namely to the image of ω' under the above map. It is convenient to use the notion $f^*\omega$ for both ω' and ω''; however to avoid confusion, we will always specify the module, either $f^*\Omega^1_{Y/S}$ or $\Omega^1_{X/S}$, when we talk about the section $f^*\omega$.

The canonical sequences between modules of differentials, as given above, can immediately be globalized to the case of differentials over schemes; cf. [EGA IV$_4$], 16.4:

Proposition 1. *Let* $f : X \longrightarrow Y$ *be an S-morphism. Then the canonical sequence of* $\mathcal{O}_X$*-modules*

$$f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact.

Proposition 2. *Let* $j : Y \hookrightarrow X$ *be an immersion of S-schemes. Let* $\mathscr{J}$ *be the sheaf of ideals defining* Y *as a subscheme of* X*. Then the canonical sequence of* $\mathcal{O}_Y$*-modules*

$$\mathscr{J}/\mathscr{J}^2 \xrightarrow{\delta} j^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow 0$$

is exact.

Furthermore, we cite that the formation of sheaves of relative differentials commutes with base change and products:

Proposition 3. *Let X and S' be S-schemes. Let $X' = X \times_S S'$ be the S'-scheme obtained by base change, and let $p: X' \longrightarrow X$ be the projection. Then the canonical map*

$$p^*\Omega^1_{X/S} \xrightarrow{\sim} \Omega^1_{X'/S'}$$

is an isomorphism.

Proposition 4. *Let X_1 and X_2 be S-schemes. If $p_i: X_1 \times_S X_2 \longrightarrow X_i$ are the projections for $i = 1, 2$, the canonical map*

$$p_1^*\Omega^1_{X_1/S} \oplus p_2^*\Omega^1_{X_2/S} \xrightarrow{\sim} \Omega^1_{X_1 \times_S X_2/S}$$

is an isomorphism.

2.2 Smoothness

In this section we want to explain the basic concept of unramified, étale, and smooth morphisms from the viewpoint of differential calculus. Our approach differs from the one given in [EGA IV$_4$], 17, in so far as we have chosen the Jacobi criterion as point of departure. In the following, let S be a base scheme.

Definition 1. *A morphism of schemes $f: X \longrightarrow S$ is called unramified at a point $x \in X$ if there exist an open neighborhood U of x and an S-immersion*

$$j: U \hookrightarrow \mathbb{A}^n_S$$

of U into some linear space $\mathbb{A}^n_S$ over S such that the following conditions are satisfied:

(a) *locally at $j(x)$ (i.e., in an open neighborhood of $j(x)$), the sheaf of ideals $\mathscr{I}$ defining $j(U)$ as a subscheme of $\mathbb{A}^n_S$ is generated by finitely many sections,*

(b) *the differential forms of type dg with sections g of $\mathscr{I}$ generate $\Omega^1_{\mathbb{A}^n_S/S}$ at $j(x)$.*

The morphism $f: X \longrightarrow S$ is called unramified if it is unramified at all points of X.

Condition (a) says that unramified morphisms are locally of finite presentation. Obviously, an immersion which is locally of finite presentation is unramified. It can easily be shown that the class of unramified morphisms is stable under base change, under composition, and under the formation of products. We give some equivalent characterizations of unramified morphisms:

Proposition 2. *Let $f: X \longrightarrow S$ be locally of finite presentation, let x be a point of X, and set $s = f(x)$. Then the following conditions are equivalent:*

(a) *f is unramified at x.*

(b) *$\Omega^1_{X/S,x} = 0$*

(c) *The diagonal morphism $\Delta: X \longrightarrow X \times_S X$ is a local isomorphism at x.*

(d) *The $k(s)$-scheme $X_s = X \times_S \operatorname{Spec} k(s)$ is unramified over $k(s)$ at x.*

(e) *The maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ is generated by the maximal ideal $\mathfrak{m}_s$ of $\mathcal{O}_{S,s}$, and $k(x)$ is a (finite) separable extension of $k(s)$.*

Proof. The equivalence of conditions (a) and (b) follows from the exact sequence 2.1/2. The equivalence of (b) and (c) is seen by using the identity

$$\Omega^1_{X/S} = \Delta^*(\mathscr{I}/\mathscr{I}^2),$$

where $\mathscr{I}$ is the sheaf of ideals defining the diagonal in $X \times_S X$, and by applying Nakayama's lemma. Furthermore, since unramified morphisms are preserved by any base change, condition (a) implies condition (d). Conversely, if (d) is satisfied, we know already

$$\Omega^1_{X_s/k(s),x} = 0.$$

Let $\mathfrak{m}_s$ be the maximal ideal of $\mathcal{O}_{S,s}$. Then, since the formation of sheaves of differentials is compatible with base change, we have

$$\Omega^1_{X_s/k(s),x} = \Omega^1_{X/S,x}/\mathfrak{m}_s\Omega^1_{X/S,x},$$

and Nakayama's lemma yields $\Omega^1_{X/S,x} = 0$. So condition (b) is satisfied, and we see that conditions (a) to (d) are equivalent.

In order to show that the equivalence extends to condition (e), we may assume that S is the spectrum of a field k. Then the implication (e) $\Longrightarrow$ (b) is an elementary algebraic fact, because $\Omega^1_{X/S,x} = \Omega^1_{k(x)/k}$ in this case. Conversely, let us show that condition (c) implies condition (e). We may assume that X is affine, say $X = \operatorname{Spec} A$, and that the diagonal morphism $\Delta: X \longrightarrow X \times_k X$ is an open immersion. Let $\bar{k}$ be the algebraic closure of k. It suffices to prove that $A \otimes_k \bar{k}$ is a finite direct sum of fields isomorphic to $\bar{k}$; then A will be a finite direct sum of separable field extensions of k. To do this we may assume that k is algebraically closed. For a closed point z of X, let $h_z: X \longrightarrow X$ be the constant morphism mapping X to z, and consider the morphism

$$(id_X, h_z): X \longrightarrow X \times_k X.$$

Since Δ is an open immersion,

$$(id_X, h_z)^{-1}(\Delta(X)) = \{z\}$$

is open in X. Hence each closed point of X is open, and X consists of a finite number of isolated points. In particular, A is a finite-dimensional vector space over k. Shrinking X if necessary, we may assume that X consists of only one point. Then the same is true for $X \times_k X$. Since Δ is an open immersion, the corresponding morphism $\Delta^*: A \otimes_k A \longrightarrow A$ is an isomorphism and, by comparing vector space dimensions, we see $A = k$. □

If follows from condition (e) above that the relative dimension of an unramified morphism is zero. More generally, one can show that the relative dimension $\dim_x f = \dim_x f^{-1}(f(x))$ at a point x of an S-subvariety $X \subset \mathbb{A}^n_S$ with structural morphism $f: X \longrightarrow S$ is r if, locally at x, the subvariety is defined by sections $g_{r+1}, \ldots, g_n$ of $\mathcal{O}_{\mathbb{A}^n_S}$ and if the differentials $dg_{r+1}(x), \ldots, dg_n(x)$ are linearly independent in $\Omega^1_{\mathbb{A}^n_S/S} \otimes k(x)$. Namely, this follows from the result above and the fact that the relative dimension decreases at most by 1 if one goes over from an S-scheme to a subscheme defined by a single equation.

