For the *proof of Theorem* 1, we will use the canonical S-ample sheaf $\mathscr{L}(X/S)$ which was constructed in Proposition 4 for smooth curves X over S. Namely, due to Theorem 9.3/7 and the explanation following it, we know already that $\mathrm{Pic}^0_{X/S}$ is a scheme locally for the étale topology on S. Thus, we are concerned with a problem of descent. It suffices to verify the assertion concerning the canonical S-ample invertible sheaf $\mathscr{L}(X/S)$. Due to 6.1/7, it is enough to give the definition of $\mathscr{L}(X/S)$ after étale surjective base extension. Moreover, it suffices to look at the universal case. Since the base of the versal deformations of a fibre of X is smooth over $\mathbb{Z}$ (cf. Deligne and Mumford [1], Cor. 1.7), we may assume that S is regular. In this situation, we have to construct $\mathscr{L}(X/S)$. Denote by S_0 the open subscheme of S where X is smooth over S; note that S_0 is dense in S. Due to Proposition 4, there is a canonical line bundle $\mathscr{L}(X_0/S_0)$ on $\mathrm{Pic}^0_{X_0/S_0}$. Since S is regular, we can extend $\mathscr{L}(X_0/S_0)$ to a line bundle $\mathscr{L}(X/S)$ on $\mathrm{Pic}^0_{X/S}$ such that the pull-back of $\mathscr{L}(X/S)$ under the unit section is trivial on S. Since the geometric fibres of $\mathrm{Pic}^0_{X/S}$ are connected, the extension is unique. Then it follows from Raynaud [4], Thm. XI.1.13, page 170, that $\mathscr{L}(X/S)$ is S-ample, since the restriction of $\mathscr{L}(X/S)$ to S_0 is S_0-ample and since, for all points $s \in S$ of codimension 1, the restriction of $\mathrm{Pic}^0_{X/S}$ to $\mathrm{Spec}(\mathscr{O}_{S,s})$ is the identity component of the Néron model of its generic fibre; cf. 7.4/3 and 9.2/8. □

Finally we want to sketch the *proof of Theorem* 2. Denote the generic point of S by η and the closed point of S by s. Let P be the open subfunctor of $\mathrm{Pic}_{X/S}$ consisting of all line bundles of total degree zero.

Let $Y \hookrightarrow X$ be a rigidificator for $\mathrm{Pic}_{X/S}$; cf. 8.1/6. Then, due to 8.3/3, the functor $(\mathrm{Pic}_{X/S}, Y)$ is an algebraic space over S. Denote by (P, Y) the open subfunctor of $(\mathrm{Pic}_{X/S}, Y)$ consisting of all line bundles of total degree zero. Due to 8.4/2, (P, Y) is smooth over S. Let

$$r : (P, Y) \longrightarrow P$$

be the canonical morphism. There is a largest separated quotient Q of P (in the sense of sheaves for the fppf-topology), and one knows that Q is a smooth and separated S-group scheme; cf. 9.5/3. Let

$$q : P \longrightarrow Q$$

be the canonical morphism. It is clear that r and q are epimorphisms of sheaves with respect to the fppf-topology.

We want to show that q induces an isomorphism of P^0 to Q^0. Note that q_η is an isomorphism. First we want to see that $q \times_S S'$ admits a section over Q^0 where S' is a strict henselization of S. We may assume $S = S'$. Due to 9.1/12, there exists a universal line bundle $\mathscr{L}_\eta$ on $(X \times_S \mathrm{Pic}_{X/S})_\eta$. Let $(\mathscr{M}, \alpha)$ be the universal line bundle of (P, Y). Since $\mathscr{L}_\eta$ induces the universal line bundle of P_η, the line bundles $(\mathrm{id}_X \times q \circ r)^*\mathscr{L}_\eta$ and $\mathscr{M}_\eta$ define the same homomorphism to P_η. So, due to 8.1/4, there exists a line bundle $\mathscr{N}_\eta$ on $(P, Y)_\eta$ such that

$$(\mathrm{id}_X \times q \circ r)^*\mathscr{L}_\eta \cong \mathscr{M}_\eta \otimes f^*(\mathscr{N}_\eta)$$

Since (P, Y) is smooth over S and since S is regular, $\mathscr{N}_\eta$ extends to a line bundle $\mathscr{N}$ on (P, Y). After replacing $\mathscr{M}$ by $\mathscr{M} \otimes f^*\mathscr{N}$, we may assume that $\mathscr{M}$ extends $(\mathrm{id}_X \times q \circ r)^*\mathscr{L}_\eta$. By computing the associated divisor, one can show that, over the identity component $(P, Y)^0$, the line bundle $\mathscr{M}|_{X \times_S (P,Y)^0}$ descends to a line bundle $\mathscr{L}$ on $X \times_S Q^0$. Namely, as X is normal, $\mathscr{M}$ is determined by a Weil divisor D on $X \times (P, Y)^0$. Since $\mathscr{M}_\eta$ descends to $\mathscr{L}_\eta$, we may assume that D_η descends, too. So it suffices to look at "vertical" Weil divisors on $X \times (P, Y)^0$ with support contained in the special fibre. To treat the latter we remark that the sets of vertical Weil divisors (with support contained in special fibres) on X, on $X \times (P, Y)^0$, or, on $X \times Q^0$ are in one-to-one correspondence under the pull-back maps. Then $\mathscr{L}$ gives rise to a morphism $\lambda : Q^0 \longrightarrow P^0$. Since Q is separated and since $(q \circ \lambda)_\eta = \mathrm{id}_{Q^0_\eta}$, it follows that $q \circ \lambda = \mathrm{id}_{Q^0}$. Moreover, one shows easily that λ is a group homomorphism.

Next we claim that P is an algebraic space over S. Due to 8.3/1, it remains to see that f is cohomologically flat in dimension zero. By what we have said at the beginning of this section, it suffices to show that

$$\dim_{k(s)} H^1(X_s, \mathscr{O}_{X_s}) = \dim_{k(\eta)} H^1(X_\eta, \mathscr{O}_{X_\eta}) .$$

Due to 8.4/1, we know that $\dim_{k(s)} H^1(X_s, \mathscr{O}_{X_s})$ is equal to the dimension of $\mathrm{Pic}_{X_s/k(s)} = (\mathrm{Pic}_{X/S})_s$. Moreover we have $\dim P_\eta = \dim Q_\eta = \dim Q_s$. The latter holds, since Q is flat over S. So it remains to see that the canonical map $q_s : P_s \longrightarrow Q_s$ is locally quasi-finite or, that the kernel of $q_s|_{P^0_s}$ is finitely generated as an abstract group. Indeed, a group scheme of finite type over a field whose group of geometric points is finitely generated is finite; so the morphism $q_s|_{P^0_s}$ is quasi-finite, since P^0_s is of finite type over $k(s)$. The kernel of $q_s|_{P^0_s}$ is smooth over $k(s)$ since, due to the existence of the section λ_s, it is a quotient of the smooth group P^0_s. So, assuming that S is strictly henselian, it remains to see that the set of $k(s)$-rational points of the kernel is finitely generated. Since the map $(P, Y)_s \longrightarrow P_s$ is smooth, the rational points of P_s are induced by rational points of $(P, Y)_s$. Since (P, Y) is smooth over S, the rational points of $(P, Y)_s$ are induced by S-valued points of (P, Y); in particular, by line bundles on X. Due to the existence of the section λ which is defined by a line bundle, we see that the $k(s)$-rational points of the kernel of $q_s|_{P^0_s}$ are induced by line bundles on X which are trivial on the generic fibre. Due to the assumption on X, such a line bundle $\mathscr{L}$ is associated to a Cartier divisor D having support on the special fibre only; hence $\mathscr{L} \cong \mathscr{O}_X(D)$. Thus we see that the kernel of the morphism $q_s|_{P^0_s}$ is finitely generated as an abstract group; namely, the group of Cartier divisors having support only on the special fibre is a subgroup of the free group generated by the irreducible components of the special fibre of X.

Now it is easy to complete the proof. In order to show that $q : P^0 \longrightarrow Q^0$ is an isomorphism, we may assume that S is strictly henselian. Recall that q is unramified and an isomorphism on generic fibres. Now look at the commutative diagram

$$\begin{tikzcd} Q^0 \arrow[rr, "\lambda"] \arrow[dr, "\mathrm{id}"'] & & P^0 \arrow[dl, "q"] \\ & Q^0 & \end{tikzcd}$$

It follows from 2.2/9 that λ is étale. Then it is clear that λ and, hence, q are isomorphisms. □

Finally we want to mention that, in the case where X is regular, there is a direct proof of the cohomological flatness in Artin and Winters [1] which uses the intersection form.

9.5 Picard Functor and Néron Models of Jacobians

Let $S = \operatorname{Spec} R$ be a base scheme consisting of a discrete valuation ring R. As usual we denote by K the field of fractions of R and by k the residue field of R. In the following we will fix a proper and flat curve X over S; its generic fibre X_K is assumed to be *normal* as well as *geometrically irreducible*. Let $J_K = \operatorname{Pic}^0_{X_K/K}$ be the Jacobian of X_K. It is a smooth and connected K-group scheme of finite type and we can ask if there is a Néron model J of J_K. The purpose of the present section is to describe J, if it exists, in terms of the relative Picard functor $\operatorname{Pic}_{X/S}$. Thereby we will obtain a second method to construct Néron models, which is largely independent of the original method involving the smoothening process.

The key point of the whole construction is the fact that the relative Picard functor $\operatorname{Pic}_{X/S}$ satisfies a mapping property which is similar to the one enjoyed by Néron models. To explain this point, assume that X is regular and that X_K admits a section. Furthermore, consider a smooth S-scheme T and a K-morphism $u_K : T_K \longrightarrow \operatorname{Pic}_{X_K/K}$. Then, using 8.1/4, u_K corresponds to a line bundle ξ_K on $X_K \times_K T_K$, and the latter extends to a line bundle ξ on $X \times_S T$ since $X \times_S T$ is regular; see 2.3/9. Thus it follows that u_K extends to an S-morphism $u : T \longrightarrow \operatorname{Pic}_{X/S}$, where u is unique if $\operatorname{Pic}_{X/S}$ is separated. The same mapping property holds for $\operatorname{Pic}^0_{X/S}$ if the special fibre X_k is geometrically irreducible; use 9.1/2 and 9.2/13. So if, in addition, we know that $\operatorname{Pic}^0_{X/S}$ is a smooth and separated S-group scheme, for example if we are in the situation of Grothendieck's theorem 9.3/1, it follows that $\operatorname{Pic}^0_{X/S}$ is a Néron model of $J_K = \operatorname{Pic}^0_{X_K/K}$. In the latter case the assumption on X to have a section is not really necessary. Namely, if the special fibre of X is geometrically reduced (as is required in 9.3/1), then the smooth locus of X is faithfully flat over S by 2.2/16. Working over a strict henselization R^{sh} of R, it follows from 2.3/5 that $X \otimes_R R^{sh}$ admits a section. So, due to the fact that Néron models descend from R^{sh} to R by 6.5/3, we can state the following result.

Theorem 1. *Let X be a flat projective curve over S which is regular and which has geometrically reduced and irreducible fibres. Then $\operatorname{Pic}^0_{X/S}$ is a Néron model of its generic fibre; i.e., of the Jacobian J_K of X_K. In particular, the special fibre of the Néron model of J_K is connected.*

Before we construct Néron models of Jacobians J_K of a more general type, let us state the mapping property of the relative Picard functor $\operatorname{Pic}_{X/S}$ in the form we will need it later. The curve X is as mentioned at the beginning of this section.

Lemma 2. *Assume either that X_K admits a section or that K is the field of fractions of a henselian discrete valuation ring R with algebraically closed residue field k. Then each element of $\operatorname{Pic}_{X/S}(K)$ is represented by a line bundle on X_K. In particular, if X is regular, the canonical map $\operatorname{Pic}_{X/S}(R) \longrightarrow \operatorname{Pic}_{X/S}(K)$ is surjective.*

Proof. Let K' be the direct image of $\mathcal{O}_{X_K}$ with respect to the structural morphism $X_K \longrightarrow \operatorname{Spec} K$. Since X_K is geometrically irreducible, K' is a field and the extension K'/K is finite and purely inseparable. If X_K admits a section, K' coincides with K and the first assertion of the lemma follows from 8.1/4. On the other hand, if R is henselian and k is algebraically closed, there is a classical result of Lang saying that the cohomological Brauer group $\operatorname{Br}(K)$ vanishes (see Grothendieck [3], 1.1, or Milne [1], Chap. III, 2.22). In the same way we can show that $\operatorname{Br}(K')$ vanishes. Namely, K' can be viewed as the field of fractions of the integral closure R' of R in K' and R' is a discrete henselian valuation ring with algebraically closed residue field k; use 2.3/1′ or 2.3/4 (d) to show that R' is henselian. Thereby we see that there are no obstructions to representing elements of $\operatorname{Pic}_{X/S}(K)$ by line bundles on X_K; cf. 8.1/4.

If X is regular, each line bundle on X_K extends to a line bundle on X and the second assertion is clear also. □

If X is more general than in Theorem 1, but say, still regular, $\operatorname{Pic}^0_{X/S}$ might not be representable by a scheme or by an algebraic space. Moreover, even if $\operatorname{Pic}^0_{X/S}$ exists as a scheme and, thus, is a smooth scheme by 8.4/2 (for example, if X admits a section), the canonical map $\operatorname{Pic}^0_{X/S} \longrightarrow J$ to a possible Néron model J of J_K is not necessarily surjective. To remedy this, we replace $\operatorname{Pic}^0_{X/S}$ by the open and closed subsheaf $P \subset \operatorname{Pic}_{X/S}$ consisting of all line bundles of total degree 0 and pass to the biggest separated quotient Q of P. As we will see, the latter is a good candidate for a Néron model of J_K.

The subfunctor $P \subset \operatorname{Pic}_{X/S}$ may be viewed as the kernel of the degree morphism $\deg : \operatorname{Pic}_{X/S} \longrightarrow \mathbb{Z}$ and is formally smooth since the same is true for $\operatorname{Pic}_{X/S}$; cf. 8.4/2. Furthermore, the fibres of P over S are representable by smooth schemes (8.2/3 and 8.4/2) and, on the generic fibre, P coincides with $\operatorname{Pic}^0_{X/S}$ so that $P_K = J_K$.

In order to pass to the biggest separated quotient of P, we extend the notion of separatedness from S-schemes to contravariant functors $(\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ by using the valuative criterion as a definition; thus a contravariant functor $F : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ is called separated if, for any discrete valuation ring R' over R with field of fractions K', the canonical map $F(\operatorname{Spec} R') \longrightarrow F(\operatorname{Spec} K')$ is injective. If F is representable by a scheme or by an algebraic space and if the latter are locally of finite type over S (which, for algebraic spaces, is automatically the case by our definition), then the separatedness in terms of functors coincides with the usual notion of separatedness for schemes or algebraic spaces.

Now consider the quotient $Q = P/E$ (say, in the sense of fppf-sheaves) where E is the schematic closure in P of the unit section $S_K \longrightarrow \operatorname{Pic}_{X_K/K}$; then E is a subgroup functor of P. To define E if $\operatorname{Pic}_{X/S}$ is not necessarily representable by a scheme (or by an algebraic space), consider the sub-fppf-sheaf of $\operatorname{Pic}_{X/S}$ which is generated by all morphisms $Z \longrightarrow \mathrm{Pic}_{X/S}$ in $\mathrm{Pic}_{X/S}(Z)$ where Z is flat over S and where $Z_K \longrightarrow (\mathrm{Pic}_{X/S})_K = \mathrm{Pic}_{X_K/K}$ factors through the unit section of $\mathrm{Pic}_{X_K/K}$. Since the latter is a closed immersion, one recovers the usual notion of schematic closure if $\mathrm{Pic}_{X/S}$ exists as a scheme or as an algebraic space. Likewise, one can extend the notion of schematic closure in $\mathrm{Pic}_{X/S}$ to any closed subscheme of the generic fibre of $\mathrm{Pic}_{X/S}$. For example, we can view P as the schematic closure in $\mathrm{Pic}_{X/S}$ of the Jacobian $\mathrm{Pic}^0_{X_K/K} = J_K$.

Proposition 3. *As before, let X be a flat proper curve over S such that X_K is normal and geometrically irreducible. Then the quotient $Q = P/E$ is representable by a smooth and separated S-group scheme; it is the biggest separated quotient of P. Furthermore, the projection $P \longrightarrow Q$ is an isomorphism on generic fibres and, thus, the generic fibre of Q coincides with the Jacobian J_K of X_K.*

Proof. Instead of just dealing with the most general case, we will explain how to proceed depending on what is known about $\mathrm{Pic}_{X/S}$. That $P \longrightarrow Q$ is an isomorphism on generic fibres is due to the fact that, by the definition of E, the generic fibre E_K coincides with the generic fibre of the unit section $S \longrightarrow P$ since the generic fibre of P is separated. Furthermore, it is clear that Q is the biggest separated quotient of P if Q is representable by a separated scheme.

1st case: $\mathrm{Pic}_{X/S}$ *is a scheme.* In this situation P is a smooth group scheme whose identity component P^0 is separated by [SGA 3_{I}], Exp. $\mathrm{VI_B}$, 5.5. So the intersection of E with P^0 is trivial and it follows that E is étale over S. More precisely, $E \longrightarrow S$ is a local isomorphism with respect to the Zariski topology. Then it is easily seen that the quotient $Q = P/E$ is representable by a smooth scheme and that the projection $P \longrightarrow Q$ is a local isomorphism with respect to the Zariski topology.

2nd case: $\mathrm{Pic}_{X/S}$ *is an algebraic space.* Since the unit section of P is locally closed, E is still étale over S, and it is clear that the quotient $Q = P/E$ exists as an algebraic S-group space which is smooth and separated. Furthermore, it follows from 6.6/3 that Q is an S-group scheme.

3rd case: $\mathrm{Pic}_{X/S}$ *is not necessarily representable by a scheme or by an algebraic space.* Then we can apply 8.1/6 and choose a rigidificator $Y \subset X$ of the structural morphism $f: X \longrightarrow S$. Associated to it is a sequence

$$0 \longrightarrow V_X^* \hookrightarrow V_Y^* \longrightarrow (\mathrm{Pic}_{X/S}, Y) \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow 0$$

which is exact with respect to the étale topology; cf. 8.1/11. Considering only line bundles of total degree 0, this sequence restricts to a sequence

$$0 \longrightarrow V_X^* \hookrightarrow V_Y^* \longrightarrow (P, Y) \longrightarrow P \longrightarrow 0$$

which, again, is exact with respect to the étale topology. One knows from 8.3/3 and 8.4/2 that $(\mathrm{Pic}_{X/S}, Y)$ and, hence, (P, Y) is an algebraic space which is smooth over S.

Consider the exact sequence

$$V_Y^* \longrightarrow (P, Y) \xrightarrow{r} P \longrightarrow 0\,,$$

and let H be the schematic closure of the kernel of r_K. Then H is an algebraic subgroup space of (P, Y); it contains the kernel of r, as is easily seen by using the fact that V_Y^* is flat over S. Furthermore, the quotient $(P, Y)/H$ exists as an algebraic space by 8.3/9 since H is flat over S; it is separated due to the definition of H. We claim that $(P, Y)/H$ is canonically isomorphic to $Q = P/E$. To see this, we mention that, by continuity, r maps H into E. So r induces a morphism $\bar{r}: (P, Y)/H \longrightarrow P/E$. On the other hand, one concludes from $\ker(r) \subset H$ that the projection $(P, Y) \longrightarrow (P, Y)/H$ splits into morphisms

$$(P, Y) \xrightarrow{r} P \xrightarrow{q} (P, Y)/H\,.$$

Since $(P, Y)/H$ is separated and, thus, $E \subset \ker q$, we thereby obtain a morphism $\bar{q}: P/E \longrightarrow (P, Y)/H$ which is an inverse of $\bar{r}$. So Q is isomorphic to $(P, Y)/H$ and therefore is an algebraic group space. But then Q is a separated group scheme by 6.6/3, which is smooth by the analogue of [SGA 3_{I}], Exp. $\mathrm{VI_B}$, 9.2, for algebraic group spaces. □

In order to show that the smooth and separated S-group scheme Q of Proposition 3 is, in fact, a Néron model of J_K, we have to work under conditions like the ones given in Lemma 2 assuring that each K-valued point of Q extends to an R-valued point of Q (assuming R to be strictly henselian). Also we have to show that Q is of finite type over S.