Definition 3. *A morphism $f: X \longrightarrow S$ is called smooth at a point x of X (of relative dimension r) if there exist an open neighborhood U of x and an S-immersion*

$$j: U \hookrightarrow \mathbb{A}_S^n$$

of U into some linear space $\mathbb{A}_S^n$ over S such that the following conditions are satisfied:

(a) *locally at $y := j(x)$, the sheaf of ideals defining $j(U)$ as a subscheme of $\mathbb{A}_S^n$ is generated by $(n-r)$ sections $g_{r+1}, \ldots, g_n$, and*

(b) *the differentials $dg_{r+1}(y), \ldots, dg_n(y)$ are linearly independent in $\Omega^1_{\mathbb{A}_S^n/S} \otimes k(y)$.*

A morphism is called smooth if it is smooth at all points. Furthermore, a morphism is said to be étale (at a point) if it is smooth (at the point) of relative dimension 0.

Note that, as we have explained above, the integer r is indeed the relative dimension of f at x and that, due to its definition, the smooth locus of a morphism which is locally of finite presentation is open. It is an elementary task to verify that the class of smooth (resp. étale) morphisms is stable under base change, under composition, and under the formation of products. It is clear that open immersions are étale. Furthermore, étale morphisms are unramified, but the converse is not true as is seen by the following lemma.

Lemma 4. *An immersion $f: X \longrightarrow S$ is étale if and only if f is an open immersion.*

Proof. The if-part is obvious. For the only-if-part, it suffices to consider the special case where f is a closed immersion. Furthermore we may assume that, as an S-scheme, X has been realized as a closed subscheme of an affine open subscheme $V \subset \mathbb{A}_S^n$, in such a way that X is defined by n sections $g_1, \ldots, g_n$ of $\mathcal{O}_{\mathbb{A}_S^n}$ on V, where the differentials $dg_1, \ldots, dg_n$ generate $\Omega^1_{\mathbb{A}_S^n/S}|_V$. Since $f: X \longrightarrow S$ is a closed immersion, we may assume that the coordinate functions $T_1, \ldots, T_n$ of $\mathbb{A}_S^n$ vanish on X. Then we have relations

$$T_j = \sum_i a_{ij} g_i$$

with $a_{ij} \in \mathcal{O}_{\mathbb{A}_S^n}(V)$ for $i, j = 1, \ldots, n$. Taking the differentials of these equations shows that the matrix (a_{ij}) is invertible in a neighborhood of X. Due to Cramer's rule, the sheaves of ideals generated by $(T_1, \ldots, T_n)$ and $(g_1, \ldots, g_n)$ coincide in this neighborhood. This implies that f is an open immersion. □

More generally, one can show that étale morphisms are flat and, hence, open (cf. 2.4); in fact, a morphism is étale if and only if it is flat and unramified, see 2.4/8. In particular, if S is the spectrum of a field k, the notions *étale* and *unramified* coincide. In this case, each étale S-scheme X consists of isolated reduced points such that the residue field $k(x)$ of each point $x \in X$ is a finite separable extension of k.

Proposition 5. *Let $f: X \longrightarrow Y$ be a smooth morphism of schemes. Then:*

(a) *$\Omega^1_{X/Y}$ is locally free. Its rank at $x \in X$ is equal to the relative dimension of f at x.*

(b) *If f is a smooth morphism of smooth S-schemes, the canonical sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact and locally split. (Actually, the assumption on X and Y to be smooth over S is unnecessary; cf. [EGA IV$_4$], 17.2.3.)

Proof. Since $\Omega^1_{\mathbb{A}_Y^n}$ is free of rank n, assertion (a) follows immediately from the definition of smoothness if one uses 2.1/2. In the situation (b) we know from 2.1/1 that the canonical sequence

$$f^*\Omega^1_{Y/S} \xrightarrow{\alpha} \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact. Due to (a), the three $\mathcal{O}_X$-modules are locally free of finite rank. Hence, for all $x \in X$, the $\mathcal{O}_{X,x}$-module $(f^*\Omega^1_{Y/S})_x$ is isomorphic to the direct sum of $\ker \alpha_x$ and $\operatorname{im} \alpha_x$ both of which are free. Counting the ranks, one sees $\ker \alpha = 0$. □

It is an easy consequence of (a) that, for a smooth morphism $f: X \longrightarrow S$, the map $x \longmapsto \dim_x f$ is locally constant. Next we want to characterize smoothness by the infinitesimal lifting property for morphisms.

Proposition 6. *Let $f: X \longrightarrow S$ be locally of finite presentation. The following conditions are equivalent:*

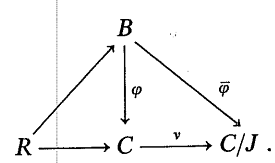
(a) *f is unramified (resp. smooth, resp. étale).*

(b) *For all S-schemes Y which are affine and for all closed subschemes Y_0 of Y defined by sheaves of ideals $\mathcal{J}$ of $\mathcal{O}_Y$ with $\mathcal{J}^2 = 0$, the canonical map*

$$\operatorname{Hom}_S(Y, X) \longrightarrow \operatorname{Hom}_S(Y_0, X)$$

is injective (resp. surjective, resp. bijective).

Proof. First we want to treat the characterization of unramified morphisms. In this situation, conditions (a) and (b) are local on X and S, so we may assume that X and S are affine, say $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$. Let C be an R-algebra, let J be an ideal of C with $J^2 = 0$, and consider a commutative diagram



One easily shows that the map

$$\{\psi \in \operatorname{Hom}_R(B, C) ; \nu \circ \psi = \bar{\varphi}\} \longrightarrow \operatorname{Der}_R(B, J), \qquad \psi \longmapsto \psi - \varphi,$$

between the set of liftings of $\bar{\varphi}$ and the B-module of R-derivations is bijective. Notice that J is a C/J-module and, hence, a B-module via $\bar{\varphi}$.

If X is unramified over S, we know $\Omega^1_{B/R} = 0$ from Proposition 2 so that $\operatorname{Der}_R(B, J) = 0$ in this case. Thus, the implication (a) $\Longrightarrow$ (b) is clear. In order to verify the implication (b) $\Longrightarrow$ (a), set $C := (B \otimes_R B)/I^2$, where I is the kernel of the map

$$m : B \otimes_R B \longrightarrow B\,, \qquad \sum x_i \otimes y_i \longmapsto \sum x_i y_i\,.$$

Furthermore, set $J = I/I^2$. The considerations above show $\mathrm{Der}_R(B, J) = 0$. Since $J \cong \Omega^1_{B/R}$, the implication (b) $\Longrightarrow$ (a) follows.