Theorem 4. *Let X be a proper and flat curve over $S = \mathrm{Spec}\, R$ whose generic fibre is geometrically irreducible. Assume that, in addition, X is regular and either that the residue field k of R is perfect or that X admits an étale quasi-section. Then:*

(a) *If P denotes the open subfunctor of $\mathrm{Pic}_{X/S}$ given by line bundles of total degree 0 and if E is the schematic closure in P of the unit section $S_K \longrightarrow P_K$, then $Q = P/E$ is a Néron model of the Jacobian J_K of X_K.*

(b) *Let $X_1, \ldots, X_n$ be the irreducible components of the special fibre X_k and let δ_i be the geometric multiplicity of X_i in X_k; cf. 9.1/3. Assume that the greatest common divisor of the δ_i is 1. Then $\mathrm{Pic}^0_{X/S}$ is a separated scheme and, consequently, the projection $P \longrightarrow Q$ gives rise to an isomorphism $\mathrm{Pic}^0_{X/S} \xrightarrow{\sim} Q^0$. Thus, in this case, $\mathrm{Pic}^0_{X/S}$ coincides with the identity component of the Néron model of J_K.*

Remark 5. In the situation of the theorem, the assumption that X admits an étale quasi-section is automatically satisfied if the special fibre X_k is geometrically reduced or, more generally, if X_k contains an irreducible component which has geometric multiplicity 1 in X_k. Namely, then the smooth part of X must meet such a component and, passing to a strict henselization of S, we have a section by 2.3/5. On the other hand, if X admits an étale quasi-section over S, say a true section after we have replaced S by an étale extension, then, X being regular, this section factors through the smooth locus of X; see 3.1/2. In particular, there are irreducible components which have geometric multiplicity 1 in X_k so that the condition in Theorem 4 (b) is automatically satisfied.

Now let us start with the *proof of Theorem* 4. The main part will be to show that Q is of finite type over S. We will use the remainder of the present section to establish this fact; see Lemmata 7 and 11 below. But let us first explain how to obtain assertions (a) and (b) if we know that Q is of finite type.

The formation of the schematic closure E is compatible with flat extensions of valuation rings. Likewise, the regularity of X remains invariant under étale base change by 2.3/9. Thus, in order to show that Q is a Néron model of J_K, we may assume that R is strictly henselian.

It is already known from Proposition 3 that Q is a smooth and separated S-group scheme with generic fibre J_K. Furthermore, it follows from Lemma 2 and 9.1/2 that the canonical map $P(R) \longrightarrow P(K)$ is surjective. So we see that the canonical map $Q(R) \longrightarrow Q(K)$ is surjective and, hence, bijective since Q is separated. Thus, if Q is of finite type, it is a Néron model of J_K by the criterion 7.1/1. This verifies assertion (a). Using the representability result 9.4/2 for $\mathrm{Pic}^0_{X/S}$, assertion (b) is a consequence of assertion (a).

It remains to show that the quotient $Q = P/E$ is of finite type over S. We will present two methods to obtain this result. The first one is based on the existence theorem for Néron models 10.2/1 and uses the fact that the Néron-Severi group of the special fibre of $\mathrm{Pic}_{X/S}$ is finitely generated. But it works only under the additional assumption that the generic fibre X_K is geometrically reduced (which is the case if X admits an étale quasi-section; see 3.1/2). Relying on the existence of a Néron model J of J_K, there is a canonical morphism $Q \longrightarrow J$ and it is to show that the latter is an isomorphism. The second method is independent of the theory of Néron models and uses the intersection form which is associated to the irreducible components of the special fibre X_k. It works in the general situation of Theorem 4 and, as we will see in Section 9.6, provides a means of computing the group of connected components (of the special fibre) of the Néron model J of J_K.

Q is of finite type, a first proof via the existence of a Néron model J of J_K. We start by translating the existence theorem for Néron models 10.2/1 to our situation, a result which we will prove in Chapter 10 and which is independent of Chapter 9.

Proposition 6. *Let X_K be a proper curve over K which is geometrically reduced and irreducible. Let J_K be its Jacobian. Then J_K admits a Néron model J of finite type over S if any of the following conditions is satisfied*:

(a) *X_K is smooth,*
(b) *$X_K \times_K \hat{K}$ is normal, where $\hat{K}$ is the completion of K,*
(c) *X_K is normal and R is excellent.*

Proof. If X_K is smooth, J_K is an abelian variety by 9.2/3. So J_K has a Néron model J of finite type.

If only condition (b) is known, J_K is not necessarily an abelian variety. However, condition (b) is compatible with separable extensions of the field $\hat{K}$. So, for any separable field extension L over $\hat{K}$, we know from 9.2/4 that J_L does not contain subgroups of type $\mathbb{G}_a$ or $\mathbb{G}_m$. Therefore we can conclude from 10.2/1 that J_K has a Néron model J of finite type.

Finally, condition (c) implies condition (b) since $\hat{K}$ is separable over K in this case. □

Let us apply Proposition 6 in order to show that, in the situation of Theorem 4 and under the additional assumption of X_K being geometrically reduced, the Jacobian J_K of X_K admits a Néron model of finite type. Since X is proper over S, all closed points of X belong to the special fibre X_k. Therefore, if $\hat{R}$ is the completion of R, the local rings at closed points of $X_{\hat{R}}$ may be viewed as completions of local rings of X and, thus, the hypothesis on the regularity of X remains unchanged if we replace R by its completion $\hat{R}$. So, in particular, $X_{\hat{R}}$ is regular and, thus, J_K admits a Néron model J of finite type by Proposition 6. Now it is quite easy to prove that Q is of finite type.

Lemma 7. *In the situation of Theorem 4, assume that X_K is geometrically reduced. Then $Q = P/E$ is of finite type over S.*

Proof. As we have just seen, J_K admits a Néron model J. Since the formation of Q and of J is compatible with étale base change, we may assume that the base ring R is strictly henselian. Furthermore, recall that Q is a smooth and separated S-group scheme such that the canonical map $Q(R) \longrightarrow Q(K)$ is bijective. It is enough to show that the canonical morphism $v : Q \longrightarrow J$ restricts to an isomorphism $Q^0 \xrightarrow{\sim} J^0$. Namely, using the bijectivity of $Q(R) \longrightarrow J(R)$, this implies that the groups $Q(R)/Q^0(R)$ and $J(R)/J^0(R)$, which by 2.3/5 can be interpreted as the groups of connected components of the special fibres of Q and J, coincide and thus are finite. Consequently, Q will be of finite type.

So let us show that v induces an isomorphism $Q^0 \longrightarrow J^0$. The group of connected components $Q(R)/Q^0(R) = Q(k)/Q^0(k)$ may be viewed as a quotient of a subgroup of the Néron-Severi group of the special fibre of $\mathrm{Pic}_{X/S}$ and, thus, is finitely generated (in the sense of abstract groups); see 9.2/14. Since the map $v : Q \longrightarrow J$ is surjective on R-valued points and, hence, on k-valued points, it follows that the quotient $J^0_k/v(Q^0_k)$ is a connected smooth algebraic group over k whose group of k-valued points is finitely generated. However, then $J^0_k/v(Q^0_k)$ must be of dimension zero and, thus, is trivial as is easily seen by considering the multiplication with an integer n not divisible by char k. Therefore $Q^0 \longrightarrow J^0$ is surjective and quasi-finite. But then, being an isomorphism on generic fibres, it must be an isomorphism by Zariski's Main Theorem 2.3/2′ so that the desired assertion on Q follows. □

Q is of finite type, a second proof via the intersection form associated to the special fibre X_k. This approach requires a detailed analysis of divisors on X which have support on the special fibre X_k only.

Lemma 8. *Let X be a proper flat curve over $S = \operatorname{Spec} R$ such that X is normal and such that X_K is geometrically irreducible. Assume that R is a strictly henselian discrete valuation ring. Let D be the group of Cartier divisors on X which have support on the special fibre X_k, let D_0 be the subgroup of all divisors in D which are principal, and let E be as in Theorem 4. Then the canonical map $D/D_0 \longrightarrow E(R)$ is bijective.*

Proof. The injectivity of the map follows from 8.1/3. To show the surjectivity, we consider the Stein factorization

$$X \xrightarrow{g} Y \xrightarrow{h} S$$

of the structural morphism $f: X \longrightarrow S$, where $g_*(\mathcal{O}_X) = \mathcal{O}_Y$ and where $h: Y \longrightarrow S$ is finite. Then Y is the spectrum of a normal ring R' which is finite over R. Since X_K is geometrically irreducible and since X is normal, it follows that $K' = R' \otimes_R K$ is a finite purely inseparable field extension of K and that R' is the integral closure of R in K'. So, similarly as in the proof of Lemma 2, it is seen that R' is a strictly henselian discrete valuation ring and that each $a \in E(R)$ is represented by a line bundle $\mathscr{L}$ on X.

Now fix a point $a \in E(R)$ and a representing line bundle $\mathscr{L}$ on X. Since the restriction of $\mathscr{L}$ to the generic fibre X_K is trivial, $\mathscr{L}$ is of the form $\mathcal{O}_X(\Delta)$ where Δ is a Cartier divisor on X having support on the special fibre of X. Thus a is represented by $\Delta \in D$. □

Let $(X_i)_{i \in I}$ be the family of reduced irreducible components of the special fibre X_k. As in 9.1/3, we write d_i for the multiplicity of X_i in X_k and e_i for the geometric multiplicity of X_i. Then e_i is a power of the characteristic of k and $\delta_i = d_i e_i$ is the geometric multiplicity of X_i in X_k; cf. 9.1/4.

For any line bundle $\mathscr{L}$ on X, one can consider its degree $\deg_i(\mathscr{L})$ on the component X_i; it is a multiple of the geometric multiplicity e_i of X_i; cf. 9.1/8. In particular, we can consider the map

$$\rho: \mathrm{Pic}(X) \longrightarrow \mathbb{Z}^I, \qquad \mathscr{L} \longmapsto (e_i^{-1} \cdot \deg_i(\mathscr{L}))_{i \in I}$$

which, composed with the canonical map $D \longrightarrow \mathrm{Pic}(X)$ yields a map $\alpha: D \longrightarrow \mathbb{Z}^I$, where D is as in Lemma 8.

Lemma 9. *Let R, X, D, D_0, and E be as in Lemma 8. Then there is a canonical complex*

$$0 \longrightarrow D_0 \hookrightarrow D \xrightarrow{\alpha} \mathbb{Z}^I \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0$$

where β is given by $\beta(a_1, \ldots, a_r) := \sum a_i \delta_i$. The latter gives rise to a surjection

$$\sigma: \ker \beta / \mathrm{im}\, \alpha \longrightarrow Q(S)/Q^0(S)$$

which is bijective if $P \longrightarrow Q = P/E$ induces a surjection

$$\mathrm{Pic}^0_{X/S}(S) \longrightarrow Q^0(S)$$

between S-valued points of the identity components of $\mathrm{Pic}_{X/S}$ and Q. Furthermore, if $\mathrm{im}\, \alpha$ has rank $\mathrm{card}(I) - 1$, then $\ker \beta / \mathrm{im}\, \alpha$ and, thus, also $Q(S)/Q^0(S)$ is finite.

Proof. To begin with, recall that divisors in D have total degree 0 and that therefore $\beta \circ \alpha = 0$ by 9.1/4 and 9.1/5. So the sequence in question is a complex. Furthermore, the map $\rho: \mathrm{Pic}(X) \longrightarrow \mathbb{Z}^I$ is surjective by 9.1/10. Since R is strictly henselian and since $\mathrm{Pic}_{X/S}$ can be defined by using the étale topology in place of the fppf-topology, we can interpret $\mathrm{Pic}(X)$ as $\mathrm{Pic}_{X/S}(S)$. So $P(S)$ is mapped surjectively onto $\ker \beta$ and, due to 9.2/13, we have the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_{X/S}(S) \longrightarrow P(S) \longrightarrow \ker \beta \longrightarrow 0 .$$

Using Lemma 8 we can interpret $\mathrm{im}\, \alpha$ as the image of $E(S)$ under the map $\rho: \mathrm{Pic}(X) \longrightarrow \mathbb{Z}^I$. Therefore we have a canonical isomorphism

$$P(S)/(\mathrm{Pic}^0_{X/S}(S) + E(S)) \xrightarrow{\sim} \ker \beta / \mathrm{im}\, \alpha .$$

Taking the above isomorphism as an identification, we define σ as the canonical map

$$(*) \qquad P(S)/(\mathrm{Pic}^0_{X/S}(S) + E(S)) \longrightarrow Q(S)/Q^0(S) .$$

To show that it is surjective, it is enough to show that the canonical map

$$(**) \qquad P(S)/P^0(S) \longrightarrow Q(S)/Q^0(S)$$

is surjective. We will prove the latter fact by relating (**) to the canonical map

$$(***) \qquad P_k(k)/P_k^0(k) \longrightarrow Q_k(k)/Q_k^0(k) .$$

The map (***) is surjective. Namely, k is separably closed, and P_k is smooth, as follows from the formal smoothness of P. Thus, (***) may be interpreted as mapping connected components of P_k to connected components of Q_k. So it is surjective, due to the surjectivity of $P_k \longrightarrow Q_k$.

Since we know already from Proposition 3 that Q is a smooth group scheme and since the base S is strictly henselian, it follows from 2.3/5 that the restriction map

$$Q(S)/Q^0(S) \longrightarrow Q_k(k)/Q_k^0(k)$$

is bijective. The same is true for

$$P(S)/P^0(S) \longrightarrow P_k(k)/P_k^0(k)$$

if P is a scheme or an algebraic space which is locally of finite type over S. Namely, then the formal smoothness of P says that P is, in fact, smooth. So (**) will be surjective in this case.

In the general case, we must work with a rigidificator Y and consider the associated exact sequence

$$0 \longrightarrow V_X^* \hookrightarrow V_Y^* \longrightarrow (P, Y) \longrightarrow P \longrightarrow 0$$

of 8.1/11. It is enough to show that

$$P(S)/P^0(S) \longrightarrow P_k(k)/P_k^0(k)$$

is surjective, or, that the composition

$$(P, Y)(S) \longrightarrow (P, Y)_k(k) \longrightarrow P_k(k)$$

is surjective. The first map $(P, Y)(S) \longrightarrow (P, Y)_k(k)$ is surjective by 2.3/5 since (P, Y) is smooth (8.4/2). Furthermore, $(P, Y)_k$ is an extension of the smooth group scheme P_k by the quotient $(V_Y^*)_k/(V_X^*)_k$. The latter is smooth since V_Y^* is smooth; cf. [SGA 3_{I}], Exp. $\mathrm{VI_B}$, 9.2. Thus, by the same reference, we see that the morphism $(P, Y)_k \longrightarrow P_k$ is smooth and it follows, again from 2.3/5, that $(P, Y)_k(k) \longrightarrow P_k(k)$ is surjective. This shows that the map (**) is surjective.

The injectivity of σ under the assumption that $\mathrm{Pic}^0_{X/S}(S) \longrightarrow Q^0(S)$ is surjective is easily derived from the exact sequence

$$0 \longrightarrow E(S) \longrightarrow P(S) \longrightarrow Q(S) .$$

Finally, the submodule $\ker\beta \subset \mathbb{Z}^I$ has rank $\operatorname{card}(I) - 1$. If the same is true for $\operatorname{im}\alpha$, it follows that $\ker\beta/\operatorname{im}\alpha$ and, thus, also $Q(S)/Q^0(S)$ is finite. □

Let us assume now that X is regular. Under this assumption we can give an explicit description of the $\mathbb{Z}$-submodule $\operatorname{im}\alpha \subset \mathbb{Z}^I$ considered in the preceding lemma. To do so we introduce the intersection matrix $((X_i \cdot X_j))_{i,j\in I}$ where the intersection number $(X_i \cdot X_j)$ is defined as the degree on X_j of the line bundle which is associated to X_i as a Cartier divisor on X. Thereby we obtain a symmetric bilinear intersection pairing $D \times D \longrightarrow \mathbb{Z}$ on the group $D \simeq \mathbb{Z}^I$ of divisors on X which have support on the special fibre X_k; see also [SGA 7_{II}], Exp. X, 1.6. The map α is closely related to the intersection pairing; namely, $\alpha: D \simeq \mathbb{Z}^I \longrightarrow \mathbb{Z}^I$, as a $\mathbb{Z}$-linear map, is described by the matrix $(e_i^{-1}(X_i \cdot X_j))_{i,j\in I}$ which is called the *modified intersection matrix*.

Lemma 10. *Let R, X, and D be as in Lemmata* 8 *and* 9 *and assume that, in addition, X is regular. Let d_i be the multiplicity of X_i in X_k, i.e., the multiplicity of X_i in the divisor (π) =* "special fibre of X", *and let d be the greatest common divisor of the d_i, $i \in I$. Then, for any divisor $\sum n_i X_i \in D$, we have*

$$(\sum n_i X_i)^2 = -\sum_{i<j} \frac{1}{d_i d_j}(n_i d_j - n_j d_i)^2 (X_i \cdot X_j) .$$

Therefore the intersection form $D \times D \longrightarrow \mathbb{Z}$ is negative semi-definite and its kernel is generated by the divisor $\Delta = \sum d_i d^{-1} X_i \in D$. Furthermore, the $\mathbb{Z}$-module $\operatorname{im}\alpha$ of Lemma 9 *is isomorphic to $D/\mathbb{Z}\Delta$ and thus has rank* $\operatorname{card}(I) - 1$.

Proof. Tensoring with $\mathbb{Q}$, we can extend the bilinear pairing $D \times D \longrightarrow \mathbb{Z}$ to a bilinear pairing $D \otimes \mathbb{Q} \times D \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$. Therefore we may work with rational coefficients. Set $Y_i = d_i X_i$ and $m_i = n_i d_i^{-1}$. Since $(\pi) = \sum d_j X_j = \sum Y_j$ and since $(Y_i \cdot (\pi)) = (X_i \cdot (\pi)) = 0$ for all i, we can write

$$\begin{aligned}
\Big(\sum_i n_i X_i\Big)^2 = \Big(\sum_i m_i Y_i\Big)^2 &= \sum_i m_i\Big(Y_i \cdot \sum_j m_j Y_j\Big) \\
&= \sum_i m_i\Big(Y_i \cdot \sum_j m_j Y_j - m_i \sum_j Y_j\Big) \\
&= \sum_i m_i\Big(Y_i \cdot \sum_{j\neq i} (m_j - m_i) Y_j\Big) \\
&= \sum_{i\neq j} m_i(m_j - m_i)(Y_i \cdot Y_j) \\
&= \sum_{i<j} (m_i - m_j)(m_j - m_i)(Y_i \cdot Y_j) \\
&= -\sum_{i<j} (m_i - m_j)^2 (Y_i \cdot Y_j) \\
&= -\sum_{i<j} \frac{1}{d_i d_j}(n_i d_j - n_j d_i)^2 (X_i \cdot X_j)
\end{aligned}$$

All assertions of the lemma follow easily from this computation since the special fibre of X is connected. The latter is due to the fact that X is proper over S and that the generic fibre of X is connected. □

Now it is easy to complete the proof of Theorem 4 and to show that the group scheme Q is of finite type over R.

Lemma 11. *Assume that X is a flat proper curve over R which is regular and which has geometrically irreducible generic fibre X_K. Then the smooth and separated S-group scheme $Q = P/E$ is of finite type.*

Proof. We may assume that R is strictly henselian. Then it follows from Lemmata 9 and 10 that $\ker\beta/\operatorname{im}\alpha$ and thus $Q(S)/Q^0(S)$ are finite. The latter implies that Q is of finite type since it is locally of finite type; cf. [SGA 3_{I}], Exp. $\mathrm{VI_B}$, 3.6. □

Remark 12. In the assertion of Theorem 4, we may replace the condition that X be regular by the condition that all local rings of $X \times_R \operatorname{Spec}(R^{sh})$ are factorial (R^{sh} being a strict henselization of R); only this is needed for the proof of Lemma 2. In particular, it is enough to require the strict henselizations of all local rings of X to be factorial.

Remark 13. The above approach to the proof of Theorem 4 via the relative Picard functor and via the intersection form provides a second method of constructing Néron models, which is fairly independent of the one presented in earlier chapters. However, if one starts with a proper and smooth curve X_K over K, say under the assumption that R is excellent and that its residue field k is perfect, then in order to apply Theorem 4 to the Jacobian J_K of X_K, one first has to construct a proper R-model X of X_K which is regular; i.e., one has to use the process of desingularization for curves over R; see Abhyankar [1] or Lipman [1]. Alternatively, for a smooth curve X_K, one can apply the semi-stable reduction theorem and thereby construct a semi-abelian Néron model of J_K, after replacing R by its integral closure in a finite extension of K. Then the technique of Weil restriction leads to a Néron model of J_K over R; cf. 7.2/4. Proceeding either way, one constructs Néron models for Jacobians of smooth curves and eventually for general abelian varieties. But it should be kept in mind that the original construction of Néron models which we have given in Chapters 3 and 4 is more elementary in the sense that it uses just the smoothening process and not the theory of Picard functors as well as the existence of desingularizations or semi-stable reductions.

9.6 The Group of Connected Components of a Néron Model

In the following we assume that the base scheme $S = \operatorname{Spec} R$ consists of a *strictly henselian discrete valuation ring* R. Then, if J is an R-group scheme which is a Néron model of its generic fibre J_K, we can talk about the group $J(R)/J^0(R)$ of connected components of J or, more precisely, of the special fibre of J. The purpose of the present section is to give explicit computations for this group in the situation of Theorem 9.5/4, where we deal with Néron models J of Jacobians and where J can be described in terms of the relative Picard functor of a proper and flat S-curve X. As a key ingredient, we will use Lemma 9.5/9 of the previous section.