Next we turn to the characterization of smooth morphisms. Starting with the implication (a) $\Longrightarrow$ (b), let us first consider a special case which corresponds to the local situation of a smooth morphism. So let S be affine, say $S = \mathrm{Spec}\, R$, and let $X = \mathrm{Spec}\, B$ be a closed subscheme of an affine open subscheme $V = \mathrm{Spec}\, A$ of $\mathbb{A}^n_S$. Let I be the ideal of A defining X. Assume that there are $g_1, \ldots, g_n \in A$ such that $dg_1, \ldots, dg_n$ form a basis of $\Omega^1_{A/R}$ and such that, for some r, the ideal $I \subset A$ of X is generated by $g_{r+1}, \ldots, g_n$. Then, since I/I^2 is generated over $B = A/I$ by the residue classes of these elements, the canonical sequence

$$(*) \qquad 0 \longrightarrow I/I^2 \longrightarrow \Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow 0$$

is easily seen to be split exact.

Now let $Y = \mathrm{Spec}\, C$ be an affine S-scheme, and fix a closed subscheme $Y_0 \subset Y$ defined by an ideal J of C with $J^2 = 0$. To verify condition (b), we have to show that each R-morphism $\bar{\varphi} : B \longrightarrow C/J$ lifts to an R-morphism $\varphi : B \longrightarrow C$. Due to the universal property of a polynomial ring, we can lift $\bar{\varphi}$ to an R-morphism $\psi : A \longrightarrow C$ such that the diagram

$$\begin{array}{ccccc} & & A & \longrightarrow & B = A/I \\ & \nearrow & \downarrow \psi & & \downarrow \bar{\varphi} \\ R & \longrightarrow & C & \longrightarrow & C/J \end{array}$$

is commutative. Since $\psi(I) \subset J$, the map ψ gives rise to a B-linear map

$$\psi' : I/I^2 \longrightarrow J\,.$$

Since the sequence $(*)$ is split exact, the B-linear map ψ' extends to a B-linear map ψ'' as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega^1_{A/R} \otimes_A B & \longrightarrow & \Omega^1_{B/R} & \longrightarrow & 0 \\ & & & \searrow^{\psi'} & \downarrow \psi'' & & & & \\ & & & & J & & & & \end{array}$$

Hence, ψ'' induces an R-derivation $\delta : A \longrightarrow J$ satisfying $\psi|_I = \delta|_I$. Then $(\psi - \delta) : A \longrightarrow C$ is an R-morphism inducing a lifting $\varphi : B \longrightarrow C$ of $\bar{\varphi}$.

It remains to reduce the general case of an arbitrary smooth morphism $f : X \longrightarrow S$ to the special case treated above. This can be done by showing that condition (b) is a local condition on X. So, as before, let $Y = \mathrm{Spec}\, C$ be an affine S-scheme, and let Y_0 be a closed subscheme of Y defined by a sheaf of ideals $\mathcal{J}$ of $\mathcal{O}_Y$ with $\mathcal{J}^2 = 0$. Let $\bar{\varphi} : Y_0 \longrightarrow X$ be an S-morphism. Due to the special case

discussed above, there exists an open covering $\{Y_\alpha\}_\alpha$ of Y such that $\bar{\varphi}|_{Y_\alpha \cap Y_0}$ lifts to a morphism $\varphi'_\alpha : Y_\alpha \longrightarrow X$. The obstruction for (φ'_α) to define a morphism from Y to X is a cocycle with values in $\mathcal{H}om_{\mathcal{O}_{Y_0}}(\bar{\varphi}^*\Omega^1_{X/S}, \mathcal{J})$; see also [SGA 1], Exp. III, 5.1. Since this sheaf is a quasi-coherent $\mathcal{O}_{Y_0}$-module, its first cohomology group vanishes on the affine scheme Y_0. So there exist liftings $\varphi_\alpha : Y_\alpha \longrightarrow X$ of $\bar{\varphi}|_{Y_\alpha \cap Y_0}$ such that (φ_α) gives rise to a morphism $\varphi : Y \longrightarrow X$ lifting $\bar{\varphi}$. This establishes the implication (a) $\Longrightarrow$ (b) for smooth morphisms.

In order to show the converse, we may assume that X is a closed subscheme of a linear space $\mathbb{A}^n_S$ which is defined by a finitely generated sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{\mathbb{A}^n_S}$. Then it suffices to show that the canonical sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega^1_{\mathbb{A}^n_S/S} \otimes \mathcal{O}_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

is locally split exact. We will prove this in a more general situation where $\mathbb{A}^n_S$ is replaced by a smooth S-scheme Z. In order to do this, we may assume that S and Z are affine, say $S = \mathrm{Spec}\, R$ and $Z = \mathrm{Spec}\, A$, and that $X = \mathrm{Spec}\, B$ is defined by a finitely generated ideal $I \subset A$; in particular, we have $B = A/I$. Due to condition (b), the map

$$\bar{\varphi} = \mathrm{id} : A/I \longrightarrow A/I = (A/I^2)/(I/I^2)$$

lifts to an R-morphism $\varphi : A/I \longrightarrow A/I^2$. Then the exact sequence of R-modules

$$0 \longrightarrow I/I^2 \xrightarrow{\ \iota\ } A/I^2 \xrightarrow{\ \nu\ } A/I \longrightarrow 0$$

splits; namely, φ is a section of ν, and $\mathrm{id}_{A/I^2} - \varphi \circ \nu$ defines an R-linear map

$$\tau : A/I^2 \longrightarrow I/I^2$$

which is a section of the inclusion ι. Since $\tau(a) \cdot \tau(b) = 0$ for all $a, b \in A/I^2$, we have

$$\tau(ab) = ab - \varphi \circ \nu(ab) + (a - \varphi \circ \nu(a))(b - \varphi \circ \nu(b)) = a\tau(b) + b\tau(a)\,.$$

Hence τ is an R-derivation giving rise to an A-homomorphism $\Omega^1_{A/R} \longrightarrow I/I^2$. Consequently, the sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_{A/R} \otimes_A B \longrightarrow \Omega^1_{B/R} \longrightarrow 0$$

is split exact.

Finally, the characterization of étale morphisms follows from what has been shown for smooth and unramified morphisms, since a morphism is étale if and only if it is smooth and unramified. □

In the definition of smoothness it is required that a smooth S-scheme X can locally be realized as a subscheme of a suitable linear space $\mathbb{A}^n_S$ such that the associated sheaf of ideals satisfies certain conditions. Now we will see that these conditions are fulfilled for each immersion of X into a smooth S-scheme.

Proposition 7. (Jacobi Criterion). *Let X and Z be S-schemes, and let $j : X \hookrightarrow Z$ be a closed immersion which is locally of finite presentation. Let $\mathcal{J}$ be the sheaf of ideals of $\mathcal{O}_Z$ which defines X as a subscheme of Z. Let x be a point of X, and set $z = j(x)$. Assume that, as an S-scheme, Z is smooth at z of relative dimension n. Then the following conditions are equivalent:*

(a) *As an S-scheme, X is smooth at x of relative dimension r.*
(b) *The canonical sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow j^*\Omega^1_{Z/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

is split exact at x, and $r = \operatorname{rank}(\Omega^1_{X/S} \otimes k(x))$.

(c) *If* $dz_1, \ldots, dz_n$ *is a basis of* $(\Omega^1_{Z/S})_z$, *and if* $g_1, \ldots, g_N$ *are local sections of* $\mathcal{O}_Z$ *generating* $\mathcal{I}_z$, *there exists a re-indexing of the* $z_1, \ldots, z_n$ *and of the* $g_1, \ldots, g_N$ *such that* $g_{r+1}, \ldots, g_n$ *generate* $\mathcal{I}$ *at z and such that* $dz_1, \ldots, dz_r, dg_{r+1}, \ldots, dg_n$ *generate* $(\Omega^1_{Z/S})_z$.

(d) *There exist local sections* $g_{r+1}, \ldots, g_n$ *of* $\mathcal{O}_Z$ *generating* $\mathcal{I}_z$ *such that the differentials* $dg_{r+1}(z), \ldots, dg_n(z)$ *are linearly independent in* $\Omega^1_{Z/S} \otimes k(z)$.

Proof. The implication (a) $\Longrightarrow$ (b) follows from the preceding proposition. Namely, if condition (a) is satisfied, X has the lifting property, and, as shown in the last part of the proof of Proposition 6, the canonical exact sequence of (b) is split exact. Furthermore, $(\Omega^1_{X/S})_x$ is free of rank r by Proposition 5.

The implication (b) $\Longrightarrow$ (c) follows from Nakayama's lemma, whereas (c) $\Longrightarrow$ (d) is clear. Finally, the implication (d) $\Longrightarrow$ (a) is easily checked by using a local representation of Z at z as required for $Z \longrightarrow S$ to be smooth at z. □

Condition (d) can also be stated in terms of matrices. Namely, considering a representation

$$dg_j = \sum_{i=1}^{n} \frac{\partial g_j}{\partial z_i} dz_i$$

of the differential forms $dg_{r+1}, \ldots, dg_n$ with respect to a basis $dz_1, \ldots, dz_n$ of $(\Omega^1_{Z/S})_z$, condition (d) says that $\mathcal{I}_z$ is generated by the $(n-r)$ elements g_j and that there exists an $(n-r)$-minor of the matrix $(\partial g_j/\partial z_i)$ which does not vanish at z. So we see that Proposition 7 corresponds to the Jacobi Criterion in differential geometry. We want to derive a second version of it (see [EGA IV$_4$], 17.11.1 for a further generalization).

Proposition 8. *Let* $f : X \longrightarrow Y$ *be an S-morphism. Let x be a point of X, and set* $y = f(x)$. *Assume that X is smooth over S at x and that Y is smooth over S at y. Then the following conditions are equivalent:*
(a) *f is smooth at x.*
(b) *The canonical homomorphism* $(f^*\Omega^1_{Y/S})_x \longrightarrow (\Omega^1_{X/S})_x$ *is left-invertible (i.e., is an isomorphism onto a direct factor).*
(c) *The canonical homomorphism* $(f^*\Omega^1_{Y/S}) \otimes k(x) \longrightarrow \Omega^1_{X/S} \otimes k(x)$ *is injective.*

Proof. The implication (a) $\Longrightarrow$ (b) is a direct consequence of Proposition 5; the implication (b) $\Longrightarrow$ (c) is trivial. Concerning the implication (c) $\Longrightarrow$ (a), we will first treat the case where $Y = \mathbb{A}^s_S$. Then the morphism f is given by global sections $\bar{f}_1, \ldots, \bar{f}_s$ of $\mathcal{O}_X$, and condition (c) means that $d\bar{f}_1(x), \ldots, d\bar{f}_s(x)$ are linearly independent. Furthermore, we may assume that X is a subscheme of $\mathbb{A}^m_S$ of relative dimension r and that the sheaf of ideals defining X is generated by sections $h_{r+1}, \ldots, h_m$ such that $dh_{r+1}(x), \ldots, dh_m(x)$ are linearly independent. Let us consider the graph embedding

$$X \hookrightarrow X \times_S Y \hookrightarrow \mathbb{A}^m_S \times_S \mathbb{A}^s_S , \qquad x \longmapsto (x, f(x)) .$$

We can lift the sections $\bar{f}_i$ to sections f_i defined in a neighborhood of x in $\mathbb{A}^m_S$. Then, locally at $(x, f(x))$, we have realized X as the subscheme of $\mathbb{A}^{m+s}_S = \mathbb{A}^m_Y$ which is given by

$$h_{r+1}, \ldots, h_m , \qquad T_1 - f_1, \ldots, T_s - f_s ,$$

where $T_1, \ldots, T_s$ denote the coordinate functions of $\mathbb{A}^s_S = Y$. This yields a local representation of X as a subscheme of $\mathbb{A}^m_Y$ as required.

In order to handle the general case, let Y be smooth at y of relative dimension s over S. Let $g_1, \ldots, g_s$ be local sections at y of $\mathcal{O}_Y$ such that $dg_1, \ldots, dg_s$ induce a basis of $(\Omega^1_{Y/S})_y$. After shrinking X and Y, we may assume that $g_1, \ldots, g_s$ are global sections. Due to condition (c), there exist local sections $h_{s+1}, \ldots, h_r$ at x of $\mathcal{O}_X$ such that

$$f^*dg_1, \ldots, f^*dg_s , \qquad dh_{s+1}, \ldots, dh_r$$

is a basis of $\Omega^1_{X/S,x}$ where r is the relative dimension at x of X over S. Again, we may assume that $h_{s+1}, \ldots, h_r$ are global sections of $\mathcal{O}_X$. Setting

$$g = (g_1, \ldots, g_s) : Y \longrightarrow \mathbb{A}^s_S ,$$
$$h = (h_{s+1}, \ldots, h_r) : X \longrightarrow \mathbb{A}^{r-s}_S ,$$

we obtain the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{(f,h)} & Y \times_S \mathbb{A}^{r-s}_S & \xrightarrow{p} & Y \\ & \searrow^{(g \circ f, h)} & \downarrow{\scriptstyle g \times \mathrm{id}} & & \\ & & \mathbb{A}^r_S & & \end{array}$$

By the special case above, the maps $(g \circ f, h)$ and $g \times \mathrm{id}$ are étale at x and y, respectively. Hence, due to Lemma 9 below, the morphism (f, h) is étale at x. Then, $f = p \circ (f, h)$ is a composition of smooth morphisms and, hence, smooth at x. □

Lemma 9. *Let* $X \longrightarrow S$ *be unramified (resp. smooth, resp. étale), and let* $Y \longrightarrow S$ *be unramified. Then each S-morphism* $X \longrightarrow Y$ *is unramified (resp. smooth, resp. étale).*

Proof. The assertion follows from Proposition 6. Namely, one verifies immediately that $X \longrightarrow Y$ satisfies the lifting property (b) of this proposition. □

Let us state the assertion of Proposition 8 for the special case of étale morphisms.