The notations will be as in 9.5/4. So X is a flat proper curve over S which is regular and whose generic fibre is geometrically irreducible. Furthermore, let $(X_i)_{i\in I}$ be the family of reduced irreducible components of the special fibre X_k, and let d_i (resp. e_i, resp. $\delta_i = d_i e_i$) be the multiplicity of X_i in X_k (resp. the geometric multiplicity of X_i, resp. the geometric multiplicity of X_i in X_k); cf. 9.1/3. Usually we will set $I = \{1, \ldots, r\}$. Also recall that the intersection number $(X_i \cdot X_j)$ between irreducible components of X_k has been defined as the degree on X_i of the line bundle given by X_j as a Cartier divisor on X; it is divisible by the multiplicity e_i.

Theorem 1. *Let S be the spectrum of a strictly henselian discrete valuation ring R and, as in 9.5/4, let X be a flat proper curve over S which is regular and whose generic fibre is geometrically irreducible. Furthermore, assume either that the residue field k of R is perfect (and, thus, algebraically closed) or that X admits an étale quasi-section (and, thus, a true section).*

Let J_K be the Jacobian of X_K, and let $(X_i)_{i\in I}$ be the family of (reduced) irreducible components of X_k. Then, considering the maps

$$D \simeq \mathbb{Z}^I \xrightarrow{\alpha} \mathbb{Z}^I \xrightarrow{\beta} \mathbb{Z}$$

of 9.5/9, where α is given by the modified intersection matrix $(e_i^{-1}(X_i \cdot X_j))_{i,j\in I}$ and where $\beta(a_1, \ldots, a_r) = \sum a_i \delta_i$, the group of connected components $J(R)/J^0(R)$ of the Néron model J of J_K is canonically isomorphic to the quotient $\ker\beta/\operatorname{im}\alpha$.

Proof. It follows from 9.5/4 that the Néron model J of J_K exists and coincides with the quotient $Q = P/E$, where P is the kernel of the degree morphism $\deg: \mathrm{Pic}_{X/S} \longrightarrow \mathbb{Z}$ and where E is the schematic closure of the generic fibre of the unit section $S \longrightarrow \mathrm{Pic}_{X/S}$. Furthermore, Lemma 9.5/9 provides a canonical surjection

$$\ker\beta/\operatorname{im}\alpha \longrightarrow Q(S)/Q^0(S) = J(S)/J^0(S)$$

which we have to show is bijective. As stated in 9.5/9, the bijectivity will follow if the canonical map

$$(*) \qquad \mathrm{Pic}^0_{X/S}(S) \longrightarrow Q^0(S)$$

is surjective. So let us prove the latter fact.

The easiest case is the one where X admits a section or, more generally (see 9.5/5), where the gcd of the geometric multiplicities δ_i of the components X_i in X_k is 1. Then it follows from 9.5/4 (b) that $\mathrm{Pic}^0_{X/S}$ is a separated scheme and that the canonical morphism $\mathrm{Pic}^0_{X/S} \longrightarrow Q^0$ is an isomorphism. So the bijectivity of $(*)$ is trivial in this case.

It remains to treat the case where the residue field k is algebraically closed. To do this, we may assume that, in addition to our assumptions, the base ring R is complete. Namely, the assumptions of the theorem are not changed if R is replaced by its completion; for the regularity of X this has been explained after 9.5/6. Furthermore, note that the special fibre X_k remains the same if R is replaced by its completion and that the formation of Q is compatible with such a base change since it commutes with flat extensions of discrete valuation rings.

The canonical morphism $P \longrightarrow Q$ is an isomorphism on generic fibres. Furthermore, the map $P(S) \longrightarrow P(K)$ is surjective by 9.5/2 and $Q(S) \longrightarrow Q(K)$ is bijective since Q is a Néron model of its generic fibre. So the canonical map

$$P(S) \longrightarrow Q(S)$$

is seen to be surjective. In order to derive the surjectivity of $(*)$ from this fact, we will use the Greenberg functor; see Greenberg [1]. Having no information on the representability of P at hand, it is necessary to work within the context of rigidificators.

Therefore, choose a rigidificator $Y \subset X$, and let (P, Y) be the open and closed subfunctor of the Picard functor of rigidified line bundles $(\mathrm{Pic}_{X/S}, Y)$ which equals the kernel of the degree morphism. We claim that the canonical map $(P, Y)(S) \longrightarrow P(S)$ is surjective. Namely, each element of $P(S)$ is given by a line bundle $\mathscr{L}$ on X and the pull-back of $\mathscr{L}$ to Y is trivial. The latter is true because Y is finite over S and because S is a local scheme. Hence, the composite map $(P, Y)(S) \longrightarrow Q(S)$ is surjective. For our purposes, it is enough to show that it restricts to a surjection $(P, Y)^0(S) \longrightarrow Q^0(S)$. Then, a fortiori, $P^0(S) \longrightarrow Q^0(S)$ will be surjective. Therefore, using the fact that (P, Y) is a smooth algebraic space (see 8.3/3 and 8.4/2) and that $(P, Y)(S)/(P, Y)^0(S)$ can be viewed as a quotient of a subgroup of the Néron-Severi group of the special fibre of X and, thus, is of finite type by 9.2/14, we have reduced the problem to showing the following assertion:

Lemma 2. *Let R be a complete discrete valuation ring with algebraically closed residue field k. Let $G \longrightarrow H$ be an R-morphism of smooth commutative algebraic R-group spaces with the property that $G(R)/G^0(R)$ is finitely generated (in the sense of abstract groups). Then, if $G(R) \longrightarrow H(R)$ is surjective, the same is true for $G^0(R) \longrightarrow H^0(R)$.*

By means of the Greenberg functor, we will be able to reduce the assertion to the corresponding one where R is replaced by the algebraically closed field k and where we consider a k-morphism $G \longrightarrow H$ of smooth commutative k-group schemes of finite type such that $G(k)/G^0(k)$ is finitely generated. Then, if $G(k) \longrightarrow H(k)$ is surjective, it is easy to see that the map $G^0(k) \longrightarrow H^0(k)$ is surjective. Namely, proceeding indirectly, assume that $G^0(k) \longrightarrow H^0(k)$ is not surjective. Then $G^0 \longrightarrow H^0$ cannot be an epimorphism since we are working over an algebraically closed field k. So the image of G^0 in H^0 is a closed subgroup M such that H^0/M is of positive dimension. Its group of k-valued points may be viewed as a quotient of a subgroup of $G(k)/G^0(k)$ and thus, by our assumption on $G(k)/G^0(k)$, is finitely generated. However, then H^0/M cannot have positive dimension as is easily seen by considering the multiplication on H^0/M by an integer which is not divisible by char k. Hence we have derived a contradiction and it follows that $G^0(k) \longrightarrow H^0(k)$ is surjective as claimed.

Next let us recall some basic facts on the Greenberg functor from Greenberg [1]; see also Serre [3], § 1. Let π be a uniformizing element of R and set $R_n := R/(\pi^n)$. Then the Greenberg functor Gr_n of level n associates to each R_n-scheme Y_n of locally finite type a k-scheme $\mathfrak{Y}_n = \mathrm{Gr}_n(Y_n)$ of locally finite type in such a way that, functorially in Y_n, we have $Y_n(R_n) = \mathfrak{Y}_n(k)$. For example, in the equal characteristic case, R_n may be viewed as a finite-dimensional k-algebra and the Greenberg functor Gr_n associated to R_n is just the Weil restriction functor (see 7.6) with respect to the morphism $\operatorname{Spec} R_n \longrightarrow \operatorname{Spec} k$. Weil restrictions are always representable by schemes in this case, due to the fact that R_n is an artinian local ring with residue field k.

In the unequal characteristic case, R_n cannot be viewed as a k-algebra and the notion of Weil restriction has to be generalized. Then, k being perfect, R is canonically an algebra of module-finite type over the ring of Witt vectors $W(k)$ and $W(k)$ is a complete discrete valuation ring of mixed characteristic, just as R is; see Bourbaki [2], Chap. 9, §§ 1 and 2, in particular, § 1, n°7, Prop. 8, and § 2, n°5, Thm. 3. So, in terms of $W(k)$-modules, R_n is a direct sum of rings of Witt vectors of finite length over k. Using the definition of Witt vectors, we can identify the set of R_n with a product k^m in such a way that the ring structure of R_n corresponds to a ring structure on k^m which is given by polynomial maps. Thereby it is immediately clear that we may interpret R_n as the set of k-valued points of a ring scheme $\mathscr{R}_n$ over k where, as a k-scheme, $\mathscr{R}_n$ is isomorphic to $\mathbb{A}_k^m$.

Similarly as in the case of Weil restrictions, one defines $\mathrm{Gr}_n(Y_n)$ for any R_n-scheme Y_n on a functorial level before one tries to prove its representability by a k-scheme. Namely, consider the functor h^* which associates to any k-scheme T the locally ringed space $h^*(T)$ consisting of T as a topological space and of $\mathscr{H}om_k(T, \mathscr{R}_n)$ as structure sheaf. Then

$$h^*(\operatorname{Spec} A) = \operatorname{Spec}(R_n \otimes_{W(k)} W(A))$$

for any k-algebra A. In particular, taking $A = k$, we see that $h^*(T)$ is a locally ringed space over $\operatorname{Spec} R_n$. It is shown in Greenberg [1] that, for R_n-schemes Y_n of locally finite type, the contravariant functor

$$\mathrm{Gr}_n(Y_n) : (\mathrm{Sch}/k) \longrightarrow (\mathrm{Sets}) , \qquad T \longmapsto \operatorname{Hom}_{R_n}(h^*(T), Y_n)$$

is representable by a k-scheme $\mathfrak{Y}_n$ which, again, is locally of finite type. So $\mathfrak{Y}_n = \mathrm{Gr}_n(Y_n)$ is characterized by the equation

$$\operatorname{Hom}_k(T, \mathfrak{Y}_n) = \operatorname{Hom}_{R_n}(h^*(T), Y_n)$$

and, in particular, setting $T := \operatorname{Spec} k$, we obtain $\mathfrak{Y}_n(k) = Y_n(R_n)$, the property of the Greenberg functor Gr_n we have mentioned at the beginning.

The canonical projection $R_{n+1} \longrightarrow R_n$ gives rise to a functorial transition morphism $\mathrm{Gr}_{n+1} \longrightarrow \mathrm{Gr}_n$. Furthermore, the Greenberg functor Gr_n respects closed immersions, open immersions, and fibred products. In fact, by establishing the first two of these compatibility properties, the representability of $\mathfrak{Y}_n = \mathrm{Gr}_n(Y_n)$ is reduced to the trivial case where $Y_n = \mathbb{A}^m_{R_n}$ and where $\mathfrak{Y}_n = (\mathscr{R}_n)^m$. Furthermore, it can be shown that the Greenberg functor respects smooth and étale morphisms. So this functor extends in a natural way from schemes to algebraic spaces. Working with group objects in the sense of algebraic spaces, we see that $\mathfrak{Y}_n$ will be an algebraic group space and, thus, by 8.3, a group scheme over k if Y is an algebraic group space over R_n. Moreover, for smooth group objects, the Greenberg functor respects identity components.

After this digression, let us turn to the *proof of Lemma* 2. Let $R_n = R/(\pi^n)$ be as above. Applying the base change $R \longrightarrow R_n$ and then the Greenberg functor of level n, we can associate to $G \longrightarrow H$ a morphism of k-group schemes of locally finite type $\mathfrak{G}_n \longrightarrow \mathfrak{H}_n$ such that the maps

$$G(R_n) \longrightarrow H(R_n) , \qquad \mathfrak{G}_n(k) \longrightarrow \mathfrak{H}_n(k)$$

can be identified. Since $G(R) \longrightarrow H(R)$ is surjective by our assumption and since $H(R) \longrightarrow H(R_n)$ is surjective by the lifting property 2.2/6 characterizing smoothness, we see that $G(R_n) \longrightarrow H(R_n)$ and, thus, $\mathfrak{G}_n(k) \longrightarrow \mathfrak{H}_n(k)$ is surjective. Furthermore, it follows that $\mathfrak{G}_n(k)/\mathfrak{G}_n^0(k)$, as a quotient of $G(R)/G^0(R)$, is finitely generated. Thus, as we have explained before, $\mathfrak{G}_n^0(k) \longrightarrow \mathfrak{H}_n^0(k)$ and therefore also $G^0(R_n) \longrightarrow H^0(R_n)$ must be surjective.

The map $G^0(R) \longrightarrow H^0(R)$ can be interpreted as the projective limit of the surjective maps $\mathfrak{G}_n^0(k) \longrightarrow \mathfrak{H}_n^0(k)$, $n \in \mathbb{N}$. In order to show the surjectivity of

$$\varprojlim \mathfrak{G}_n^0(k) \longrightarrow \varprojlim \mathfrak{H}_n^0(k) ,$$

it is enough to show that the system $(\mathfrak{N}_n)$, where $\mathfrak{N}_n$ is the kernel of the morphism $\mathfrak{G}_n^0 \longrightarrow \mathfrak{H}_n^0$, satisfies the Mittag-Leffler condition. However, this is clear since each $\mathfrak{G}_n^0$ is a k-scheme of finite type and, thus, satisfies the noetherian chain condition. So we have finished the proof of Lemma 2 and thereby also the proof of Theorem 1. □

The assertion of Theorem 1 reduces the computation of the group of connected components $J(R)/J^0(R)$ to a problem of linear algebra. In the remainder of the present section, we want to give some formulas for the order of $J(R)/J^0(R)$ as well as determine this group explicitly in some special cases. Let us start with some easy consequences of Theorem 1.

Corollary 3. *Assume that the conditions of Theorem* 1 *are satisfied. Set* $I = \{1, \ldots, r\}$ *and let* $n_1, \ldots, n_{r-1}, 0$ *be the elementary divisors of the modified intersection matrix* $A = (e_i^{-1}(X_i \cdot X_j))_{i,j \in I}$. *Then the group of connected components* $J(R)/J^0(R)$ *of the Néron model* J *of* J_K *is isomorphic to* $\mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_{r-1}\mathbb{Z}$. *Its order is the greatest common divisor of all* $(r-1) \times (r-1)$*-minors of* A.

Proof. Since the image of $\beta : \mathbb{Z}^r \longrightarrow \mathbb{Z}$ has no torsion and, thus, is free of rank 1, it follows that $\ker \beta$ is a direct factor in $\mathbb{Z}^r$, free of rank $r - 1$. We know from 9.5/10 that the submodule $\operatorname{im} \alpha \subset \ker \beta$ is of rank $r - 1$ also and, thus, can be described by non-zero elementary divisors $n_1, \ldots, n_{r-1}$. But then $n_1, \ldots, n_{r-1}, 0$ are the elementary divisors of $\operatorname{im} \alpha$ viewed as a submodule of $\mathbb{Z}^r$ and the assertions of the corollary are clear. □

If, in the above situation, all geometric multiplicities e_i are trivial, i.e., if $e_i = 1$ for all i, then the modified intersection matrix A coincides with the intersection matrix $((X_i \cdot X_j))_{i,j \in I}$. Considering the associated intersection pairing on the group $D \simeq \mathbb{Z}^I$ of all Cartier divisors on X which have support on the special fibre X_k, we know from 9.5/10 that the pairing is negative semi-definite and has a kernel $\Delta \cdot \mathbb{Z}$ of rank 1, where $\Delta = \sum d_i d^{-1} X_i$, as a divisor in D; the element d is the gcd of the multiplicities d_i. Dividing out the kernel, we get a quadratic form on $D/\Delta\mathbb{Z} \simeq \mathbb{Z}^I/\ker\alpha$ whose discriminant yields the order of the group of connected components $J(R)/J^0(R)$.

Corollary 4 (Lorenzini [1], 2.1.2). *Assume that the conditions of Theorem 1 are satisfied and that, in addition, all geometric multplicities e_i, $i \in I$, are equal to 1. Let $I = \{1, \ldots, r\}$. Then, for all indices $i, j \in I$, the absolute value of*

$$a_{ij}^* \, (\gcd(d_1, \ldots, d_r))^2 d_i^{-1} d_j^{-1} ,$$

where a_{ij}^* is the $(r-1) \times (r-1)$-minor of index (i, j) of the intersection matrix $A = ((X_i \cdot X_j))$, is independent of i and j. It equals the order of the group of connected components $J(R)/J^0(R)$.

The *proof* is by establishing a lemma from linear algebra (see Lemma 5 below) which allows to compute the gcd of the $(r-1) \times (r-1)$-minors of the intersection matrix A. To apply it, set $d_i' := d_i d^{-1}$. Then the assertion of Corollary 4 follows from Corollary 3. For the purposes of the lemma, we will use an exponent "t" to denote transposition of matrices.

Lemma 5. *Let $A = (a_{ij}) \in \mathbb{Z}^{r \times r}$ define a semi-definite quadratic form of rank $r - 1$. Let its kernel be generated over $\mathbb{Z}$ by the vector $d' = (d_1', \ldots, d_r')^t \in \mathbb{Z}^r$ and let $A^* = (a_{ij}^*)$ be the adjoint matrix of A. Then there exists a positive integer v such that*

$$A^* = \pm v \cdot d' \cdot d'^t .$$

Furthermore, v is the gcd of all $(r-1) \times (r-1)$-minors of A.

Proof. Since $\gcd(d_1', \ldots, d_r') = 1$, the assertion on the greatest common divisor of the $(r-1) \times (r-1)$-minors of A follows from the formula for A^*. So it is enough to establish this formula. To do this, note that the kernel of A as a semi-definite quadratic form on $\mathbb{Z}^r$ coincides with the kernel of A as a $\mathbb{Z}$-linear map $\mathbb{Z}^r \longrightarrow \mathbb{Z}^r$. Then, using the equation

$$A \cdot A^* = \det(A) \cdot \text{unit matrix} = 0 ,$$

we see that all columns of A^* belong to the kernel of A. So there is a vector $c = (c_1, \ldots, c_r)^t \in \mathbb{Z}^r$ satisfying $A^* = d \cdot c^t$. Since A^* is symmetric, we have $c \cdot d'^t = d' \cdot c^t$ and, thus, $A \cdot c \cdot d'^t = 0$. This implies $A \cdot c = 0$ since $d' \neq 0$ so that c belongs to the kernel of A. Hence there is an element $v \in \mathbb{Z}$ satisfying $c = v \cdot d'$. Replacing v by its absolute value if it is negative, we have $A^* = \pm v \cdot d' \cdot d'^t$ as required. □

If one wants to prove more specific assertions on the group of connected components $J(R)/J^0(R)$, it is important to have information on the configuration of the components X_i of the special fibre X_k. The latter can be described using graphs. There are several possibilities to associate a graph to X_k depending on how

multiple intersections of components as well as multiplicities of intersection points are treated. We will deal with two cases, the one where the graph of X_k, in the weakest possible sense, is a tree and the one where X is a semi-stable curve. As a general assumption, we require that we are in the situation of Theorem 1 and that, in addition, the multiplicities e_i, $i \in I$, are equal to 1. For example, the latter is the case if k is algebraically closed. The index set I will always be the set $\{1, \ldots, r\}$.

The case where the graph of X_k is a tree (cf. Lorenzini [1]). The graph Γ we want to associate to X_k is constructed in the following way: the vertices of Γ are the components X_i of X_k, and a vertex X_i is joined to a vertex X_j different from X_i if the intersection number $(X_i \cdot X_j)$ is non-zero. In particular, the precise number of intersection points in $X_i \cap X_j$ is not reflected in the graph Γ. We define the multiplicity s_i of X_i, as a vertex of Γ, as the number of edges joining X_i; so

$$s_i = \operatorname{card}\{ j \in I ; i \neq j \quad \text{and} \quad (X_i \cdot X_j) \neq 0 \} .$$

Furthermore, we need the multiplicity d_i of X_i in the special fibre X_k (which coincides with the geometric multiplicity δ_i of X_i in X_k since $e_i = 1$), the number $d = \gcd(d_1, \ldots, d_r)$, and the quotients $d_i' = d_i d^{-1}$ which are relatively prime.

Proposition 6. *In the situation of Theorem 1, assume that the graph Γ is a tree and that the geometric multiplicities e_i are equal to 1. Then, writing $a_{ij} = (X_i \cdot X_j)$, the group of connected components $J(R)/J^0(R)$ has order*

$$\prod_{a_{ij} \neq 0, i<j} a_{ij} \cdot \prod_{i=1}^{r} (d_i')^{s_i - 2} .$$

Furthermore, if all d_i' are equal to 1, we have

$$J(R)/J^0(R) \simeq \prod_{a_{ij} \neq 0, i<j} \mathbb{Z}/a_{ij}\mathbb{Z} .$$

The assertion will be reduced to Corollary 3 by means of the following result:

Lemma 7. *Let $A = (a_{ij}) \in \mathbb{Z}^{r \times r}$ be a symmetric matrix, which is negative semi-definite of rank $r - 1$, and let the vector $(d_1', \ldots, d_r')^t \in \mathbb{Z}^r$ with positive entries d_i' generate the kernel of A. Furthermore, let Γ be the graph associated to A in the manner we have described for intersection matrices above. Then, if Γ is a tree, the greatest common divisor of all $(r-1) \times (r-1)$-minors of A is given by the product*

$$\prod_{a_{ij} \neq 0, i<j} a_{ij} \cdot \prod_{i=1}^{r} (d_i')^{s_i - 2} .$$

Furthermore, if $d_i' = 1$ for all i, the elements a_{ij} occurring in the first factor constitute the non-zero elementary divisors of A.