Corollary 10. *Let* $f : X \longrightarrow Y$ *be an S-morphism. Let x be a point of X, and set* $y = f(x)$. *Assume that X is smooth over S at x and that Y is smooth over S at y. Then the following conditions are equivalent:*
(a) *f is étale at x.*
(b) *The canonical homomorphism* $(f^*\Omega^1_{Y/S})_x \longrightarrow (\Omega^1_{X/S})_x$ *is bijective.*

Thinking of the classical inverse function theorem, the corollary suggests an analogy between the notions of étale morphisms in algebraic geometry and in differential geometry. But note that, in algebraic geometry, if one wants to view étale morphisms as local isomorphisms, the Zariski topology has to be replaced by the so-called étale topology (cf. 2.3/8). In differential geometry, the implicit function theorem shows that, locally, smooth morphisms are fibrations by open subsets of linear spaces. Up to localization by étale morphisms, the same is true in algebraic geometry:

Proposition 11. *Let $f: X \longrightarrow S$ be a morphism, and let x be a point of X. Then the following conditions are equivalent:*

(a) *f is smooth at x of relative dimension n.*

(b) *There exists an open neighborhood U of x and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{g} & \mathbb{A}^n_S \\ & {\scriptstyle f|_U}\searrow & \downarrow{\scriptstyle p} \\ & & S \end{array}$$

where g is étale and p is the canonical projection.

Proof. That condition (b) implies condition (a) is clear, since the composition of smooth morphisms is smooth. To show the converse, choose local sections $g_1, \ldots, g_n$ of $\mathcal{O}_X$ such that $dg_1, \ldots, dg_n$ generate $\Omega^1_{X/S}$ at x. Due to Corollary 10, the latter is equivalent to the fact that $g_1, \ldots, g_n$ define an étale map from an open neighborhood U of x to $\mathbb{A}^n_S$. □

Remark 12. If X is a smooth S-scheme and if $g_1, \ldots, g_n$ are local sections of $\mathcal{O}_X$ at a point $x \in X$, then, by Nakayama's lemma, the differentials $dg_1, \ldots, dg_n$ generate $\Omega^1_{X/S}$ at x if and only if the differentials $dg_1(x), \ldots, dg_n(x)$ form a basis of the $k(x)$-vector space $\Omega^1_{X/S,x} \otimes k(x)$. Furthermore, as we have mentioned in the preceding proof, this condition is equivalent to the fact that $g_1, \ldots, g_n$ define an étale morphism from an open neighborhood U of x to $\mathbb{A}^n_S$. If $g_1, \ldots, g_n$ satisfy these equivalent conditions, they will be called a *system of local coordinates at x* (over S). This terminology is justified since, up to an étale morphism, $g_1, \ldots, g_n$ indeed behave like a set of coordinates of the affine n-space $\mathbb{A}^n_S$.

As a consequence of Proposition 11, we obtain the following useful fact.

Corollary 13. *If X is a smooth scheme over a field k, the set of closed points x of X such that $k(x)$ is a separable extension of k is dense in X.*

Proof. For each point x_0 of X, there exists an open neighborhood U of x_0 and a factorization

$$U \xrightarrow{g} \mathbb{A}^n_k \xrightarrow{p} \operatorname{Spec} k$$

where g is étale. Then, if x is a point of U, the extension $k(x)$ of $k(g(x))$ is finite and separable. Hence it is enough to show $g(U)$ contains a closed point y such that $k(y)$ is a separable extension of k. The set of closed points y such that $k(y)$ is separable over k is dense in $\mathbb{A}^n_k$. Namely, this is clear if k is perfect. If k is not perfect, it contains infinitely many elements so that the set of k-valued points is dense in $\mathbb{A}^n_k$. Thus it suffices to show that $g(U)$ contains a non-empty open subset. However, the latter is clear by reasons of dimension, since $g(U)$ is constructible (cf. [EGA IV_1], 1.8.4). (Actually, $g(U)$ is open, because an étale map is flat and hence open.) □

Next we apply Proposition 7 in order to construct étale sections of smooth morphisms.

Proposition 14. *Let $f: X \longrightarrow S$ be a smooth morphism. Let s be a point of S, and let x be a closed point of the fibre $X_s = X \times_S \operatorname{Spec} k(s)$ such that $k(x)$ is a separable extension of $k(s)$. Then there exist an étale morphism $g: S' \longrightarrow S$ and a point $s' \in S'$ above s such that the morphism $f': X \times_S S' \longrightarrow S'$ obtained from f by the base change $S' \longrightarrow S$ admits a section $h: S' \longrightarrow X \times_S S'$, where $h(s')$ lies above x, and where $k(s') = k(x)$.*

Proof. Let n be the relative dimension of X over S at x. Let $\mathcal{J} \subset \mathcal{O}_{X_s}$ be the sheaf of ideals associated to the closed point x of X_s. Since $\operatorname{Spec} k(x) \longrightarrow \operatorname{Spec} k(s)$ is étale, the ideal $\mathcal{J}_x$ is generated by n elements $\bar{g}_1, \ldots, \bar{g}_n$ such that their differentials $d\bar{g}_1, \ldots, d\bar{g}_n$ generate $\Omega^1_{X/S} \otimes k(x)$, as seen by the Jacobi criterion (Proposition 7). Now we lift $\bar{g}_1, \ldots, \bar{g}_n$ to sections $g_1, \ldots, g_n$ of $\mathcal{O}_X$ defined on an open neighborhood of x in X. Then let S' be the subscheme of X defined by $g_1, \ldots, g_n$. Again by Proposition 7, the scheme S' is étale over S at x. After shrinking S' we may assume that $S' \longrightarrow S$ is étale. Then the tautological section $h': S' \longrightarrow X'$ is a section as required. □

Using Proposition 7, the smoothness of a scheme X over a field k can be characterized by algebraic properties of the local rings of X. A k-scheme X which is locally of finite type is called *regular* if, for each closed point x of X, the local ring $\mathcal{O}_{X,x}$ is regular. (One knows then that $\mathcal{O}_{X,x}$ is regular also for non-closed points $x \in X$; cf. [EGA 0_{IV}], 17.3.2).