Proof. Let us first assume $d_i' = 1$ for all i. Then, since the vector $(d_1', \ldots, d_r')$ belongs to the kernel of the intersection matrix $A = (a_{ij})$, it follows that the sum of all columns of A is zero. The same is true for the sum of all rows of A since A is symmetric. Consider a terminal edge C of Γ; i.e., an edge with attached vertices, say

X_1 and X_2, such that $s_1 = 1$ and $s_2 = \cancel{2}$ beliebig. Then the intersection matrix A has the following form where $a_{ij} = a_{ji}$ and where empty space indicates zeros:

$$\begin{bmatrix} a_{11} & a_{12} & & & & \\ a_{21} & a_{22} & * & \cdot & \cdot & * \\ & * & * & \cdot & \cdot & * \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & * & * & \cdot & \cdot & * \end{bmatrix}$$

Now add the first column to the second column and, likewise, the first row to the second row. Using the fact that the sum of the columns or rows in A vanishes, we have $a_{11} = -a_{12} = -a_{21}$. Thus, we see that this operation kills the entries a_{12} and a_{21} so that the resulting matrix is of the form

$$\begin{bmatrix} -a_{12} & & & & & \\ & a'_{22} & * & \cdot & \cdot & * \\ & * & * & \cdot & \cdot & * \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & * & * & \cdot & \cdot & * \end{bmatrix}$$

where $a'_{22} = a_{22} + a_{21}$. Let Γ' be the graph obtained from Γ by removing the terminal edge C we are considering as well as the vertex X_1. Then Γ' is a tree again and it can be viewed as a graph which corresponds to the lower bloc, call it A', of the above matrix, where A' has again the property that the sum of its columns or rows vanishes. Thus we can proceed with A' and Γ' in the same way as we have done before with A and Γ. Since Γ is a tree, the procedure of removing terminal edges and vertices stops after finitely many steps with a graph which is reduced to a single vertex and with an associated (1×1)-matrix which is zero. At the same time we have converted A by means of elementary column and row operations into a diagonal matrix; the diagonal elements, except for the last entry which is zero, consist of all elements $-a_{ij}$, $i < j$, such that X_i is joined to X_j by an edge of Γ. This verifies the assertion of the proposition in the case where all d'_i are equal to 1.

In order to verify the remaining assertion on the greatest common divisor of all $(r-1) \times (r-1)$-minors of A in the general case, we consider the matrix $B = (a_{ij} d'_i d'_j)$. It is negative semi-definite of rank $r-1$ again and has the property that the sum of its columns or rows is zero. So, using the graph Γ, we can determine its elementary divisors as before. In particular, the gcd of all $(r-1) \times (r-1)$-minors of B equals the product

$$\mu := \prod_{a_{ij} \neq 0, i<j} a_{ij} \cdot \prod_{i=1}^{r} (d'_i)^{s_i} .$$

Let ν be the gcd of all $(r-1) \times (r-1)$-minors of A. Writing A_{11} and B_{11} for the matrices obtained from A and B by removing the first column and the first row, we see from Lemma 5 that

$$\det A_{11} = \pm \nu (d'_1)^2 , \qquad \det B_{11} = \pm \mu .$$

Thus

$$\mu = \pm \det B_{11} = \pm (d'_2 \ldots d'_r)^2 \det A_{11} = \pm (d'_1 \ldots d'_r)^2 \nu ,$$

and the desired assertion follows from the above equation for μ. □

Remark 8. The graph Γ associated to the special fibre X_k of a curve X as above is a tree if the Néron model J of the Jacobian J_K of X_K has potential abelian reduction or, more generally, if the special fibre J_k does not contain a non-trivial torus. Namely, using the notation of 9.5/4, we have $J_k = P_k / E_k$, where E_k^0 is a unipotent group by Raynaud [6], 6.3/8. So if J_k does not contain a non-trivial torus, the same is true for P_k and, thus, for $\mathrm{Pic}_{X_k/k}$. Then the configuration of the components X_i of X_k is "tree-like" by 9.2/12. However, it should be noted that the graph Γ as we have defined it can be a tree also in some cases where the configuration of the components of X_k is not "tree-like". For example, X_k can be a semi-stable curve consisting of two components which intersect each other in several points. In this case, it follows from 9.2/10 again that J_k contains a non-trivial torus.

We want to apply Proposition 6 in order to show that the order of the group of connected components $J(R)/J^0(R)$ is bounded if J_K has potential good reduction. See Lorenzini [1] for more precise bounds and McCallum [1] for a generalization to abelian varieties.

Theorem 9. *Let R be a strictly henselian discrete valuation ring with algebraically closed residue field k and with field of fractions K. Furthermore, let X_K be a proper smooth curve over K, which is geometrically connected, has a Jacobian J_K with potential good reduction, and admits a regular minimal model X over R.*

Then, for each integer $g > 0$, there exists a bound $M(g)$ such that, for each choice of R, K, and k, and for each curve X_K of genus g as above, the order of the group of connected components $J(R)/J^0(R)$ of the Néron model J of J_K is bounded by $M(g)$.

Proof. We will use the methods of Artin and Winters [1]; the notation is as before. The connected components of X_k are denoted by X_i, and d_i is the multiplicity of X_i in X_k. Furthermore, let d be the gcd of the d_i and set $d'_i = d_i d^{-1}$. Let X'_k be the scheme given by $\sum d'_i X_i$, the latter being viewed as a Cartier divisor on X. Then

$$(*) \qquad H^0(X'_k, \mathcal{O}_{X'_k}) = k$$

by Artin and Winters [1], Lemma 2.6, since the gcd of the d'_i is 1.

We want to compute the arithmetic genus of X'_k. Let $\mathfrak{K}$ be a relative canonical divisor on X. Then we can compute the Euler-Poincaré characteristic of $\mathcal{O}_{X'_k}$ as

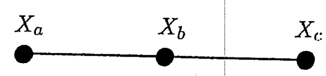
$$-\chi(\mathcal{O}_{X'_k}) = (X'_k \cdot (X'_k + \mathfrak{K}))/2 = (X'_k \cdot \mathfrak{K})/2 = (X_k \cdot \mathfrak{K})/2d = (g-1)/d ;$$

the last equality is due to the fact that the degree of $\mathfrak{K}$ is the same on the generic and on the special fibre of X. So, using the equality (*), the arithmetic genus g' of X'_k is given by

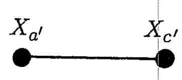
$$g' = 1 - \chi(\mathcal{O}_{X'_k}) = 1 + (X'_k \cdot \mathfrak{K})/2 = 1 + (g-1)/d\,.$$

In particular, g' coincides with the abstract genus introduced by Artin and Winters [1], 1.3, and we have $g' \leq g$. If $H^0(X_k, \mathcal{O}_{X_k}) \neq k$, which may be the case if $d > 1$, and if we compute the arithmetic genus of X_k, it can happen that the latter is greater than g. This is the reason why one has to introduce the curve X'_k.

Now, in order to determine the order of the group of connected components $J(R)/J^0(R)$, one applies Corollary 3 and determines the greatest common divisor of all $(r-1) \times (r-1)$-minors of the intersection matrix $((X_i \cdot X_j))$; let us denote it by v. The intersection matrix is the same for X_k and for X'_k. Thus, also the graph Γ is the same for both curves, and it follows from our explanations given in Remark 8 that Γ is a tree since J_K has potential good reduction. We want to show that the integer v remains invariant if we contract an exceptional curve C of the second kind in the sense of Artin and Winters [1], 1.4, in X_k. Such a curve C corresponds to the middle edge of a chain



in Γ such that $d'_a = d'_b = d'_c$ and $(X_a \cdot X_b) = (X_b \cdot X_c) = 1$ and such that $s_b = 2$; i.e., there is no ramification at the vertex X_b. Contracting X_b modifies Γ to the extent that we have to replace the above chain by



where now $d'_{a'} = d'_a$, $d'_{c'} = d'_c$, and $(X_{a'} \cdot X_{c'}) = 1$, all other intersection numbers remaining untouched. It follows from the formula in Lemma 7 that the integer v remains unchanged under such a contraction process. In a similar way one shows that contractions of exceptional curves of the first kind, as considered in Artin and Winters [1], Lemma 1.18, cannot cause v to increase.

We now use the fact proved in Artin and Winters [1], Thm. 1.6, that, up to contraction of exceptional curves of the first and second kind, there are only finitely many possible types of graphs and intersection matrices for a given genus g' and, thus, for the finitely many genera $g' < g$. So there are only finitely many possible values for the integer v and, hence, for the order of the group of connected components $J(R)/J^0(R)$. □

The case of semi-stable curves. In the following we will assume that all geometric multiplicities $\delta_i = d_i e_i$ are equal to 1. So, in addition to $e_i = 1$, we have $d_i = 1$ for all $i \in I$. We do not require from the beginning that the special fibre X_k of the curve X is semi-stable; we will restrict ourselves to this case later. The graph we want to consider here is the so-called intersection graph Γ of X_k. Its vertices are given by the irreducible components X_i of the special fibre X_k as before, whereas, different from the graph used above, its edges correspond to the intersection points of such components; i.e., X_i and X_j, $i \neq j$, are joined by as many edges as there are irreducible components in the intersection $X_i \cap X_j$.

We want to compute the group $J(R)/J^0(R)$ of connected components of the Néron model of the Jacobian J_K of X_K by describing the group $\ker \beta / \operatorname{im} \alpha$ of Theorem 1 in terms of the graph Γ. To do this, choose an orientation on Γ and consider the (augmented) simplicial homology complex

$$0 \longrightarrow C_1(\Gamma, \mathbb{Z}) \xrightarrow{\partial_1} C_0(\Gamma, \mathbb{Z}) \xrightarrow{\partial_0} \mathbb{Z}$$

of Γ with coefficients in $\mathbb{Z}$. Then $\operatorname{im} \partial_1 = \ker \partial_0$ since Γ is connected. Identifying $C_0(\Gamma, \mathbb{Z})$ with $\mathbb{Z}^I$, the map ∂_0 coincides with $\beta : \mathbb{Z}^I \longrightarrow \mathbb{Z}$. Thus, if M is any $\mathbb{Z}$-submodule of $C_1(\Gamma, \mathbb{Z})$ lifting $\operatorname{im} \alpha$, i.e., whose image under ∂_1 coincides with $\operatorname{im} \alpha \subset \mathbb{Z}^I \simeq C_0(\Gamma, \mathbb{Z})$, we see that

$$J(R)/J^0(R) \simeq \ker \beta / \operatorname{im} \alpha \simeq C_1(\Gamma, \mathbb{Z})/(M + H_1(\Gamma, \mathbb{Z}))\,,$$

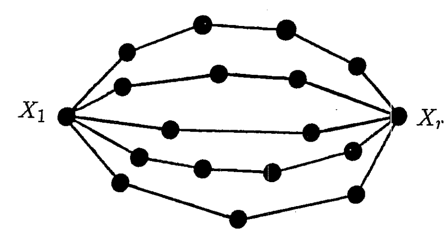
where the first cohomology group $H_1(\Gamma, \mathbb{Z})$ is the kernel of the map ∂_1.

A canonical lifting M of $\operatorname{im} \alpha$ can be obtained by choosing canonical liftings ζ_i of the generators $\xi_i = ((X_i \cdot X_j))_{j \in I}$, $i \in I$, of $\operatorname{im} \alpha$. Namely, define ζ_i as a sum $\sum_\rho c_{i\rho} \eta_{i\rho}$ where the $c_{i\rho}$ are integers which will be specified below and where the $\eta_{i\rho}$ vary over all edges joining the vertex X_i with a second vertex X_j. Up to its sign, the multiplicity $c_{i\rho}$ is the local intersection number of X_i and X_j at the irreducible component x of $X_i \cap X_j$ which corresponds to $\eta_{i\rho}$. The sign of $c_{i\rho}$ is "+" or "−" depending on the orientation of $\eta_{i\rho}$. We use "+" if $\eta_{i\rho}$ originates at X_i and ends at X_j and "−" otherwise. Then, since X_k, as a Cartier divisor on X, is principal, we have $\sum_{j \in I}(X_i \cdot X_j) = 0$ for all $i \in I$ and we see that $M := \sum_{i \in I} \zeta_i \mathbb{Z}$ is a lifting of $\operatorname{im} \alpha$ so that

(*) $$J(R)/J^0(R) \simeq C_1(\Gamma, \mathbb{Z})/(M + H_1(\Gamma, \mathbb{Z}))\,.$$

We want to give an explicit example.

Proposition 10. *Let X be a proper and flat curve over S, which is regular and has a geometrically irreducible generic fibre X_K as well as a geometrically reduced special fibre X_k. Assume that X_k consists of the irreducible components $X_1, \ldots, X_r$ and that the local intersection numbers of the X_i are 0 or 1 (the latter is the case if different components intersect at ordinary double points). Furthermore, assume that the intersection graph Γ is of the type*



I. e., Γ consists of l arcs of edges starting at X_1 and ending at X_r. For each $\lambda = 1, \ldots, l$, let the λ-th arc consist of the edges $\eta_{\lambda 1}, \ldots, \eta_{\lambda m_\lambda}$, where m_λ is its length. Then the group $J(R)/J^0(R)$ has order

$$\sigma_{l-1}(m_1,\dots,m_l) := \sum_{\lambda=1}^{l} \prod_{\mu\neq\lambda} m_\mu .$$

More precisely, $J(R)/J^0(R)$ is trivial if $l = 1$. For $l \geq 2$ it is isomorphic to the group

$$(\mathbb{Z}/g_1\mathbb{Z}) \oplus (\mathbb{Z}/g_2g_1^{-1}) \oplus \dots \oplus (\mathbb{Z}/g_{l-2}g_{l-3}^{-1}\mathbb{Z}) \oplus (\mathbb{Z}/\sigma_{l-1}(m_1,\dots,m_l)g_{l-2}^{-1}\mathbb{Z})$$

where g_i is the greatest common divisor of all summands occurring in the i-th elementary symmetric polynomial

$$\sigma_i(m_1,\dots,m_l) , \qquad i = 1,\dots,l-2 .$$

Proof. We use the formula (*). A basis of $C_1(\Gamma,\mathbb{Z})$ is given by the elements

$$\begin{array}{ll} \eta_{11} , & \dots, \ \eta_{l1} \\ \eta_{12}-\eta_{11} , & \dots, \ \eta_{l2}-\eta_{l1} \\ \vdots & \\ \eta_{1m_1}-\eta_{1m_1-1} , & \dots, \ \eta_{lm_l}-\eta_{lm_l-1} \end{array}$$

Next we write down generators for the canonical lifting M of $\operatorname{im}\alpha$:

$$\begin{array}{ll} \sum_{\lambda=1}^{l} \eta_{\lambda 1} , & \\ \eta_{12}-\eta_{11} , & \dots, \ \eta_{l2}-\eta_{l1} \\ \vdots & \\ \eta_{1m_1}-\eta_{1m_1-1} , & \dots, \ \eta_{lm_l}-\eta_{lm_l-1} \\ -\sum_{\lambda=1}^{l} \eta_{\lambda m_\lambda} , & \end{array}$$

and for $H_1(\Gamma,\mathbb{Z})$:

$$\sum_{j=1}^{m_\lambda} \eta_{\lambda j} - \sum_{j=1}^{m_1} \eta_{1j} ; \qquad \lambda = 2,\dots, l .$$

Using the above generators for $C_1(\Gamma,\mathbb{Z})$, M, and $H_1(\Gamma,\mathbb{Z})$, as well as the fact that

$$\eta_{\lambda j} = \eta_{\lambda 1} + (\eta_{\lambda 2}-\eta_{\lambda 1}) + \dots + (\eta_{\lambda j}-\eta_{\lambda j-1}) ,$$

if follows that $J(R)/J^0(R) \simeq C_1(\Gamma,\mathbb{Z})/(M + H_1(\Gamma,\mathbb{Z}))$ is isomorphic to the quotient of the free $\mathbb{Z}$-module generated by $\eta_{11},\dots,\eta_{l1}$, divided by the submodule generated by the relations

$$\sum_{\lambda=1}^{l} \eta_{\lambda 1} , \qquad m_\lambda\eta_{\lambda 1} - m_1\eta_{11} ; \qquad \lambda = 2,\dots,l .$$

... described by the matrix

$$A = \begin{bmatrix} 1 & -m_1 & -m_1 & \cdots & -m_1 \\ 1 & m_2 & 0 & \cdots & 0 \\ 1 & 0 & m_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \vdots & & \vdots \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & \cdots & m_l \end{bmatrix} .$$

Computing the determinant of A by developing it via the first column, we get

$$\det A = \sigma_{l-1}(m_1,\dots,m_l) .$$

Thus, by the theory of elementary divisors, this is already the group order of $J(R)/J^0(R)$. To determine the elementary divisors of A explicitly, we use the criterion involving the gcd of minors; cf. Bourbaki [1], Chap. 7, § 4, n°5, Prop. 4.

The gcd of all coefficients of A is 1; so this is the first elementary divisor. For $1 < \lambda < l$, the gcd of all $(\lambda \times \lambda)$-minors is the gcd of all products occurring as summands in the $(\lambda-1)$-st elementary symmetric polynomial $\sigma_{\lambda-1}(m_1,\dots,m_l)$; hence it is $g_{\lambda-1}$. Therefore the elementary divisors of A are

$$1, g_1, g_2g_1^{-1}, \dots, g_{l-2}g_{l-3}^{-1}, \sigma_{l-1}(m_1,\dots,m_l)g_{l-2}^{-1}$$

and, consequently, $J(R)/J^0(R)$ is as claimed. □

Corollary 11. *Let X be a flat proper curve over S. Assume that the generic fibre X_K is smooth and that the special fibre X_k is geometrically reduced and consists of two irreducible components X_1 and X_2 which intersect transversally at l rational points $x_1,\dots,x_l$. Thus, for each $\lambda = 1,\dots,l$, the curve X is, up to étale localization at x_λ, described by an equation of type $uv = \pi^{m_\lambda}$. If X has no other singularities, then, just as in the situation of Proposition 9, the group of connected components of the Néron model J of the Jacobian J_K of X_K is isomorphic to the group*

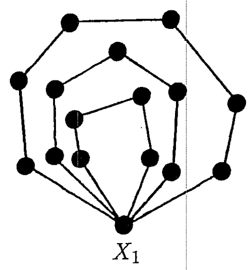
$$(\mathbb{Z}/g_1\mathbb{Z}) \oplus (\mathbb{Z}/g_2g_1^{-1}\mathbb{Z}) \oplus \dots \oplus (\mathbb{Z}/g_{l-2}g_{l-3}^{-1}\mathbb{Z}) \oplus (\mathbb{Z}/\sigma_{l-1}(m_1,\dots,m_l)g_{l-2}^{-1}\mathbb{Z})$$

where g_i is the greatest common divisor of all summands occurring in the i-th elementary symmetric polynomial

$$\sigma_i(m_1,\dots,m_l) , \qquad i = 1,\dots,l-2 .$$

The assertion is a direct consequence of the preceding proposition since the minimal desingularization of X is of the type considered in Proposition 10. Curves of this type occur within the context of modular curves; see the appendix by Mazur and Rapoport to the article Mazur [1].

Remark 12. If in the situation of Proposition 10 the graph Γ of the special fibre of X is of type



i.e., consists of l loops of length $m_1, \ldots, m_l$ starting at X_1 each, the group of connected components of J can be computed as exercised in the proof of Proposition 10. One shows

$$J(R)/J^0(R) = \mathbb{Z}/m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/m_l\mathbb{Z}\,.$$

Thereby one obtains an analogue of Corollary 11 for curves X whose special fibre is irreducible and has at most ordinary double points as singularities.

9.7 Rational Singularities

Let $S = \operatorname{Spec} R$ be a base scheme consisting of a discrete valuation ring R. As usual, K is the field of fractions and k is the residue field of R. Starting with a proper and flat S-curve X which is normal and has geometrically irreducible generic fibre, we want to relate the fact that a Néron model J of the Jacobian J_K of X_K exists and that the canonical morphism $\operatorname{Pic}^0_{X/S} \longrightarrow J^0$ is an isomorphism to the fact that X has singularities of a certain type, namely *rational singularities*. To explain the latter terminology, assume that X admits a desingularization $f\colon X' \longrightarrow X$ (which, by Abhyankar [1] or Lipman [1] exists at least in the case where R is excellent). There are only finitely many points where X is not regular. X is said to have rational singularities if $R^1f_*(\mathcal{O}_{X'}) = 0$. It can be shown that the latter condition is independent of the chosen desingularization.

Theorem 1. *Let X be a flat proper curve over S which is normal and which has geometrically irreducible generic fibre X_K. Let $X_1, \ldots, X_n$ be the irreducible components of the special fibre X_k. Assume that X admits a desingularization $f\colon X' \longrightarrow X$ and, furthermore, that the following conditions are satisfied:*

(i) *The residue field k of R is perfect or X admits an étale quasi-section.*

(ii) *The greatest common divisor of the geometric multiplicities δ_i of X_i in X_k (cf. 9.1/3) is 1.*

Then, by (i), the Jacobian J_K of X_K admits a Néron model J of finite type and, by (ii), the identity component $\operatorname{Pic}^0_{X/S}$ of the relative Picard functor is a scheme. Furthermore, the canonical morphism $\operatorname{Pic}^0_{X/S} \longrightarrow J^0$ is an isomorphism if and only if X has rational singularities.

Proof. It is easily seen that conditions (i) and (ii) carry over from X to X'. For example, if X admits an étale quasi-section over S, the same is true for X' by the valuative criterion of properness since $f\colon X' \longrightarrow X$ is proper. Thus it follows from condition (i) and from 9.5/4 that J_K, which is also the Jacobian of X'_K, has a Néron model J of finite type. Furthermore, the canonical morphism $\bar{v}\colon P'/E' \longrightarrow J$ is an isomorphism where P' is the subfunctor of $\operatorname{Pic}_{X'/S}$ given by line bundles of total degree 0 and where E' is the schematic closure of the generic fibre of the unit section of $\operatorname{Pic}_{X'/S}$.

On the other hand, using 9.4/2, condition (ii) implies that $\operatorname{Pic}^0_{X/S}$ and $\operatorname{Pic}^0_{X'/S}$ are represented by separated schemes. So we get canonical maps between S-group schemes

$$\operatorname{Pic}^0_{X/S} \longrightarrow \operatorname{Pic}^0_{X'/S} \xrightarrow{\sim} J^0,$$

the latter map being an isomorphism by 9.5/4. So $\operatorname{Pic}^0_{X/S} \longrightarrow J^0$ is an isomorphism if and only if $\operatorname{Pic}^0_{X/S} \longrightarrow \operatorname{Pic}^0_{X'/S}$ is an isomorphism and the latter is the case if and only if $\operatorname{Lie}(\operatorname{Pic}^0_{X/S}) \longrightarrow \operatorname{Lie}(\operatorname{Pic}^0_{X'/S})$ is an isomorphism. Writing $R[\varepsilon]$ for the algebra of dual numbers over R, we can interpret $\operatorname{Lie}(\operatorname{Pic}^0_{X/S})$ as the subfunctor of $\operatorname{Hom}_S(\operatorname{Spec} R[\varepsilon], \operatorname{Pic}^0_{X/S})$ consisting of all morphisms which modulo ε reduce to the unit section of $\operatorname{Pic}^0_{X/S}$. Then, as we have seen in the proof of 8.4/1, it follows that $\operatorname{Lie}(\operatorname{Pic}^0_{X/S})$ can be identified with the cohomology group $H^1(X, \mathcal{O}_X)$. Proceeding in the same way with X', we see that $\operatorname{Lie}(\operatorname{Pic}^0_{X/S}) \longrightarrow \operatorname{Lie}(\operatorname{Pic}^0_{X'/S})$ is an isomorphism if and only if the canonical map $H^1(X, \mathcal{O}_X) \longrightarrow H^1(X', \mathcal{O}_{X'})$ is an isomorphism.

Now let us look at the Leray sequence associated to $f\colon X' \longrightarrow X$. It starts as follows:

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X', \mathcal{O}_{X'}) \longrightarrow H^0(X, R^1f_*(\mathcal{O}_X)) \longrightarrow H^2(X, \mathcal{O}_X)$$

Since X is a curve, we have, in fact, a short exact sequence

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X', \mathcal{O}_{X'}) \longrightarrow H^0(X, R^1f_*(\mathcal{O}_X)) \longrightarrow 0.$$

So $H^1(X, \mathcal{O}_X) \longrightarrow H^1(X', \mathcal{O}_{X'})$ is an isomorphism if and only if $H^0(X, R^1f_*(\mathcal{O}_X)) = 0$. Since $R^1f_*(\mathcal{O}_X)$ is concentrated at a finite number of closed points of X, the latter is equivalent to $R^1f_*(\mathcal{O}_X) = 0$; i.e., to the fact that X has rational singularities. This establishes the desired equivalence. □

For semi-stable curves over S (cf. 9.2/6), assumptions (i) and (ii) of Theorem 1 are automatically satisfied. So, using 9.2/8, we see:

Corollary 2. *Let X be a semi-stable curve over S which is proper, flat, and normal, and which has a geometrically irreducible generic fibre X_K. Then the Jacobian J_K of X_K has a Néron model J and the canonical morphism $\operatorname{Pic}^0_{X/S} \longrightarrow J^0$ is an isomorphism. In particular, J has semi-abelian reduction.*

In the situation of the theorem we can say that $\mathrm{Pic}^0_{X/S}$ is independent of the choice of the S-model X of X_K as long as we limit ourselves to proper, normal, and flat S-curves which have rational singularities. Namely, then $\mathrm{Pic}^0_{X/S}$ coincides with the identity component of the Néron model J of the Jacobian J_K of X_K.

We want to give an application to the modular curve $X_0(N)$. To recall the description of this curve, let N be a positive integer and write U_N for the open subscheme of $\operatorname{Spec}\mathbb{Z}$ where N is invertible. Then $X_0(N)|_{U_N}$ is a proper and smooth curve over U_N; it is the compactified coarse moduli space associated to the stack of couples (E, C) of the following type: E is an elliptic curve over some U_N-scheme S and C is a subgroup scheme of E which is finite, étale, and cyclic of order N. For $N = 1$ one obtains the projective line P over $\mathbb{Z}$, to be interpreted as the compactification of the affine line where the j-invariant of elliptic curves serves as a parameter.

Writing $X_0(N)$ for the normalization of P in $X_0(N)|_{U_N}$, the curve $X_0(N)$ is proper over $\mathbb{Z}$ and extends the curve we had already over U_N. For example, if p is a prime strictly dividing N, the curve $X_0(N)$ has semi-stable reduction at p. More precisely, the fibre of $X_0(N)$ over p consists of two smooth components which intersect transversally at the supersingular points; cf. Deligne and Rapoport [1], Chap. VI, Thm. 6.9, or the appendix by Mazur and Rapoport to Mazur [1], Thm. 1.1.

If p^2 divides N, the geometry of fibres is more complicated and certain components have non-trivial multiplicities. In this case one can use the modular interpretation à la Drinfeld which yields information on $X_0(N)$, particularly at bad places. Namely, $X_0(N)$ is the coarse moduli space associated to a certain modular stack which is relatively representable and regular over $\mathbb{Z}$; cf. Katz and Mazur [1], 5.1.1. Then, if x is a closed point of $X_0(N)$, the henselization at x is a quotient of a regular local ring by a finite group whose order divides 12. From this one deduces by means of a norm argument that the singularities of the fibres of $X_0(N)$ over any prime $p > 3$ are rational. Furthermore, over each prime p, there are irreducible components which have geometric multiplicity 1 in the fibre over p; cf. Katz and Mazur [1], 13.4.7. So, using 9.4/2, and Theorem 1, as well as a globalization argument of the type provided in 1.2/4, we obtain:

Proposition 3. *The modular curve $X_0(N)$ is cohomologically flat over $\mathbb{Z}$ and $\mathrm{Pic}^0_{X_0(N)/\mathbb{Z}}$ is representable by a group scheme. Furthermore, outside $p = 2$ and 3, it is the identity component of the Néron model of the Jacobian of $X_0(N) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Chapter 10. Néron Models of Not Necessarily Proper Algebraic Groups

For this last chapter we introduce a new type of Néron models, so-called Néron lft-models. To define them, we modify the definition of Néron models by dropping the condition that they are of finite type. Then, due to the smoothness, Néron lft-models are locally of finite type. This is the reason why we use the abbreviation "lft". For example, tori do admit Néron lft-models whereas, for non-zero split tori, Néron models (in the original sense) do not exist.

We begin by collecting basic properties of Néron lft-models and by explaining some examples. Then, for the local case, we prove a necessary and sufficient condition for a smooth algebraic K-group G_K to admit a Néron model (resp. a Néron lft-model). In the special case where the valuation ring is strictly henselian and excellent, it states that G_K admits a Néron model (resp. a Néron lft-model) if and only if G_K does not contain a subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$ (resp. of type $\mathbb{G}_a$). In the last section, we attempt to globalize our results for excellent Dedekind schemes. An example of Oesterlé shows that one cannot expect a local-global-principle for the existence of Néron models. However, in the case of Néron lft-models, we feel that such a principle is true and formulate it as a conjecture: G_K admits a Néron lft-model if G_K does not contain a subgroup of type $\mathbb{G}_a$. Finally, admitting the existence of desingularizations, we are able to show that the existence of Néron models (in the original sense) is related to the fact that G_K does not contain a non-trivial unirational subvariety.

10.1 Generalities

If R is a discrete valuation ring with field of fractions K, the set of K-valued points of the multiplicative group $\mathbb{G}_{m,K}$ is not bounded in $\mathbb{G}_{m,K}$. Thus $\mathbb{G}_{m,K}$ does not have a Néron model of finite type over R. We will see, however, that there exists a unique R-model of $\mathbb{G}_{m,K}$ which is a smooth R-group scheme and satisfies the Néron mapping property, but which is not of finite type. This is one of the reasons why we want to generalize the notion of Néron models.

Definition 1. *Let S be a Dedekind scheme with ring of rational functions K. Let X_K be a smooth K-scheme. A smooth and separated S-model X is called a Néron* lft*-model of X_K if X satisfies the Néron mapping property*; *cf.* 1.2/1.

Since we do not require X to be of finite type over S, such models are just locally of finite type (lft) over S. As in the case of Néron models, it follows from the Néron mapping property that Néron lft-models are unique and that their formation is compatible with localization and étale base change. In particular, the analogue of 1.2/4 remains valid: an S-scheme X which is locally of finite type is a Néron lft-model of X_K over S if and only if $X \otimes_S \mathcal{O}_{S,s}$ is a Néron lft-model of X_K over $\operatorname{Spec} \mathcal{O}_{S,s}$ for each closed point $s \in S$. The Néron lft-model X of a group scheme X_K is a group scheme again. In this case the identity component X^0 is of finite type. Namely, locally on S, there exists an S-dense open affine subscheme U of X^0 and the map $U \times_S U \longrightarrow X^0$ induced by the group law is surjective. Furthermore, it follows from 6.4/1 that any finite set of points of a fibre of X is contained in an affine open subscheme of X.

In the following we want to generalize certain results on Néron models to the case of Néron lft-models. Let us start with the criterion 7.1/1.

Proposition 2. *Let R be a discrete valuation ring and let G be a smooth and separated R-group scheme. Then the following conditions are equivalent:*

(a) *G is a Néron* lft-*model of its generic fibre.*

(b) *Let $R \longrightarrow R'$ be a local extension of discrete valuation rings where R' is essentially smooth over R. Then, if K' is the field of fractions of R', the canonical map $G(R') \longrightarrow G(K')$ is surjective. (Recall that R' is said to be essentially smooth over R if it is the local ring of a smooth R-scheme).*

Proof. The implication (a) $\Longrightarrow$ (b) is a consequence of the Néron mapping property. For the implication (b) $\Longrightarrow$ (a), consider a smooth R-scheme Z and a K-morphism $Z_K \longrightarrow G_K$ of the generic fibres. Due to the assumption, this map extends to an R-rational map $Z \dashrightarrow G$ and, hence, to an R-morphism $Z \longrightarrow G$ by Weil's extension theorem 4.4/1. Thus we see that G satisfies the Néron mapping property. □

Note that, in Proposition 2, it is not sufficient to ask the extension property for étale integral points, as it is in 7.1/1 in the case of Néron models. Next we want to formulate 7.2/1 (ii) for Néron lft-models; the second proof we have given in Section 7.2 carries over without changes.

Proposition 3. *Let R be a discrete valuation ring and let $R \longrightarrow R'$ be an extension of ramification index 1 with fields of fractions K and K'. Assume that G_K is a smooth K-group scheme. If G is a Néron* lft-*model of G_K over R, then $G \otimes_R R'$ is a Néron* lft-*model of $G_K \otimes_K K'$ over R'.*

Moreover, there is an analogue of 7.2/4.

Proposition 4. *Let $S' \longrightarrow S$ be a finite flat extension of Dedekind schemes with rings of rational functions K' and K. Let G_K be a smooth K-group scheme and denote by $G_{K'}$ the K'-group scheme obtained by base change. Let H_K be a closed subgroup of G_K which is smooth. Assume that $G_{K'}$ admits a Néron* lft-*model G' over S'. Then the Néron* lft-*model of H_K over S exists and can be constructed as a group smoothening of the schematic closure of H_K in the Weil restriction $\mathfrak{R}_{S'/S}(G')$.*

Proof. Since any finite set of points of G' is contained in an affine open subscheme of G', the Weil restriction $\mathfrak{R}_{S'/S}(G')$ is represented by an S-scheme which is separated and smooth; cf. 7.6/4 and 7.6/5. By functoriality it is clear that $\mathfrak{R}_{S'/S}(G')$ is the Néron lft-model of $\mathfrak{R}_{K'/K}(G'_{K'})$ over S; cf. 7.6/6. There is a canonical closed immersion

$$\iota : H_K \longrightarrow \mathfrak{R}_{K'/K}(G'_{K'}) .$$

Denote by $\bar{H}$ the schematic closure of H_K in $\mathfrak{R}_{S'/S}(G')$. Then $\bar{H}$ is flat over S. Similarly as exercised in Section 7.1 by applying the smoothening process to the closed fibres of $\bar{H}$, we get a morphism $H \longrightarrow \bar{H}$ from a smooth R-group scheme H to $\bar{H}$ by successively blowing up subgroup schemes in the closed fibres. Indeed, $\bar{H} \cap \mathfrak{R}_{S'/S}(G')^0$ is of finite type over S, since the identity component $\mathfrak{R}_{S'/S}(G')^0$ of $\mathfrak{R}_{S'/S}(G')$ is of finite type over S. So $\bar{H} \cap \mathfrak{R}_{S'/S}(G')^0$ has at most finitely many non-smooth fibres over S. Using translations, one sees that the same is true for $\bar{H}$ and, furthermore, that the non-smooth locus of $\bar{H}$ is invariant under translations. Then it is clear that the process of group smoothenings will work as in the finite type case, since it suffices to control the defect of smoothness over $\bar{H} \cap \mathfrak{R}_{S'/S}(G')^0$. As in 7.1/6, one verifies that H is the Néron lft-model of H_K over R. □

Example 5. *Let S be a Dedekind scheme with ring of rational functions K. The multiplicative group $\mathbb{G}_{m,K}$ over K admits a Néron* lft-*model G over S. Its identity component is isomorphic to $\mathbb{G}_{m,S}$.*

Proof. In order to give a precise description of G, one proceeds as follows.. Let s be a closed point of S and let π_s be a generator of the ideal corresponding to the closed point $s \in S$ over an open neighborhood $U(s)$ of s. So, for each $\nu \in \mathbb{Z}$, we can view π_s^ν as a $(U(s) - \{s\})$-valued point of $\mathbb{G}_{m,S}$. Then, let $\pi_s^\nu \cdot \mathbb{G}_{m,S}$ be a copy of $\mathbb{G}_{m,S} \times_S U(s)$, viewed as the translate of $\mathbb{G}_{m,S}$ by π_s^ν in the Néron lft-model we want to construct. The translations by the sections π_s^ν, $\nu \in \mathbb{Z}$, define gluing data between $\mathbb{G}_{m,S}$ and the $\pi_s^\nu \cdot \mathbb{G}_{m,S}$ over $U(s) - \{s\}$ in a canonical way. So we can define

$$G = \bigcup_{s \in |S|} \bigcup_{\nu \in \mathbb{Z}} (\pi_s^\nu \cdot \mathbb{G}_{m,S})$$

as the result of the gluing of $\mathbb{G}_{m,S}$ with the copies $(\pi_s^\nu \cdot \mathbb{G}_{m,S})$ where $|S|$ is the set of closed points of S.

In order to show that G is a Néron lft-model of $\mathbb{G}_{m,K}$ over S, note first that G is a smooth and separated S-group scheme with generic fibre $\mathbb{G}_{m,K}$. So we have only to verify the Néron mapping property for G. Since the construction of G is compatible with localization of S, we may assume that S consists of a discrete valuation ring R; cf. the analogue of 1.2/4. Due to Proposition 2, it suffices to show for any extension $R \longrightarrow R'$ of ramification index 1 that each K'-valued point extends to an R'-valued point of G. Since the construction of G is compatible with such ring extensions, we may assume $R = R'$. But then it is clear that the canonical map $G(R) \longrightarrow G(K)$ is bijective, so that we are done. □

The example we have just given can be generalized to tori over K.

Proposition 6. *Let S be a Dedekind scheme with ring of rational functions K. Any torus T_K over K admits a Néron* lft-*model over S.*

Proof. We may assume that S is affine and that it consists of a Dedekind ring R. If the torus is split, the assertion follows from the above example. In the general case, there exists a finite separable field extension K'/K such that $T_{K'} = T_K \otimes_K K'$ is split. If R' is the integral closure of R in K', then $T_{K'}$ admits a Néron lft-model over R'. Now the assertion follows from Proposition 4. □

Also we can handle the case of extensions of certain algebraic K-groups by tori. For technical reasons we will restrict ourselves to split tori, although this restriction is unnecessary as can be seen by using 10.2/2.

Proposition 7. *Let S be a Dedekind scheme with ring of rational functions K. Let G_K be a smooth connected algebraic K-group which is an extension of a smooth algebraic K-group H_K by a split torus T_K. Assume that* $\mathrm{Hom}(H_K, \mathbb{G}_{m,K}) = 0$; *for example, the latter is the case if H_K is an extension of an abelian veriety by a unipotent group. Then, if H_K admits a Néron* lft-*model over S, the same is true for G_K.*

Proof. Since T_K is a split torus, say of rank r, the extension G_K of H_K by T_K is given by primitive line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_r$ on H_K; cf. Serre [1], Chap. VII, n°15, Thm. 5. Although Serre considers only the case where H_K is an abelian variety, the result extends to our situation, since each homomorphism of H_K to $\mathbb{G}_{m,K}$ is constant. A line bundle $\mathscr{L}$ on a group scheme G is called primitive if there is an isomorphism

$$m^*\mathscr{L} \cong p_1^*\mathscr{L} \otimes p_2^*\mathscr{L}$$

where m is the group law of G and where $p_i : G \times G \longrightarrow G$ are the projections, $i = 1, 2$. Since the local rings of the Néron model H of H_K are factorial, the line bundles $\mathscr{L}_\rho$, $\rho = 1, \ldots, r$, extend to primitive line bundles on the identity component H^0 of H. Thus, they give rise to an extension

$$1 \longrightarrow T^0 \longrightarrow G^0 \longrightarrow H^0 \longrightarrow 1$$

whose generic fibre is the extension we started with. Then G^0 will be the identity component of the Néron lft-model G of G_K whereas G itself has to be constructed by gluing "translates" of G^0.