Proposition 15. *Let X be locally of finite type over a field k. Let x be a point of X. Then the following conditions are equivalent:*

(a) *X is smooth over k at x.*

(b) *$(\Omega^1_{X/k})_x$ is generated by $\dim_x X$ elements (and hence free).*

(c) *There exist an open neighborhood U of x and a perfect field extension k' of k such that $U \otimes_k k'$ is regular.*

(d) *There exists an open neighborhood U of x such that $U \otimes_k k'$ is regular for all field extensions k' of k.*

Proof. We start with the implication (a) $\Longrightarrow$ (d). Due to Proposition 11, there exists an étale morphism $g: U \longrightarrow \mathbb{A}^n_k$, defined on an open neighborhood $U \subset X$ of x. Then Proposition 2 shows for each $y \in U$ that the maximal ideal $\mathfrak{m}_y$ is generated by $\mathfrak{m}_{g(y)}$. So $\mathfrak{m}_y$ is generated by $n = \dim U$ elements because $\mathbb{A}^n_k$ is regular; hence U is regular. Since the situation remains essentially the same after extending the field k to k', the assertion follows.

The implication (d) $\Longrightarrow$ (c) is trivial. So let us consider the implication (c) $\Longrightarrow$ (b). We may assume $k = k'$ and $X = U$. Moreover, it suffices to show for each closed point $y \in X$ that $(\Omega^1_{X/k})_y$ is generated by $\dim \mathcal{O}_{X,y}$ elements. For such a point y, the field $k(y)$ is separable over k. Hence $\Omega^1_{k(y)/k} = 0$, and the exact sequence of 2.1/2 yields an exact sequence

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \longrightarrow (\Omega^1_{X/k})_y \otimes k(y) \longrightarrow 0 .$$

Since $\mathfrak{m}_y/\mathfrak{m}_y^2$ is generated by $\dim \mathcal{O}_{X,y}$ elements (due to assumption (c)), the assertion follows with the help of Nakayama's lemma.

Finally, we turn to the implication (b) $\Longrightarrow$ (a). We may assume that X is a closed subscheme of an open subscheme V of $\mathbb{A}^n_k$, via the immersion $j: X \hookrightarrow \mathbb{A}^n_k$. Let $\mathcal{J}$ be the sheaf of ideals of $\mathcal{O}_V$ defining X, and let $r = \dim_x X$. Looking at the exact sequence of 2.1/2

$$(\mathcal{J}/\mathcal{J}^2)_x \longrightarrow (j^*(\Omega^1_{\mathbb{A}^n_k/k}))_x \longrightarrow (\Omega^1_{X/k})_x \longrightarrow 0 ,$$

we see that there exist local sections $g_{r+1}, \ldots, g_n$ of $\mathcal{J}$ at x such that $dg_{r+1}, \ldots, dg_n$ generate a free direct factor of $(\Omega^1_{\mathbb{A}^n_k/k})_x$ of rank $(n-r)$. We may assume that $g_{r+1}, \ldots, g_n$ are defined on V and give rise to a smooth subscheme $X' \subset V$ of dimension r. So X is a closed subscheme of X' and has the same dimension at x as X'. Let y be a closed point of X, which is a specialization of x. Then, by what we have already seen, $\mathcal{O}_{X',y}$ is an integral domain. Since $\dim \mathcal{O}_{X,y} \geqq r$, the surjective map $\mathcal{O}_{X',y} \longrightarrow \mathcal{O}_{X,y}$ has to be injective by reasons of dimension. This shows that X and X' coincide in a neighborhood of x. □

The property (d) of the preceding proposition gives rise to the following definition. A scheme X which is locally of finite type over a field k is called *geometrically reduced* (resp. *geometrically normal*, resp. *geometrically regular*) if $X \otimes_k k'$ is reduced (resp. normal, resp. regular) for all field extensions k' of k.

Proposition 16. *Let X be locally of finite type over a field k. If X is geometrically reduced, the smooth locus of X is open and dense in X.*

Proof. It is clear that the smooth locus is open. For the proof of the density, consider a generic point x of X. For any field extension k' of k, the algebra $k(x) \otimes_k k'$ is reduced. Then it is an elementary algebraic fact that $\Omega^1_{k(x)/k}$ is generated by n elements where n is the degree of transcendency of $k(x)$ over k; cf. Bourbaki [1], Chap. V, § 16, n°7, Thm. 5. Since n equals the dimension of X at x, Proposition 15 shows x is contained in the smooth locus of X. Thus, the smooth locus contains all generic points of X. □

2.3 Henselian Rings

In the following we want to have a closer look at the local structure of étale morphisms, in particular, we want to construct the (strict) henselization of a local

ring; references for this section are [EGA IV$_4$], 18, and Raynaud [5]. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let S be the affine (local) scheme of R, and let s be the closed point of S. From a geometric point of view, henselian and strictly henselian rings can be introduced via schemes which satisfy certain aspects of the inverse function theorem.

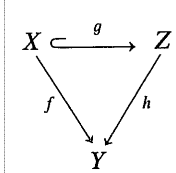
Definition 1. *The local scheme S is called henselian if each étale map $X \longrightarrow S$ is a local isomorphism at all points x of X over s with trivial residue field extension $k(x) = k(s)$. If, in addition, the residue field $k(s)$ is separably closed, S is called strictly henselian.*

Notice that if S is strictly henselian, any étale morphism $X \longrightarrow S$ is a local isomorphism at all points of X over s. Usually one introduces the notion of henselian rings in terms of properties of the local ring R; namely, one requires Hensel's lemma to be true for R. As we will explain later (cf. Proposition 4), it suffices to require a seemingly weaker condition.

Definition 1'. *The local ring R is called henselian if, for each monic polynomial $P \in R[T]$, all k-rational simple zeros of the residue class $\bar{P} \in k[T]$ lift to R-rational zeros of P. If, in addition, the residue field k is separably closed, R is called strictly henselian.*

It is easily seen that the ring R is (strictly) henselian if the scheme S is (strictly) henselian. The converse is also true, but the proof is not so easy; it is mainly a consequence of Zariski's Main Theorem. For the statement of this theorem let us recall the definition of quasi-finite morphisms. Let $f: X \longrightarrow Y$ be a morphism which is locally of finite type. Then f is said to be *quasi-finite at a point x of X* if x is isolated in the fibre $X_y = X \times_Y \operatorname{Spec} k(y)$ over the image point $y := f(x)$; the latter is equivalent to the fact that the ring $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a finite-dimensional vector space over the field $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$, cf. [EGA II], 6.2.1. For example, unramified morphisms are quasi-finite at all points. The set of points $x \in X$ such that f is quasi-finite at x is open in X, cf. [EGA IV$_3$], 13.1.4. The morphism f is called *quasi-finite* if f is quasi-finite at all points $x \in X$ and if f is of finite type. For example, a composition of a quasi-compact open immersion $X \hookrightarrow Z$ and a finite morphism $Z \longrightarrow Y$ is quasi-finite. Zariski's Main Theorem says that essentially every quasi-finite morphism is obtained in this way.