In order to do this, let us start with the construction of the local Néron lft-model at a closed point s of S. Let R_s^{sh} be a strict henselization of the local ring R_s and let K_s^{sh} be its field of fractions. Then set

$$\Lambda_s = G(K_s^{sh})/G^0(R_s^{sh}) \supset I_s = T(K_s^{sh})/T^0(R_s^{sh}) ,$$

where I_s is isomorphic to $\mathbb{Z}^r$. Due to Hilbert's Theorem 90, the quotient Λ_s/I_s is canonically isomorphic to the group $H(K_s^{sh})/H^0(R_s^{sh})$. In the case where Λ_s can be represented by a set $\{\lambda_s\}$ of K-valued points of G_K, we can, similarly as in Example 5, define a smooth and separated R-group scheme

$$G(s) = \bigcup_{\lambda_s \in \Lambda_s} (\lambda_s \cdot G^0)$$

as the result of a gluing where the gluing data are concentrated on the generic fibre and are given by the translations with the sections λ_s. Then each K-valued point of G_K extends to an R-valued point of G. Since this construction is compatible with any extension $R \longrightarrow R'$ of ramification index 1, each K'-valued point of G_K extends to an R'-valued point of G where K' is the ring of fractions of R'. Then, using Proposition 2, one shows that $G(s)$ satisfies the Néron mapping property. Hence, it is the Néron lft-model of G_K over R_s. If the sections $\{\lambda_s\}$ are not defined over R_s, one shows by means of descent that the group $G(s)$ which can be defined over a strict henselization R_s^{sh} of R_s is already defined over the given ring R_s and, hence, is a Néron lft-model of G_K over R_s. In the global case, the Néron lft-model G of G_K is given by gluing the local models $G(s)$, $s \in |S|$, where $|S|$ is the set of all closed points of S; hence

$$G = \bigcup_{s \in |S|} G(s) .$$

In order to explain the gluing procedure, consider a "component" $G(s)'$ of $G(s)$; thereby we mean an open subscheme consisting of G_K and of a connected component of $G(s)$. Then $G(s)'$ is of finite type over R_s and, hence, it extends over an open neighborhood $U(s)$ of s. Since G_K is connected, we may assume that $G(s)'$ coincides with G^0 over $U(s) - \{s\}$. So this way we obtain gluing data between G^0 and each component $G(s)'$ of $G(s)$ and, hence, between G^0 and $G(s)$. It is clear that these data give rise to gluing data for the family $(G(s); s \in |S|)$. In particular, the pull-back of G to the local scheme $\mathrm{Spec}\, \mathcal{O}_{S,s}$ is isomorphic to $G(s)$. Thus, it is clear that G satisfies the Néron mapping property and, hence, is a Néron lft-model of G_K over S. □

Unipotent K-groups may contain a subgroup of type $\mathbb{G}_a$. So they do not necessarily admit Néron lft-models as we will see by the following proposition. But we mention that, if K is not perfect, there are smooth connected unipotent groups, so-called K-wound unipotent groups, which do not contain the additive group $\mathbb{G}_{a,K}$. In Section 10.2 we will discuss the existence of Néron models for such groups.

Proposition 8. *Let S be a Dedekind scheme with ring of rational functions K. If G_K admits a Néron* lft-*model, then G_K does not contain a subgroup of type $\mathbb{G}_a$.*

Proof. Since Néron lft-models are compatible with localizations and étale extensions of the base scheme, we may assume that S consists of a strictly henselian discrete valuation ring R with uniformizing parameter π. Proceeding indirectly, we may assume by Proposition 4 that $G_K = \mathbb{G}_{a,K}$ and that G_K admits a Néron lft-model G. Let us fix a coordinate function ξ_0 for G_K, say $G_K = \mathrm{Spec}\, K[\xi_0]$. Then set $G^n = \mathrm{Spec}\, R[\xi_n]$ for $n \in \mathbb{N}$, where the ξ_n are indeterminates, and consider the morphisms

$$G^n = \mathrm{Spec}\, R[\xi_n] \longrightarrow G^{n+1} = \mathrm{Spec}\, R[\xi_{n+1}]$$

induced by sending ξ_{n+1} to $\pi \cdot \xi_n$. These morphisms induce the zero map on the special fibres. We regard each G^n as a smooth R-model of G_K via the isomorphism

$$G^n \otimes_R K \longrightarrow G_K$$

induced by the map $K[\xi_0] \longrightarrow K[\xi_n]$ sending ξ_0 to $\pi^{-n}\xi_n$. Thus, we get commutative diagrams

$$\begin{tikzcd} G_K^n \arrow[rr] \arrow[dr] & & G_K^{n+1} \arrow[dl] \\ & G_K & \end{tikzcd}$$

Due to the Néron mapping property, these diagrams extend to communtative diagrams

$$\begin{tikzcd} G^n \arrow[rr] \arrow[dr] & & G^{n+1} \arrow[dl] \\ & G & \end{tikzcd}$$

The morphisms induce the zero map on special fibres. So we see that each S-valued point of G specializes into the zero section, since such a point can be regarded as an S-valued point of some G^n. Hence, we arrive at a contradiction. □

Next we will discuss a criterion relating the existence of global Néron lft-models to the existence of local Néron lft-models.

Proposition 9. *Let S be a Dedekind scheme with ring of rational functions K. Let G_K be a smooth connected algebraic K-group. Assume that, for each closed point s of S, the local Néron* lft-*model of G_K over $\mathcal{O}_{S,s}$ exists. Then the following conditions are equivalent:*

(a) *G_K admits a global Néron* lft-*model over S.*

(b) *There exists a dense open subscheme U of S and, over U, a smooth group scheme with connected fibres which coincides with the identity component of the local Néron* lft-*model G_K for each closed point s of U.*

(c) *There exists a coherent (locally free) $\mathcal{O}_S$-module $\mathcal{L}$ which, over each local ring of S, coincides with the Lie algebra of the local Néron* lft-*model of G_K.*

Proof. The implication (a) $\Longrightarrow$ (c) is trivial. To show the implication (c) $\Longrightarrow$ (b), let $G(s)^0$, for any closed point s of S, be the identity component of the local Néron lft-model of G_K over $\mathcal{O}_{S,s}$. Since $G(s)^0$ is quasi-compact, there exist an open neighborhood $U(s)$ of s and a smooth $U(s)$-group scheme $G^0_{U(s)}$ with connected fibres such that $G^0_{U(s)}$ induces $G(s)^0$ over the local ring $\mathcal{O}_{S,s}$. Furthermore, due to the assumption (c), we may assume that the Lie algebra of $G^0_{U(s)}$ coincides with the Lie algebra of the local Néron lft-model at each point t of $U(s)$. Then, for each $t \in U(s)$, the canonical map

$$G^0_{U(s)} \times_{U(s)} \operatorname{Spec} \mathcal{O}_{S,t} \longrightarrow G(t)^0$$

is étale and, hence, an isomorphism, since it is an isomorphism on generic fibres. So condition (b) is clear.

For the implication (b) $\Longrightarrow$ (a) we will first construct the identity component of the Néron lft-model. So let G^0_U be the U-group scheme given by condition (b). If s is a closed point of S not contained in U, the identity component $G(s)^0$ of the Néron lft-model of G_K over $\mathcal{O}_{S,s}$ is of finite type over $\mathcal{O}_{S,s}$ and, hence, extends to a smooth group scheme $G^0_{U(s)}$ with connected fibres over an open neighborhood $U(s)$ of s. Since G^0_U and $G^0_{U(s)}$ coincide on the generic fibre, they coincide over an open neighborhood of s in $U \cap U(s)$. So we get gluing data and, hence, a smooth S-group scheme G^0 with connected fibres which coincides with the identity components of the local Néron lft-models at closed points of S. Now, a Néron lft-model G of G_K is obtained by gluing the local Néron lft-models $G(s)$, $s \in |S|$, where $|S|$ is the set of all closed points of S; i.e.,

$$G = \bigcup_{s \in |S|} G(s) \,.$$

The procedure is the same as in Proposition 7. Also the Néron mapping property is verified as exercised in the proof of Proposition 7. □

Since a smooth group scheme with connected fibres over a Dedekind scheme is quasi-compact, the proof of the implication (b) $\Longrightarrow$ (a) of the above proposition shows the following fact:

Corollary 10. *Let S be a Dedekind scheme with ring of rational functions K. Let G_K be a smooth connected algebraic K-group. Assume that there exists a global Néron* lft-*model of G_K over S. Then G_K admits a Néron model over S if and only if the groups of connected components of the local Néron* lft-*models are finite and, for almost all closed points of S, are trivial.*

Finally, we want to give an example showing that the existence of local Néron models does not imply the existence of a global Néron model.

Example 11 (Oesterlé [1]). Let R be an excellent Dedekind ring with field of fractions K of positive characteristic p, let K'/K be a radicial field extension of order p^n, and let R' be the integral closure of R in K'. Let G_K be the Weil restriction of the multiplicative group $\mathbb{G}_{m,K'}$ with respect to K'/K. Consider the quotient $U_K = G_K/\mathbb{G}_{m,K}$ where $\mathbb{G}_{m,K}$ is viewed as a subgroup of G_K via the canonical closed immersion

$$\mathbb{G}_{m,K} \longrightarrow G_K = \mathfrak{R}_{K'/K}(\mathbb{G}_{m,K'}) \,.$$

For each closed point s of $\operatorname{Spec} R$, we will see that the local Néron model exists and that its group of connected components is a cyclic group of order e_s where e_s is the index of ramification of the extension R'_s/R_s. Moreover, U_K admits a global Néron lft-model over R which, in general, will not be of finite type over R if R has infinitely many maximal ideals.

As a typical case, one may take for R the ring of an affine normal curve over a perfect field. In this case, the ramification index at each closed point coincides with the degree of the radicial extension $[K':K]$. In particular, U_K does not admit a global Néron model if the extension K'/K is not trivial.

So let us justify the fact on U_K we have claimed above. Due to Hilbert's Theorem 90, we have

$$U_K(K) = (K')^*/K^* \ .$$

If R is a discrete valuation ring and $R \longrightarrow R'$ is of ramification index e, the group $U_K(K)$ can be written in the form

$$(K')^*/K^* = (R')^*/R^* \times (\mathbb{Z}/e\mathbb{Z}) \ .$$

Similarly as for the generic fibre, we have a canonical map

$$\mathbb{G}_{m,R} \longrightarrow G_R := \mathfrak{R}_{R'/R}(\mathbb{G}_{m,R'}) \ ,$$

which is a closed immersion. Thus, we can define the quotient

$$U^0 = G_R/\mathbb{G}_{m,R} \ .$$

which is a smooth separated algebraic space; cf. 8.3/9. Due to 6.6/3, it even is a smooth R-group scheme. Moreover, we have

$$U^0(R) = (R')^*/R^* \ .$$

For each closed point s of Spec(R), the local Néron model $U(s)$ is obtained by gluing $U^0 \otimes_R R_s$ with e_s copies of it along the generic fibre where the gluing data are given via the translation on the generic fibre by representatives of $U(K)/U^0(R_s)$. Then, as in Example 5, it is easy to see that U_s satisfies the Néron mapping property. By Proposition 9, we see that there exists a global Néron lft-model of U_K over R.

One can show that the global Néron lft-model of U_K is isomorphic to the quotient of the Weil restriction of the Néron model of $\mathbb{G}_{m,K'}$ by the Néron model of $\mathbb{G}_{m,K}$. □

10.2 The Local Case

In the following, let R be a discrete valuation ring with field of fractions K and let G_K be a *smooth commutative algebraic K-group*. So, in particular, G_K is of finite type over K. We want to discuss criteria for the existence of a Néron model (resp. of a Néron lft-model) of G_K over R depending on its structure as algebraic group. To fix the notations, let R^{sh} be the strict henselization of R with field of fractions K^{sh}, let $\hat{R}^{sh}$ be the strict henselization of the completion $\hat{R}$ of R, and let $\hat{K}^{sh}$ be the field of fractions of $\hat{R}^{sh}$. Since certain parts of our considerations will require an excellent base ring, recall that the strict henselization of an excellent discrete valuation ring is excellent again by 3.6/2. So $\hat{R}^{sh}$ is excellent. In particular, the extension $\hat{K}^{sh}/\hat{K}$ is separable. Furthermore, if R is excellent, R^{sh} is excellent and the extension $\hat{K}^{sh}/K$ is separable.

We will first concentrate on Néron models. We know already that G_K admits a Néron model if and only if the set of its K^{sh}-valued points is bounded in G_K. Now we want to formulate a necessary and sufficient condition for the existence of a Néron model for G_K in terms of the group structure of G_K. Let us begin with some definitions. If X is a separated K-scheme of finite type, a *compactification* of X is an open immersion $X \hookrightarrow \bar{X}$ of X into a proper K-scheme $\bar{X}$ such that X is schematically dense in $\bar{X}$. The subscheme $\bar{X} - X$ will be referred to as the infinity of the compactification. Due to Nagata [1], [2], compactifications always exist. If, in addition, X and $\bar{X}$ are regular, we will call $\bar{X}$ a *regular compactification* of X. For a regular K-scheme X, there exists a regular compactification if the characteristic of K is zero or if the dimension of X is ≤ 2; cf. Hironaka [2] and Abhyankar [1].

Theorem 1. *Let R be a discrete valuation ring with field of fractions K, and let G_K be a smooth commutative algebraic K-group. Then the following conditions are equivalent:*

(a) *G_K has a Néron model over R.*
(b) *$G_K \otimes_K \hat{K}^{sh}$ contains no subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$.*
(c) *$G_K \otimes_K \hat{K}^{sh}$ admits a compactification without a rational point at infinity.*
(d) *$G_K(\hat{K}^{sh})$ is bounded in G_K.*
(e) *$G_K(K^{sh})$ is bounded in G_K.*

If, in addition, R is excellent, the above conditions are equivalent to

(b') *$G_K \otimes_K K^{sh}$ contains no subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$.*
(c') *$G_K \otimes_K K^{sh}$ admits a compactification without a rational point at infinity.*

For example, a K-wound commutative unipotent algebraic K-group admits a Néron model over R if R is excellent. Namely, such a group does not contain subgroups of type $\mathbb{G}_a$ or $\mathbb{G}_m$ and this property remains true after any separable field extension; cf. Tits [1], Chap. IV, Prop. 4.1.4.

If G_K is the Jacobian J_K of a normal proper curve X_K over K assumed to be geometrically reduced and irreducible, then, due to 9.2/4, there is no subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$ in $J_K \otimes_K L$, for any separable field extension L of K. So, if K is the field of fractions of an excellent discrete valuation ring R, our theorem implies that J_K admits a Néron model over R; cf. 9.5/6. Furthermore, there is a natural compactification of J_K without a rational point at infinity; cf. Example 9.

Before starting with the proof of Theorem 1, we want to deduce a criterion for the existence of Néron lft-models.

Theorem 2. *Let R be a discrete valuation ring with field of fractions K and let G_K be a smooth commutative algebraic K-group. Then the following conditions are equivalent:*

(a) *G_K admits a Néron lft-model over R.*
(b) *$G_K \otimes_K \hat{K}^{sh}$ contains no subgroup of type $\mathbb{G}_a$.*

If, in addition, R is excellent, these conditions are equivalent to

(b') *G_K contains no subgroup of type $\mathbb{G}_a$.*

Let us first deduce Theorem 2 from Theorem 1. The implications (a) ⟹ (b) and (a) ⟹ (b') follow from 10.1/3 and 10.1/8. Next let us show the implication (b') ⟹ (a) under the assumption that R is excellent. Let T_K be the maximal torus of G_K; cf. [SGA 3_{II}], Exp. XIV, Thm. 1.1. Then we have an exact sequence of algebraic

K-groups

$$1 \longrightarrow T_K \longrightarrow G_K \longrightarrow H_K \longrightarrow 1\ ,$$

where H_K is an extension of an abelian variety by a linear group and where the latter is an extension of a unipotent group U_K by a finite multiplicative group; cf. 9.2/1 and [SGA 3_{II}], Exp. XVII, Thm. 7.2.1. Due to [SGA 3_{II}], Exp. XVII, Thm. 6.1.1(A)(ii), the K-groups H_K and, hence, U_K do not contain a subgroup of type $\mathbb{G}_a$, since the same is true for G_K. Then it follows from Tits [1], Chap. IV, Prop. 4.1.4, that $U_K \otimes_K K'$ and, hence by [SGA 3_{II}], Exp. XVII, Lemme 2.3, that $H_K \otimes_K K'$ does not contain a subgroup of type $\mathbb{G}_a$ for any separable field extension K' of K. However, there exists a finite separable field extension K' of K such that $T_K \otimes_K K'$ is split. So, if R' is the integral closure of R in K', the K'-group $H_K \otimes_K K'$ admits a Néron model over R' by Theorem 1, since R' is excellent. Hence, $G_K \otimes_K K'$ admits a Néron lft-model over R' by 10.1/7. Then it follows from 10.1/4 that G_K admits a Néron lft-model over R. For the proof of (b) $\Longrightarrow$ (a), we may assume $R = \hat{R}^{sh}$ by 10.1/4. In particular, R is excellent now and, hence, the assertion follows from the implication (b′) $\Longrightarrow$ (a) which has just been proved. □

Now we come to the *proof of Theorem* 1. Some parts of it have already been proved:

(a) $\Longrightarrow$ (b) Néron models are compatible with base change of ramification index 1; cf. 7.2/2. Hence $G_K \otimes_K \hat{K}^{sh}$ admits a Néron model of finite type over $\hat{R}^{sh}$. So the set of $\hat{K}^{sh}$-valued points of G_K is bounded in G_K and, hence, $G_K \otimes_K \hat{K}^{sh}$ cannot contain a subgroup isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$.

(b) $\Longrightarrow$ (b′) is trivial.

(c) $\Longrightarrow$ (d) follows from 1.1/10, since $\hat{R}^{sh}$ is excellent.

(c′) $\Longrightarrow$ (e) follows from 1.1/10, since R^{sh} is excellent.

(d) $\Longrightarrow$ (e) is trivial.

(e) $\Longrightarrow$ (a); cf. Theorem 1.3/1.

The remainder of this section is devoted to the proof of the implications (b) $\Longrightarrow$ (c) and (b′) $\Longrightarrow$ (c′).

Let us first explain the meaning of conditions (c) and (c′)

Proposition 3. *Let X be a smooth and separated K-scheme of finite type. Consider the following conditions*:

(a) *There exists a compactification $\bar{X}$ of X such that there is no rational point in $\bar{X} - X$.*

(b) *For any affine smooth curve C over K with a rational point s, each K-morphism $C - \{s\} \longrightarrow X$ extends to a K-morphism $C \longrightarrow X$.*

(c) *The canonical map $X(K[[\xi]]) \longrightarrow X(K((\xi)))$ is bijective, where ξ is an indeterminate and where $K((\xi))$ is the field of fractions of $K[[\xi]]$.*

Then one has the following implications: (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c). *If, in addition, X admits a regular compactification $\bar{X}'$, conditions* (a), (b), (c) *are equivalent and, moreover, they are equivalent to*

(d) *$(\bar{X}' - X)(K)$ is empty.*

Proof. (a) $\Longrightarrow$ (b) is trivial, since such a morphism $C - \{s\} \longrightarrow X$ extends to a morphism $C \longrightarrow \bar{X}$ and since the image of s gives rise to a rational point of $\bar{X}$.

(b) $\Longrightarrow$ (c). Let R be the localization of $K[\xi]$ at the origin and let $a \in X(K((\xi)))$. If $\bar{X}$ is a compactification of X, one can view a as a $K[[\xi]]$-valued point of $\bar{X}$. Since R is excellent, it follows from 3.6/9 that there exists a local étale extension R' of R with residue field K and an R'-valued point a' of $\bar{X}$ inducing the given point a on the closed fibre. Furthermore, we may assume that the generic fibre of a' is contained in X. Rewriting the situation in terms of curves, it means that there are an étale map $\varphi : C \longrightarrow \mathbb{A}^1_K$ of an affine curve to the affine line, a rational point s of C lying above the origin, and a morphism $\alpha : C \longrightarrow \bar{X}$ such that the local ring of C at s is isomorphic to R' and such that α induces the R'-valued point a'. Due to (b), the image of α is contained in X. Thus, we see that a is a $K[[\xi]]$-valued point of X and the implication (b) $\Longrightarrow$ (c) is clear.

(c) $\Longrightarrow$ (b). The completion of the local ring of C at s is isomorphic to a formal power series ring $K[[\xi]]$. Hence the assertion follows as in 2.5/5.

(b) $\Longrightarrow$ (d). Let x be a rational point of $\bar{X}' - X$. By taking hyperplane sections, one can construct an irreducible subvariety C of $\bar{X}'$ of dimension one such that C is not contained in $\bar{X}' - X$, such that the point x lies on C, and such that C is smooth at x. We may assume that C is smooth over K. Hence, the inclusion $C \longrightarrow \bar{X}$ yields a contradiction to (b).

(d) $\Longrightarrow$ (a) is evident. □

In order to complete the proof of Theorem 1, it suffices to show that a commutative algebraic K-group G which contains no subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$ admits a G-equivariant compactification $\bar{G}$ without a rational point at infinity. A compactification $\bar{G}$ is called *G-equivariant* if G acts on $\bar{G}$ and if the action is compatible with the group law on G. Let us start with some technical definitions.

Definition 4. *Let G be an algebraic K-group which acts on a K-scheme X of finite type. A subscheme Z of X is called a K-orbit under the action of G if there exist a finite field extension K' of K and a K'-valued point x' of $Z \otimes_K K'$ such that $Z \otimes_K K'$ is the orbit of x' under $G \otimes_K K'$.*

Definition 5 (Mumford [1], Chap. 1.3). *Let G be an algebraic K-group with an action σ on a K-scheme X. Let $\pi : L \longrightarrow X$ be a line bundle on X. A G-linearization is a bundle action λ of G on L which is compatible with the G-action on X; i.e., the diagram*

$$\begin{array}{ccc} G \times_K L & \xrightarrow{\ \lambda\ } & L \\ \Big\downarrow{\scriptstyle \mathrm{id}_G \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\ G \times_K X & \xrightarrow{\ \sigma\ } & X \end{array}$$

is commutative.