Theorem 2 (Zariski's Main Theorem). *Let $f: X \longrightarrow Y$ be quasi-finite and separated. Furthermore, assume that Y is quasi-compact and quasi-separated. Then there exists a factorization*



of f, where g is an open immersion and where h is finite.

For a proof see [EGA IV$_4$], 18.12.13; a more direct argument (for the local case) can be found in Peskine [1]. For our applications we will need a weaker version which is close to Zariski's original form of the theorem, cf. [EGA IV$_3$], 8.12.10.

Theorem 2′. *Let $f: X \longrightarrow Y$ be quasi-finite and separated. Assume that X is reduced, that Y is normal, and that there exist dense open subschemes $U \subset X$ and $V \subset Y$ such that $f|_U : U \longrightarrow V$ is an isomorphism. Then f is an open immersion.*

Theorem 2 can be used to investigate the local structure of étale morphisms. In terms of the corresponding extension of algebras, an étale extension is sort of a lifting of a finite separable field extension which, due to the theorem of the primitive element, is always generated by a single element.

Proposition 3. *Let $f: X \longrightarrow Y$ be a morphism of schemes, let x be a point of X, and set $y = f(x)$. Assume that f is étale at x. Then there exist an affine open neighborhood $U = \operatorname{Spec} B$ of x, an affine open neighborhood $V = \operatorname{Spec} A$ of y with $f(U) \subset V$ and a Y-immersion $U \hookrightarrow \mathbb{A}^1_V$ such that U becomes an open subscheme of a closed subscheme $Z \subset \mathbb{A}^1_V$, where Z is defined by a monic polynomial $P \in A[T]$ and where the derivative P' of P has no zeros on the image of U. Moreover, B is isomorphic to $(A[T]/(P))_Q$ for some $Q \in A[T]$.*

A detailed proof is given in Raynaud [5], Chap. V. The idea of the proof is easy to explain. Namely, we may assume that X and Y are affine, and, due to Theorem 2, that X is an open subscheme of a scheme $X' = \operatorname{Spec} B'$ which is finite over Y. Since $k(x)$ is finite and separable over $k(y)$, there exists a non-zero element $\bar{b} \in k(x)$ such that $\bar{b}$ generates $k(x)$ over $k(y)$. Let $b \in B'$ be a lifting of $\bar{b}$ which vanishes at all points of the fibre of $X' \longrightarrow Y$ over y, except at x. Now b gives rise to a morphism $X' \longrightarrow \mathbb{A}^1_Y$. Since X' is finite over Y, one can verify that this morphism induces an open immersion of a neighborhood of x into a subscheme Z of $\mathbb{A}^1_Y$ of the required type. □

It follows immediately from Proposition 3 that the notions of henselian local rings and henselian local schemes are equivalent. This equivalence can be extended by further conditions, cf. [EGA IV$_4$], 18.5, or Raynaud [5], Chap. I.

Proposition 4. *Let R be a local ring, and set $S = \operatorname{Spec} R$. Then the following conditions are equivalent:*
(a) *R is henselian.*
(b) *S is henselian.*
(c) *For each finite R-algebra A, the canonical map*

$$\operatorname{Idempotent}(A) \longrightarrow \operatorname{Idempotent}(A \otimes_R k)$$

between the sets of idempotent elements is bijective.
(d) *Each finite R-algebra A decomposes into a product of local rings.*
(e) *For each quasi-finite morphism $X \longrightarrow S$, and for each point x above the closed point of S, there exists an open neighborhood U of x such that $U \longrightarrow S$ is finite.*

We will only sketch the *proof*, following the ideas of Grothendieck. The implications (a)$\Longrightarrow$(b) and (d)$\Longrightarrow$(e) (which are the hard ones) are clear by Proposition 3 and Theorem 2. In order to show that (b) implies (c), one has to observe that it suffices to establish (c) in the case where A is a free R-module. Then one can write down formally what the idempotent elements of A must look like, and one notices that they are represented by an étale R-scheme. So it remains to show that such an étale R-scheme admits an R-section. The proof of the remaining implications is more or less trivial. □

The main reason for us to introduce strictly henselian rings is the fact that smooth schemes over strictly henselian rings admit many sections. Due to the geometric characterization of henselian rings, this property follows directly from 2.2/13 and 2.2/14.

Proposition 5. *Let R be a local henselian ring with residue field k. Let X be a smooth R-scheme. Then the canonical map $X(R) \longrightarrow X(k)$ from the set of R-valued points of X to the set of k-valued points of X is surjective. In particular, if R is strictly henselian, the set of k-valued points of $X_k = X \otimes_R k$ which lift to R-valued points of X is dense in X_k.*

Examples of henselian rings are local rings occurring in analytic geometry such as rings of germs of holomorphic functions. Furthermore, local rings which are separated and complete with respect to the maximal-adic topology are henselian. In the latter case the condition mentioned in Definition 1′ is established by Hensel's lemma; cf. Bourbaki [2], Chap. III, §4, n°3, Thm. 1. Alternatively, using the infinitesimal lifting property 2.2/6 for étale morphisms one can verify directly that such rings fulfill Definition 1. Since a noetherian local ring R is always a subring of its maximal-adic completion $\hat{R}$, these local rings R are a priori subrings of henselian rings. The "smallest" henselian ring containing R is called the henselization of R.

Definition 6. *A henselization of a local ring R is a henselian local ring R^h together with a local morphism $i: R \longrightarrow R^h$ such that the following universal property is satisfied: For any local morphism $u: R \longrightarrow A$ from R to a henselian local ring A, there exists a unique local morphism $u^h: R^h \longrightarrow A$ such that $u^h \circ i = u$.*

If the henselization exists, it is unique up to canonical isomorphism. Moreover, the residue field of R^h must be k. In view of Definition 1, the henselization of R must be the "union" of all local rings $\mathcal{O}_{X,x}$ of étale R-schemes at points x above the closed point s of $S = \operatorname{Spec} R$, whose residue fields coincide with k. That such a "union" exists in terms of inductive limits, becomes clear by the following result:

Lemma 7. *Let S' be an étale R-scheme and let s' be a point of S' above the closed point s of $S = \operatorname{Spec} R$. Let R' be the local ring $\mathcal{O}_{S',s'}$ of S' at s' and let k' be the residue field of R'. Furthermore, let A be a local R-algebra with residue field k_A. Then all R-algebra morphisms from R' to A are local. So there is a canonical map*

$$\operatorname{Hom}_R(R', A) \longrightarrow \operatorname{Hom}_k(k', k_A)\,.$$

This map is always injective; it is bijective if A is henselian.