For example, look at the canonical action of GL_{n+1} on $\mathbb{P}^n$ and at the canonical ample line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. There is a canonical GL_{n+1}-linearization on $\mathcal{O}_{\mathbb{P}^n}(1)$, but the action of the projective linear group PGL_n cannot be lifted to a PGL_n-linearization of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Now consider a scheme T and a flat T-group scheme G of finite presentation which acts on a T-scheme X of finite presentation. Let P be a torsor under G over T. Then G acts freely on $X \times_T P$ by setting

$$g \circ (x, p) = (g \circ x, g \circ p) .$$

Denote by $(X \times_T P)/G$ the quotient (in terms of sheaves for the fppf-topology) of $X \times_T P$ with respect to the G-action. The quotient commutes with any base change $T' \longrightarrow T$. If $P \longrightarrow T$ admits a section, there is an isomorphism $(X \times_T P)/G \longrightarrow X$. So, $(X \times_T P)/G$ becomes isomorphic to X and, hence, is representable after a base change with an fppf-morphism, since $P \longrightarrow T$ is of this type. If L is a line bundle on X with a G-linearization, then $M = (L \times_T P)/G$ gives rise to a line bundle on $(X \times_T P)/G$ provided that $(X \times_T P)/G$ is a scheme. Due to 6.1/7, we have the following lemma.

Lemma 6. *If L is T-ample, then $(X \times_T P)/G$ is a T-scheme and $M = (L \times_T P)/G$ is T-ample.*

Now let T be the affine scheme of a field K and let G be a smooth K-group scheme. If, in addition, X is projective, the quotient $(X \times_K P)/G$ is always a scheme. Namely, after a finite Galois extension K'/K, there exists a K'-valued point of P. So, the quotient is representable after the extension K'/K. Since finite Galois descent is effective for quasi-projective schemes, we see that $(X \times_K P)/G$ is represented by a quasi-projective K-scheme.

The proof of the implications (b) $\Longrightarrow$ (c) and (b′) $\Longrightarrow$ (c′) in Theorem 1 will be provided by Theorem 7 below. Namely, if G is not connected, then $(\bar{G}^0 \times G)/G^0$ yields a compactification of G as required, where $\bar{G}^0$ is a compactification of the identity component G^0 as in condition (d) below.

Theorem 7. *Let K be a field and let G be a connected (not necessarily smooth) commutative algebraic K-group. Then the following conditions are equivalent:*

(a) *G contains no subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$.*

(b) *G admits a compactification $\bar{G}$ without a rational point at infinity.*

(c) *G admits a G-equivariant projective compactification $\bar{G}$ such that, for each K-torsor P under G, there is no rational point in $(\bar{G} \times_K P)/G - (G \times_K P)/G$.*

(d) *G admits a G-equivariant projective compactification $\bar{G}$ such that there is no K-orbit of $\bar{G}$ under G contained in $\bar{G} - G$.*

If, in addition, G is linear, these conditions are equivalent to

(d′) *G admits a G-equivariant compactification $\bar{G}$ together with a G-linearized ample line bundle such that there is no K-orbit of $\bar{G}$ under G contained in $\bar{G} - G$.*

Remark 8. (i) For a smooth K-wound unipotent algebraic group, the existence of an equivariant projective compactification without rational points at infinity has also been established by Tits (unpublished).

(ii) Presumably, the commutativity of G in Theorem 7 is not necessary. In particular, one can expect that a smooth algebraic K-group which does not contain a subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$ admits an equivariant projective compactification without rational points at infinity. The latter is mainly a question of linear groups. It can be answered positively if G is semi-simple; cf. Borel and Tits [1].

Before starting the proof of Theorem 7, let us have a look at Jacobians where, in certain cases, canonical compactifications exist; cf. Altman and Kleiman [1] and [2].

Example 9 (Altman and Kleiman [1], Thm. 8.5). Let X be a proper curve over a field K, assumed to be geometrically reduced and irreducible, and let $J = \mathrm{Pic}^0_{X/K}$ be its Jacobian. Let $\bar{J}$ be the fppf-sheaf induced by the functor which associates to a K-scheme S the set of isomorphism classes of modules on $X \times_K S$ which are locally of finite presentation and S-flat, and which induce torsion-free modules of rank 1 and degree 0 on the fibres of $X \times_K S$ over S. Then $\bar{J}$ is a projective K-scheme containing J as an open subscheme. If, in addition, X is normal, there is no rational point contained in $\bar{J} - J$.

Indeed, we may assume that K is separably closed, so X has a rational point. Then a rational point of $\bar{J}$ represents a torsion-free rank-1 module of degree 0 on X. Since X is a normal curve, such a module is invertible and, hence, represents a point of J. Moreover, since J is smooth, any K-orbit of $\bar{J}$ under J is smooth, too. So, by the same argument as above, it is clear that there is no K-orbit of $\bar{J}$ contained in $\bar{J} - J$.

Let X be locally planar (i.e., the sheaf of differentials is locally generated by at most two elements); for example, this is the case, if X is normal and if K admits a p-basis of length at most 1. Then J is schematically dense in $\bar{J}$ and, hence, $\bar{J}$ is a compactification of J in our sense; cf. Rego [1]. The canonical action of J on itself by left translation extends to an action of J on $\bar{J}$ and, hence, $\bar{J}$ is a J-equivariant compactification of J. In the general case, the schematic closure of J in $\bar{J}$ is an equivariant compactification in our sense.

Now let us prepare the proof of Theorem 7. The implications

$$(\mathrm{d}') \Longrightarrow (\mathrm{d}) \Longrightarrow (\mathrm{c}) \Longrightarrow (\mathrm{b}) \Longrightarrow (\mathrm{a})$$

are quite easy whereas the proof of (a) $\Longrightarrow$ (d′) (resp. of (a) $\Longrightarrow$ (d)) will be explained in the remainder of this section. If G is smooth over a perfect field K, it is an extension of an abelian variety by a smooth connected linear group L which is a product of a torus and a unipotent group, cf. 9.2/1 and 9.2/2. Furthermore, the unipotent part is a successive extension of groups of type $\mathbb{G}_a$; cf. [SGA 3_{II}], Exp. XVII, Cor. 4.1.3. Thus, condition (a) implies that the unipotent part of L is trivial and, hence, that G is an extension of an abelian variety by a torus in this case. So, when we are given a smooth K-group G, the later considerations concerning unipotent groups are only of interest in the case where the base field K is not perfect.

Due to the structure of commutative algebraic groups, we will reduce the general situation by "dévissage" to the following special cases:

—K-wound unipotent (not necessarily smooth) algebraic K-groups; i.e., connected unipotent K-groups which do not contain subgroups of type $\mathbb{G}_a$.

—anisotropic tori; i.e., tori which do not contain subgroups of type $\mathbb{G}_m$.

We will begin by discussing the K-wound unipotent case. If the group under consideration is smooth and killed by multiplication with p, one has a rather explicit description of it.

Proposition 10 (Tits [1], Chap. III, Section 3). *Let K be a field of characteristic $p > 0$ with infinitely many elements. Let G be a smooth connected commutative algebraic K-group of dimension $n - 1$ such that $p \cdot G = 0$. Then G is K-isomorphic to a closed subgroup of $\mathbb{G}_a^n$ defined by a p-polynomial*

$$F(T_1, \dots, T_n) = \sum_{i=1}^{n} \sum_{j=0}^{m_i} c_{ij} \cdot T_i^{p^j} \in K[T_1, \dots, T_n] .$$

If, in addition, G contains no subgroup of type $\mathbb{G}_a$, one can choose $F(T_1, \dots, T_n)$ in such a way that the polynomials

$$\sum_{j=0}^{m_i} c_{ij} \cdot T_i^{p^j} \in K[T_i]$$

are non-zero, $i = 1, \dots, n$, and that the principal part

$$f(T_1, \dots, T_n) = \sum_{i=1}^{n} c_{im_i} \cdot T_i^{p^{m_i}}$$

of $F(T_1, \dots, T_n)$ has no non-trival rational zero in $\mathbb{A}_K^n$.

Using the specific situation of Proposition 10, it is easy to find an equivariant compactification for smooth unipotent commutative groups which are K-wound and are killed by multiplication with p.

Proposition 11. *Let K be a field of characteristic $p > 0$. Let G be a smooth connected commutative algebraic K-group which is killed by multiplication with p. If G is K-wound, then G admits a G-equivariant compactification $\bar{G}$ together with a G-linearized ample line bundle such that there is no K-orbit of $\bar{G}$ under G in $\bar{G} - G$.*

Proof. We may assume that K has infinitely many elements; otherwise G is trivial. Keep the notations of the last proposition and assume that the exponents occurring in the principle part of the p-polymonial satisfy

$$m_1 \leq m_2 \leq \dots \leq m_n .$$

Let P be the quasi-homogeneous space over K with coordinates

$$Y_i , \qquad i = 0, 1, \dots, n ,$$

having weights

$$w_i = p^{m_n - m_i} , \qquad i = 0, \dots, n ,$$

where we have set $m_0 = m_n$. The open subspace U_0 of P where Y_0 is not zero can be viewed as the group $\mathbb{G}_a^n$ with coordinates

$$T_i = Y_i / Y_0^{w_i} , \qquad i = 1, \dots, n .$$

The action of U_0 on itself extends to an action on P by setting

$$U_0 \times_K P \longrightarrow P , \qquad ((t_i), (y_0, y_i)) \longmapsto (y_0, y_i + t_i \cdot y_0^{w_i}) .$$

We regard G as a closed subscheme of U_0 given by a p-polynomial $F(T_1, \dots, T_n)$. Now, let $X_0, \dots, X_n$ be the coordinates of the projective space $\mathbb{P}_K^n$ and let

$$u : P \longrightarrow \mathbb{P}_K^n$$

be the morphism sending X_i to $(Y_i)^{p^{m_i}}$. Denote by V_0 the open subscheme of $\mathbb{P}_K^n$ where X_0 does not vanish. We can view V_0 as the group $\mathbb{G}_a^n$ with coordinates

$$S_i = X_i / X_0 , \qquad i = 1, \dots, n .$$

The morphism u induces a morphism

$$u_0 : U_0 \longrightarrow V_0$$

of algebraic K-groups and the morphism u is equivariant. In terms of coordinates of rational points the equivariance means the commutativity of the following diagram

$$\begin{array}{ccccc} U_0 \times_K P & \longrightarrow & P & & ((t_i), (y_0, y_i)) \longmapsto (y_0, y_i + t_i \cdot (y_0)^{w_i}) \\ \downarrow{\scriptstyle u_0 \times u} & & \downarrow{\scriptstyle u} & & \\ V_0 \times \mathbb{P}_K^n & \longrightarrow & \mathbb{P}_K^n & & ((s_i), (x_0, x_i)) \longmapsto (x_0, x_i + s_i \cdot x_0) \end{array}$$

where $s_i = t_i^{p^{m_i}}$ for $i = 1, \dots, n$ and where $x_i = (y_i)^{p^{m_i}}$ for $i = 0, \dots, n$. The canonical sheaf $\mathcal{O}_{\mathbb{P}_K^n}(1)$ has a V_0-linearization. Hence, $u^*(\mathcal{O}_{\mathbb{P}_K^n}(1))$ is an ample invertible sheaf on P which has a U_0-linearization.

The schematic closure $\bar{G}$ of G in P is given by the polynomial

$$(Y_0)^{p^{m_n}} \cdot F(Y_1 / Y_0^{w_1}, \dots, Y_n / Y_0^{w_n})$$

which can be viewed as a weighted homogeneous polynomial in the variables $Y_0, \dots, Y_n$. Due to the choice of the weights, the principal part $f(Y_1, \dots, Y_n)$ of $F(Y_1, \dots, Y_n)$ is a weighted homogeneous polynomial and describes the set of the points at infinity of the compactification $\bar{G}$. So, we have

$$\bar{G} - G = \{y \in P, f(y) = 0\} .$$

Due to Proposition 10, there is no rational point in $\bar{G} - G$. Moreover, G acts trivially on $\bar{G} - G$. So $\bar{G}$ cannot contain a K-orbit under G at infinity. □

In order to generalize Proposition 11 to smooth unipotent commutative K-wound groups which are not necessarily killed by multiplication with p, we will need the following lemma.

Lemma 12. *Let G be a connected unipotent commutative algebraic K-group. Assume that G is smooth and K-wound. Then there exists a filtration*

$$0 = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_n = G$$

such that the successive quotients have the same properties as G, and, in addition, are killed by multiplication with p.

Proof. Let n be the smallest integer such that G is annihilated by p^n. We will proceed by induction on n. Let N (resp. I) be the kernel (resp. the image) of the p-multiplication on G. Then I is a smooth connected subgroup of G and, hence, K-wound. The group N is not necessarily smooth. So, consider the largest smooth subgroup M of N. Then M is K-wound as a subgroup of G and, since M is the largest smooth subgroup of N, the quotient N/M is K-wound, too. Since the image of the multiplication by p^{n-1} is contained in N and is smooth, the quotient G/M is killed by multiplication with p^{n-1}. Moreover, G/M is K-wound, since it is an extension of I by N/M both of which are K-wound. Then we can set $G_1 = M$ and the induction hypothesis is applicable to G/M. □

Proceeding by dévissage, we are now able to prove Theorem 7 for unipotent groups which are smooth. But when treating general commutative groups, we will also be concerned with unipotent groups which occur as unipotent radicals. Such unipotent groups do not need to be smooth. Therefore, we need the following lemma.

Lemma 13. *Let G be a connected unipotent commutative algebraic K-group which is not necessarily smooth.*

(a) *There exists an immersion of G into a connected unipotent commutative algebraic K-group G' which is smooth.*

(b) *If G is K-wound, one can choose G' to be K-wound, too.*

Proof. (a) We will first show that G can be embedded into a smooth unipotent commutative group. Denote by F_n the kernel of the n-fold Frobenius morphism on G. Due to [SGA 3_{I}], Exp. VII_{A}, Prop. 8.3, there exists an integer $n \in \mathbb{N}$ such that the quotient G/F_n is smooth. Thus, it suffices to show the assertion for the group F_n. So we may assume that G is a finite connected unipotent group. Hence, it is a successive extension of groups of type α_p; cf. [SGA 3_{II}], Exp. XVII, Prop. 4.2.1. Consider now the Cartier dual G^* of G, which is a successive extension of groups of type α_p also. Hence, the algebra $A = \Gamma(G^*, \mathcal{O}_{G^*})$ is local. The algebraic group U representing the group functor

$$(\text{Sch}/K)^0 \longrightarrow (\text{Groups}), \qquad T \longmapsto \Gamma(T \times_K G^*, \mathcal{O}^*_{T \times_K G^*})$$

is smooth. Interpreting the points of G as characters of G^*, one gets a morphism $G \longrightarrow U$ which is an immersion and which is closed, since G is finite. Since A is local, U is a product of the multiplicative group $\mathbb{G}_m$ and of a smooth connected unipotent group G'. Since G is unipotent, the morphism $G \longrightarrow U$ yields an embedding of G into G'.

(b) Let us start by collecting some facts on extensions of commutative unipotent algebraic groups by étale groups.

(1) If N is an étale K-group and H is an algebraic K-group, the canonical map

$$\text{Ext}(H, N) \longrightarrow \text{Ext}(H \otimes_K K', N \otimes_K K')$$

is bijective for any radicial field extension K'/K; cf. [SGA I], Exp. IX, 4.10.

(2) Let

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

be an extension of smooth commutative unipotent algebraic K-groups. Then the canonical sequence of quasi-algebraic commutative group extensions

$$1 \longrightarrow \text{Ext}(G_3, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \text{Ext}(G_2, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \text{Ext}(G_1, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow 1$$

is exact. If G_2 is killed by multiplication with p^n, one can replace $\mathbb{Q}_p/\mathbb{Z}_p$ by $\mathbb{Z}/p^n\mathbb{Z}$. Now, due to (1), we may assume that K is perfect. In this case, the result is provided by Bégueri [1], Prop. 1.21.

(3) If K is not perfect, there exists for each smooth connected commutative unipotent K-group G a commutative extension

$$1 \longrightarrow N \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

of G by a finite étale group N such that $\tilde{G}$ is K-wound.

Namely, we may assume that G is an extension

$$1 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow G_0 \longrightarrow 1$$

of a smooth connected unipotent K-group G_0 by $\mathbb{G}_a$. Proceeding by induction on the dimension of the group, we may assume, that there exists a commutative extension $\tilde{G}_0$ of G_0 by a finite étale group such that $\tilde{G}_0$ is connected and K-wound. Then, one is easily reduced to the case where G_0 is K-wound. For the group $\mathbb{G}_a$ and each element $x \in K - K^p$, consider the extension

$$1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \tilde{\mathbb{G}}_a(x) \longrightarrow \mathbb{G}_a \longrightarrow 1$$

where $\tilde{\mathbb{G}}_a(x)$ is defined as a subgroup of $\mathbb{G}_a \times \mathbb{G}_a$ by the p-polynomial

$$T_1^p + xT_2^p - T_1$$

and the map $\tilde{\mathbb{G}}_a(x) \longrightarrow \mathbb{G}_a$ is the second projection. Then, due to (2), there exists an extension $\tilde{G} \longrightarrow G$ by a finite étale group which induces $\tilde{\mathbb{G}}_a(x) \longrightarrow \mathbb{G}_a$ by restriction. Thus, $\tilde{G}$ is K-wound as an extension of K-wound groups.

Using these results, the proof of assertion (b) is easily done. Assume that K is not perfect and let G be connected, unipotent, commutative, and K-wound. Due to (a), there exists an immersion of G into a smooth unipotent commutative connected group G_1. Let H be the quotient of G_1 by G, so we have the exact sequence

$$1 \longrightarrow G \longrightarrow G_1 \longrightarrow H \longrightarrow 1 .$$

Since G_1 is smooth, H is smooth also. Due to (3), there exists a commutative extension

$$1 \longrightarrow N \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1 .$$

of H by a finite étale group N such that $\tilde{H}$ is K-wound and connected. Pulling back this extension to G_1, one gets a commutative extension

$$1 \longrightarrow N \longrightarrow \tilde{G}_1 \longrightarrow G_1 \longrightarrow 1 .$$

Note that $\tilde{G}_1$ is smooth and unipotent. Denote the identity component of $\tilde{G}_1$ by G'. Hence, one gets an exact sequence

$$1 \longrightarrow G \longrightarrow G' \longrightarrow \tilde{H} \longrightarrow 1 .$$

So the group G' is smooth, unipotent, commutative, and connected, and, as an extension of K-wound unipotent groups, it is K-wound, too. □

Next we want to discuss the compactification of tori. Let T be a torus, denote by M the group of characters of T and by N the group of 1-parameter subgroups of T. Then

$$M = \mathrm{Hom}_{\bar{K}}(T, \mathbb{G}_m) \text{ and } N = \mathrm{Hom}_{\bar{K}}(\mathbb{G}_m, T)$$

are $\mathrm{Gal}(\bar{K}/K)$-modules, where $\bar{K}$ is an algebraic closure of K. There is a perfect pairing

$$M \times N \longrightarrow \mathbb{Z} .$$

Hence, N and M are canonically dual to each other. Recall that T is anisotropic if one of the following equivalent conditions is satisfied:

(i) T does not contain a subgroup of type $\mathbb{G}_m$.
(ii) T does not admit a group of type $\mathbb{G}_m$ as a quotient.
(iii) M does not contain the unit representation.
(iv) N does not contain the unit representation.

Proposition 14. *Let T be an anisotropic torus over K. Then T admits a T-equivariant compactification $\bar{T}$ such that $\bar{T}$ is normal and projective, such that $\bar{T} - T$ does not contain a K-orbit under T, and such that there is an ample line bundle on $\bar{T}$ with a T-linearization on it.*

Proof. Equivariant compactifications of tori are closely related to rational polyhedral cone decompositions of $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. Over an algebraically closed field, this technique is well documented in the literature; cf. Kempf et al. [1], Chap. I, §§ 1 and 2. So, we will only give advice how to proceed in the case of an arbitrary field.

Consider a finite rational polyhedral cone decomposition $\{\sigma_\alpha\}$ of $N_{\mathbb{Q}}$, which is invariant under $\mathrm{Gal}(\bar{K}/K)$. The vertex of each cone is the origin of $N_{\mathbb{Q}}$. Let $\bar{T}$ be the associated T-equivariant compactification of T. The variety $\bar{T}$ is normal and projective. It has a finite number of orbits under T and these correspond bijectively to the faces of the decomposition $\{\sigma_\alpha\}$; cf. Kempf et al. [1], Chap. I, § 2, Thm. 6. Since $\{\sigma_\alpha\}$ is invariant under $\mathrm{Gal}(\bar{K}/K)$, the Galois group acts on the $\bar{K}$-variety $\bar{T}$ and, hence, by projective descent, $\bar{T}$ is defined over K.