Proof. Since the maximal ideal of R' is generated by the maximal ideal of R, all R-morphisms $R' \longrightarrow A$ are local. The injectivity of the map follows from the fact that the diagonal morphism $S' \longrightarrow S' \times_S S'$ is an open immersion. The surjectivity is due to the characterization of henselian local rings given in Definition 1. □

For the construction of the henselization of R, one considers the family $(R_i)_{i\in I^h}$ of all isomorphism classes of R-algebras which occur as local rings of étale R-schemes at points over the closed point of Spec R and which have the same residue field as R. Due to Proposition 3, the family I^h is a set and, due to Lemma 7, there is a natural partial order on I^h. Namely, one defines $i \leqq j$ for $i, j \in I^h$ if there exists an R-morphism $u_{ij}: R_i \longrightarrow R_j$. So $(R_i)_{i\in I^h}$ is an inductive system, which is seen to be directed and one easily proves that

$$R^h := \varinjlim_{i\in I^h} R_i$$

is a henselization of R (for details see Raynaud [5], Chap. VIII).

If one wants to introduce the smallest strictly henselian ring containing R, one has to be a little bit more careful. Namely, in view of Lemma 7, there may be different R-morphisms between two (local) étale R-algebras unless we require that the residue extension is trivial. One has to eliminate this ambiguity, and then one can proceed as in the case of the henselization.

Definition 6′. *A strict henselization of a local ring R is a strictly henselian local ring R^{sh}, whose residue field coincides with the separable algebraic closure k_s of k, together with a local morphism $i: R \longrightarrow R^{sh}$ such that the following universal property is satisfied: For any local morphism $u: R \longrightarrow A$ from R to a strictly henselian ring A, and for any k-morphism $\alpha: k_s \longrightarrow k_A$ from k_s to the residue field k_A of A, there exists a unique local morphism $u^{sh}: R^{sh} \longrightarrow A$ such that $u^{sh} \circ i = u$ and such that u^{sh} induces α on the residue fields.*

If R^{sh} exists, it is unique up to canonical isomorphism. For the construction of R^{sh}, let $(R_i)_{i\in I}$ be the family of all isomorphism classes of R-algebras which occur as local rings of étale R-schemes at points over the closed point of Spec R. Let I^{sh} be the set of all couples (R_i, α_{ij}) where R_i is a member of I and where $\alpha_{ij}: R_i \longrightarrow k_s$ varies over all R-morphisms into a fixed separable closure k_s of k. Due to Lemma 7, there exists a natural order on I^{sh}. So $((R_i, \alpha_{ij}))_{(i,j)\in I^{sh}}$ is a directed inductive system, and one easily verifies that

$$R^{sh} = \varinjlim_{(i,j)\in I^{sh}} (R_i, \alpha_{ij})$$

is the strict henselization of R; cf. Raynaud [5], Chap. VIII.

As an application of this construction, we want to mention some results on étale localizations of quasi-finite morphisms. Let us call $Y' \longrightarrow Y$ an *étale neighborhood of a point y in Y* if $Y' \longrightarrow Y$ is étale and if y is contained in the image of Y'.

Proposition 8. *Let $f: X \dashrightarrow Y$ be locally of finite type. Let x be a point of X, and set $y = f(x)$.*

(a) If f is quasi-finite at x, then there exists an étale neighborhood $Y' \longrightarrow Y$ of y such that the morphism $f': X' \longrightarrow Y'$, obtained from f by the base change $Y' \longrightarrow Y$, induces a finite morphism $f'|_{U'}: U' \longrightarrow Y'$, where U′ is an open neighborhood of the fibre of $X' \longrightarrow X$ above x. If, in addition, f is separated, U′ is a connected component of X′.

(b) If f is unramified at x (resp. étale at x), there exists an étale neighborhood $Y' \longrightarrow Y$ of y such that, locally at each point of X′ above x, the morphism f′ (as in (a)) is an immersion (resp. an open immersion).

Proof. Let R be a strict henselization of the local ring $\mathcal{O}_{Y,y}$ of Y at y, and set $S = \operatorname{Spec} R$. Then R is the limit of all local rings $\mathcal{O}_{Y',y'}$ which occur as local rings of étale neighborhoods Y' of $y \in Y$ at points y' above y. Using limit arguments (cf. [EGA IV$_3$], 8.10.5), it suffices to prove the assertions in the case where $Y = S$. Then (a) follows from Proposition 4, and (b) is a consequence of the fact that each finite, local, and unramified R-algebra A is a quotient of R. Namely, the assumptions yield $R/\mathfrak{m} = A/\mathfrak{m}A$, where $\mathfrak{m}$ is the maximal ideal of R, and so Nakayama's lemma applies. Finally, the case of étale morphisms is deduced from the case of unramified ones by means of 2.2/4. □

The preceding proposition justifies the interpretation of unramified, resp. étale, resp. smooth morphisms given in 2.2. Namely, Proposition 8 tells us that, up to base change by étale morphisms, unramified morphisms are immersions and étale morphisms are open immersions. So, if we look at S-schemes X only up to étale base change, as it is done within the context of the étale topology or, more generally, in the theory of algebraic spaces, we may view unramified morphisms as immersions and étale morphisms as open immersions. Furthermore, Proposition 2.2/11 says that smooth morphisms may be viewed as fibrations by open subsets of linear spaces $\mathbb{A}^n_S$.

The local structure of étale morphisms $X \longrightarrow Y$ (cf. Proposition 3) can be used to study how algebraic properties are transmitted from Y to X. By a minor calculation (cf. Raynaud [5], Chap. VII), one shows that all étale schemes over a reduced (resp. normal) base are reduced (resp. normal) again. Using the elementary fact that polynomial rings inherit such properties from the base, it follows from 2.2/11 that smooth schemes over a reduced (resp. normal) base are reduced (resp. normal) again. Finally, since polynomial rings over regular rings are regular, smooth schemes over regular schemes are regular again; use 2.2/11 and 2.2/2(e). Summarizing, we can say:

Proposition 9. *Let $X \longrightarrow Y$ be a smooth morphism. If Y is reduced (resp. normal, resp. regular), then X is reduced (resp. normal, resp. regular).*

Obviously, a directed inductive limit R of reduced (resp. normal) rings R_i is reduced (resp. normal). So we have the permanence of reducedness and normality for the (strict) henselization. Moreover, since the maximal ideal $\mathfrak{m}$ of R generates