We are going to show that $\bar{T} - T$ does not contain a K-orbit under T. So assume that there is a K-orbit in $\bar{T} - T$. It corresponds to a non-zero face σ of the decomposition $\{\sigma_\alpha\}$ which is stable under $\mathrm{Gal}(\bar{K}/K)$. Consider now the set of the extreme edges of σ which consists of a finite number of half lines $\{L_i, i \in I\}$. This set is invariant under $\mathrm{Gal}(\bar{K}/K)$. Now we can choose non-zero points $x_i \in L_i, i \in I$, such that the set $\{x_i, i \in I\}$ is invariant under $\mathrm{Gal}(\bar{K}/K)$. So the point $x = \sum_{i \in I} x_i$ is a non-zero point of σ which is invariant under $\mathrm{Gal}(\bar{K}/K)$ and, hence, gives rise to a non-zero element of N. Thus, we get a contradiction to T being anisotropic.

It remains to show that there is an ample T-linearized line bundle on $\bar{T}$. Let $\mathscr{L}$ be the ample line bundle on $\bar{T}$. Since the Picard group of $\bar{T}$ is discrete (use Kempf et al. [1]. Chap. I, § 2, Thm. 9), $\mathscr{L}$ is invariant under T. Hence, it is easy to see that a power of $\mathscr{L}$ admits a T-linearization. □

For the dévissage, we need a technique of constructing an equivariant compactification of an extension of groups with given equivariant compactifications. This part works also for not necessarily commutative groups.

So consider an exact sequence

$$1 \longrightarrow G \longrightarrow E \longrightarrow H \longrightarrow 1$$

of algebraic K-groups. In particular, $E \longrightarrow H$ is a torsor over H with respect to the H-group scheme $G_H = G \times_K H$. In order to avoid problems with representability of quotients, we will work with projective equivariant compactifications admitting ample line bundles with linearizations. We have to introduce some more notations:

Let X be a K-scheme with an action of G on X on the left and let L be an ample line bundle on X with a G-linearization. Then G_H acts on $X_H = X \times_K H$ as an H-group scheme and $L_H = L \times_K H$ is an H-ample line bundle on X_H with a G_H-linearization. G_H acts freely on $X \times_K E = X_H \times_H E$ by setting

$$g \circ (x, e) = (g \circ x, ge) .$$

Denote by $(X_H \times_H E)/G_H$ the quotient (in terms of sheaves for the fppf-topology over H) of $(X_H \times_H E)$ with respect to the G_H-action. Introduce similar notations for L instead of X. Due to Lemma 6, $(X_H \times_H E)/G_H$ is an H-scheme and $(L_H \times_H E)/G_H$ is an H-ample line bundle on $(X_H \times_H E)/G_H$.

Furthermore, there is an action of E (on the right)

$$(X \times_K E) \times_K E \longrightarrow (X \times_K E) , \qquad ((x, e), e') \longmapsto (x, ee') .$$

This action is compatible with the left action of G on X. So the E-action on $(X \times_K E)$ induces an E-action on $(X_H \times_H E)/G_H$ in a canonical way. The projection

$$(X_H \times_H E)/G_H \longrightarrow H$$

is E-equivariant where E acts on H by right translation. Similarly, the line bundle $(L_H \times_H E)/G_H$ on $(X_H \times_H E)/G_H$ has a canonical E-linearization with respect to the E-action on $(X_H \times_H E)/G_H$.

Lemma 15. *Consider the exact sequence*

$$1 \longrightarrow G \longrightarrow E \longrightarrow H \longrightarrow 1$$

of algebraic K-groups. Let $\bar{G}$ be an equivariant compactification of G and let L be an ample line bundle on $\bar{G}$ with a G-linearization. Set $Y = (\bar{G}_H \times_H E)/G_H$ and $M = (L_H \times_H E)/G_H$. Then

(a) *Y is a projective H-scheme which contains E as an open subscheme and the canonical action of E on itself by right translation extends to an action on Y and is compatible with the G-action on Y. The projection $p: Y \longrightarrow H$ is E-equivariant where E acts on H by right translation. The line bundle M has an E-linearization and is H-ample. Y is quasi-projective over K.*

If $\bar{G} - G$ does not contain a K-orbit under the action of G, then $Y - E$ does not contain a K-orbit under the action of E.

(b) *Let $\bar{H}$ be an equivariant compactification of H and let N be an ample line bundle on $\bar{H}$ with an H-linearization. Then there is a commutative cartesian diagram*

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "p"] & \bar{Y} \arrow[d, "\bar{p}"] \\ H \arrow[r, hook] & \bar{H} \end{tikzcd}$$

such that the following is satisfied: $Y \hookrightarrow \bar{Y}$ is an E-equivariant compactification and $\bar{p}$ is E-equivariant. $\bar{Y}$ is a projective K-scheme and has an ample line bundle with an E-linearization.

If $\bar{G}$-G and $\bar{H}$-H do not contain K-orbits, then $\bar{Y} - Y$ does not contain a K-orbit under the action of E.

Proof. Assertion (a) follows mainly from what has been said before. Y is quasi-projective, since H is quasi-projective. It remains to show that there is no K-orbit contained in $\bar{Y} - E$. So consider a K-orbit Z of $\bar{Y}$ under the action of E. Its image $p(Z)$ is a K-orbit of H and, hence, $p(Z) = H$. The E-action on Y induces a right action of G on the fibre over the unit element of H which is canonically isomorphic to $\bar{G}$. This action is related to the left action of G we started with by the relations

$$\bar{g} \cdot f = f^{-1} \cdot \bar{g} \,, \qquad \bar{g} \in \bar{G} \,, \qquad f \in G \,.$$

Thus we see that the intersection of Z with the fibre over the unit element of H is a K-orbit of $\bar{G}$ under the action of G. So it must be G. Then we get $Z = E$.

(b) After replacing L by $L^{\otimes n}$ for a suitable integer n, we may assume that L is very ample and, hence, that M is very H-ample. Since H admits an ample line bundle with H-linearization, it is affine. So, we may assume that M is very ample.

The K-vector space $\Gamma(Y, M)$ has an E-action induced by the E-linearization of M. Now there is a finite-dimensional subspace W of the vectorspace $\Gamma(Y, M)$ which defines an embedding of Y into its associated projective space $P = \mathbb{P}(W)$. Since the smallest subspace which is stable under E and which contains W is also of finite dimension, we may assume that W is stable under E. So E acts on P and there is an E-linearization on $\mathcal{O}_P(1)$. Due to the choice of W, there is an E-equivariant embedding $Y \longrightarrow P$ such that the pull-back of $\mathcal{O}_P(1)$ is isomorphic to M. Now consider the morphism

$$Y \longrightarrow P \times_K \bar{H}$$

induced by $Y \longrightarrow P$ and $Y \longrightarrow H \longrightarrow \bar{H}$. Let $\bar{Y}$ be the schematic image of Y in $P \times_K \bar{H}$. Then $\bar{Y}$ is projective. Since Y is proper over H, the schematic closure $\bar{Y}$ coincides with Y over H. By continuity, the action of E on Y extends to an action on $\bar{Y}$. Let

$$p_1 : \bar{Y} \longrightarrow P \,, \qquad p_2 : \bar{Y} \longrightarrow \bar{H}$$

be the projections. The restriction $\bar{M}$ of $p_1^*(\mathcal{O}_P(1))$ on $\bar{Y}$ has an E-linearization extending the given E-linearization on M and is $\bar{H}$-ample.

For $n \in \mathbb{N}$, the tensor product $p_2^*(N^{\otimes n}) \otimes \bar{M}$ has a canonical E-linearization with respect to the E-action on $\bar{Y}$ and, for large integers n, it is ample on $\bar{Y}$.

It remains to prove the assertion concerning the orbits. So let Z be a K-orbit of $\bar{Y}$ under the action of E. The projection $p_2(Z)$ is a K-orbit of $\bar{H}$ under the action of H. Due to our assumption, $p_2(Z)$ must be contained in H and, hence, is equal to H. Now we can continue as in part (a) in order to show that Z coincides with E. □

Proof of Theorem 7. We start with the implication (a) $\Longrightarrow$ (d'). Since G is linear, it is an extension of a unipotent group U by a subgroup of multiplicative type M; cf. [SGA 3_{II}], Exp. XVII, Thm. 7.2.1. Due to [SGA 3_{II}], Exp. XVII, Thm. 6.1.1 (A) (ii), the unipotent group U is K-wound. The multiplicative group M is an extension of a finite multiplicative group N by a torus T which is necessarily anisotropic since G does not contain a subgroup of type $\mathbb{G}_m$. Hence, due to Lemma 15 (b), we are reduced to prove the assertion for the groups N, T, and U. It is clear for N. Furthermore, Proposition 14 provides the assertion in the case of T. In the case of U, we may assume that K has characteristic $p > 0$ and, due to Lemma 13, that U is smooth. Using Lemma 15 and Lemma 12, we are reduced to the case where U is killed by the multiplication with p. However, this case has been dealt with in Proposition 11.

Next let us turn to the implication (a) $\Longrightarrow$ (d). It follows from the theorem of Chevalley (cf. 9.2/1) that there exists a connected linear subgroup H of G such that the quotient G/H is an abelian variety. Namely, the kernel F_n of the n-fold Frobenius morphism on G is an affine subgroup of G and, for large integers n, the quotient G/F_n is smooth, cf. [SGA 3_{II}], Exp. XVII, Prop. 4.2.1. Then the assertion follows by Lemma 15 (a) from the implication (a) $\Longrightarrow$ (d'). This concludes the proof, the remaining assertions being trivial. □

The above verification of the implication (a) $\Longrightarrow$ (d') shows that a commutative linear group G which does not contain a subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$ admits a G-equivariant compactification $\bar{G}$ together with a G-linearized ample line bundle such that there is no K-orbit contained in $\bar{G} - G$. So, due to Lemma 15 which is valid for not necessarily commutative groups, the construction carries over to the case of solvable groups G; cf. Remark 8. Namely, a K-wound solvable group admits a filtration

$$G = G_0 \supset G_1 \supset \ldots \supset G_n = \{1\}$$

such that G_i is a normal subgroup of G_{i-1} and G_{i-1}/G_i is commutative and K-wound, $i = 1, \ldots, n$; cf. Tits [1], Chap. IV, Prop. 4.1.4.

10.3 The Global Case

Let S be an excellent Dedekind scheme with infinitely many closed points and let K be its ring of rational functions. Let G_K be a smooth commutative algebraic K-group.

The existence of a Néron lft-model (resp. of a Néron model) of G_K over S implies the existence of a Néron lft-model (resp. of a Néron model) over each local ring of S. But, as we have seen in Example 10.1/11, the converse is not true when dealing with Néron models. The example was given in the case where the characteristic of K is positive.

If K has characteristic zero, we claim that the existence of a global Néron lft-model (resp. of a global Néron model) is equivalent to the existence of the local Néron lft-models (resp. of the local Néron models). Namely, due to 10.2/2, the existence of Néron lft-models over each local ring of S is equivalent to the fact that the unipotent radical of G_K is trivial. Then G_K is an extension of an abelian variety by a torus T and, hence, admits a Néron lft-model over S; the latter follows from 10.1/7 by using 10.1/4. Moreover, when the local Néron lft-models are of finite type over each local ring of S, the subtorus T of G_K is trivial. Indeed, T splits over a finite separable field extension K' of K. There exists a closed point of S at which K' is unramified. Since Néron models are compatible with localization and étale extensions, there is a closed point s' of S', where S' is the spectrum of the integral closure of $\mathcal{O}_{S,s}$ in K', such that $G_K \otimes_K K'$ admits a local Néron model at s'. Then, it follows from 10.2/1 that the torus T is trivial. Thus, we see that G_K is an abelian variety and, hence, that G_K has a Néron model over S; cf. 1.4/3.

The existence of Néron lft-models or Néron models over a global base is still an open question when K has positive characteristic. We conjecture that G_K has a Néron lft-model over S if and only if G_K has one over each local ring of S. Using Theorem 10.2/2, we can state this conjecture in the following way.

Conjecture I. *Let S be an excellent Dedekind scheme with ring of rational functions K and let G_K be a smooth commutative algebraic K-group. Then G_K admits a Néron lft-model over S if G_K contains no subgroup of type $\mathbb{G}_a$.*

As explained before, the conjecture is true if the characteristic of K is zero, but in the case of positive characteristic it is still an open question.

For the remainder of this section we want to concentrate on the existence of Néron models (of finite type). We can give a criterion for the case where G_K admits a regular compactification. Let us begin with some definitions.

A K-variety X (i.e., a separated K-scheme of finite type which is geometrically reduced and irreducible) is called *rational* (resp. *unirational*) if its field of rational functions is purely transcendental over K (resp. contained in a purely transcendental field extension of K). In geometric terms, the latter means that there is a rational map from $\mathbb{A}_K^n$ to X which is birational (resp. dominant). An algebraic K-group G_K is called rational (resp. unirational) if its underlying scheme is rational (resp. unirational). It is easy to see that unirational groups are smooth and connected. For example, tori are unirational; also the K-group of Example 10.1/11 is unirational. Each unirational subscheme of G_K which contains the origin generates a unirational subgroup of G_K. In particular, G_K contains a *largest unirational subgroup* denoted by $\mathrm{uni}(G_K)$. If G_K is an abelian variety, then $\mathrm{uni}(G_K) = 0$.

Theorem 1. *Let G_K be a smooth algebraic group over a field K, where G_K is connected and commutative. Then the following conditions are equivalent:*

(a) $\mathrm{uni}(G_K) = 0$

(b) *Each K-rational map from the projective line $\mathbb{P}_K^1$ to G_K is constant.*

(c) *For any smooth affine curve C_K over K and for any closed point x of C_K, each morphism of $C_K - \{x\}$ to G_K extends to a morphism from C_K to G_K.*

(d) *For any smooth K-scheme X_K, each K-rational map from X_K to G_K is defined everywhere.*

If, in addition, G_K admits a regular compactification $\bar{G}_K$, these conditions are equivalent to

(e) *The smooth locus of $\bar{G}_K$ coincides with G_K.*

The implications

$$(a) \Longleftrightarrow (b) \Longleftarrow (c) \Longleftarrow (d) \Longrightarrow (e)$$

are quite easy to verify and we leave them to the reader. Also it is not difficult to show the implication (e) $\Longrightarrow$ (c) (if G_K admits a regular compactification) and (c) $\Longrightarrow$ (d). Finally the implication (a) $\Longrightarrow$ (c) requires more efforts.

To start the proof, let us begin with the verification of implication (e) $\Longrightarrow$ (c). Let $\varphi : C_K - \{x\} \longrightarrow G_K$ be a K-morphism. Due to the valuation criterion of properness, φ extends to a K-morphism $\bar{\varphi} : C_K \longrightarrow \bar{G}_K$. Now consider the C_K-scheme $\bar{G}_{C_K} = \bar{G}_K \times_K C_K$ which is regular; cf. 2.3/9. Due to assumption (e), the smooth locus of $\bar{G}_{C_K}$ over C_K coincides with $G_K \times_K C_K$; cf. [EGA IV$_4$], 17.7.2. By base extension, $\bar{\varphi}$ gives rise to a section $\bar{\varphi}_{C_K}$ of $\bar{G}_{C_K}$. Now it follows from 3.1/2 that $\bar{\varphi}_{C_K}$ factors through the smooth locus of $\bar{G}_{C_K}$ and, hence, $\bar{\varphi}$ maps to G_K.

For the implication (c) $\Longrightarrow$ (d), consider a rational map $\varphi_K : X_K \dashrightarrow G_K$, where X_K is smooth and irreducible of dimension n. Since we consider K-schemes of finite type, φ_K is induced by a T-rational map $\varphi : X \dashrightarrow G$ from a smooth T-scheme X to a smooth and separated T-group scheme G, where T is an irreducible regular scheme of finite type over the ring of integers $\mathbb{Z}$. We may assume that K is the field of rational functions on T. Due to 4.4/1, the complement F of the domain of definition of φ is of pure codimension 1 and, hence, is a relative Cartier divisor. We have to show that F is empty. Proceeding indirectly, let us assume that F is not empty. Then look at the graph Γ_K of φ_K in $X_K \times_K G_K$. It is clear that the image Q_K of Γ_K under the first projection p_1 cannot contain a generic point of F_K as seen by a similar argument as used in the proof of 4.3/4. Since Q_K is constructible, we may assume, after shrinking X_K, that Q_K is disjoint from F_K. Now we will derive a contradiction by constructing a smooth curve C_K contained in X_K, but not in F_K such that C_K meets F_K at a closed point. Namely, due to assumption (c), the curve C_K must be contained in Q_K. Since F is not empty, there exists a closed point x in F. Let t be the image of x in T. The residue field of t is finite and hence perfect. So $k(x)$ is separable over $k(t)$. Then it follows from the Jacobi criterion 2.2/7 that there exist elements $f_2, \ldots, f_n$ in the maximal ideal of the local ring of X at x which, in a neighborhood of x, define an irreducible relative smooth T-curve C. We may assume that F induces a relative Cartier divisor on C. In particular, $C \cap F$ is flat over T. Hence, the generic fibre of

$C \cap F$ is not empty. Now, the induced morphism $C_K \longrightarrow G_K$ yields a contradiction to (c).

The proof of the implication (a) $\Longrightarrow$ (c) is delicate. It will follow from Corollary 3 below which makes use of the theory of Rosenlicht and Serre on rational maps from curves into commutative algebraic groups. In the following we want to sketch the main ideas of this theory.

So let X be a *proper irreducible curve* over K, assumed to be *geometrically reduced*. Denote by U the smooth locus of X, which is open and dense in X. Let G be a smooth commutative algebraic K-group. We want to study rational maps

$$\varphi : X \dashrightarrow G .$$

If V is the domain of definition of φ, then, for any $n \in \mathbb{N}$, there is a canonical morphism of the n-fold symmetric product $V^{(n)}$ to G induced by φ. We will denote it by $\varphi^{(n)} : V^{(n)} \longrightarrow G$. By restriction to $(U \cap V)^{(n)}$ we get a morphism of the set of Cartier divsors of degree n with support in $U \cap V$ to G; cf. Section 9.3. We denote this map by φ, too. A finite subscheme Y of X is called a *conductor for* φ if $\varphi(\operatorname{div}(f)) = 0$ for each rational function f of X which is defined on Y, which induces the constant function with value 1 on Y, and whose associated divisor has support in $U \cap V$.

Now let Y be a finite subscheme of X. If Y is non-empty, it is a rigidificator for $\mathrm{Pic}_{X/K}$. As introduced in Section 8.1, we denote by $(\mathrm{Pic}_{X/K}, Y)$ the rigidified Picard functor. We set $(\mathrm{Pic}_{X/K}, Y) = \mathrm{Pic}_{X/K}$ if Y is empty. Since, for a K-scheme T, any section of $(U - Y) \times_K T$ induces an effective relative Cartier divisor on $U \times_K T$ of degree 1 whose associated invertible sheaf is canonically rigidified along Y by the function 1, there exists a canonical map $(U - Y) \longrightarrow (\mathrm{Pic}_{X/K}, Y)$ and, hence, a rational map

$$\iota_Y : X \dashrightarrow (\mathrm{Pic}_{X/K}, Y) .$$

By construction Y is a conductor for ι_Y. If Y is empty, we will write ι instead of ι_Y.

For the proof of the implication (a) $\Longrightarrow$ (c) we will use the following result.

Theorem 2. *Keeping the notations of above, the following hold:*

(a) *A finite subscheme Y of X is a conductor for φ if and only if there exists a K-morphism of algebraic groups $\Phi : (\mathrm{Pic}_{X/K}, Y) \longrightarrow G$ making the following diagram commutative:*

$$\begin{tikzcd} X \arrow[r, dashed, "\iota_Y"] \arrow[dr, dashed, "\varphi"'] & (\mathrm{Pic}_{X/K}, Y) \arrow[d, "\Phi"] \\ & G \end{tikzcd}$$

Moreover, the map Φ is uniquely determined.

(b) *There exists a conductor for φ and there even is a smallest one. The latter is called the conductor of φ.*

(c) *Let $\pi : \tilde{X} \longrightarrow X$ be the normalization of X and let x be a closed point of X such that $\pi^{-1}(x)$ is contained in the smooth locus of $\tilde{X}$. If $\varphi \circ \pi$ is defined at $\pi^{-1}(x)$, then x is not contained in the support of the conductor of φ.*

(d) *If X is smooth at x and if x is not contained in the conductor of φ, then φ is defined at x.*

(e) *The conductor of φ commutes with finite separable field extensions.*

Proof. If K is algebraically closed and if X is smooth, the result is classical and is due to Rosenlicht and Serre, cf. Serre [1]; for (a) and (d) see Chap. V, n°9, Thm. 2, for (b) and (c) see Chap. III, n°3, Thm. 1. We want to give some indications on how to proceed in the general case. We may assume that X is geometrically irreducible. Namely, using assertion (e), one can easily reduce to this case.

(a) The if-part is obvious. For the only-if-part, consider first the case where Y is empty. Then the factorization follows from the construction of $\mathrm{Pic}_{X/K}$ via symmetric products à la Weil as explained in Section 9.3. The uniqueness of the factorization is due to the fact that $\mathrm{Pic}_{X/K}$ is generated by the image of ι. Now let Y be a non-empty conductor for φ. There exists a finite birational morphism $X \longrightarrow X'$ which contracts Y to a rational point Y' and which is an isomorphism outside Y and Y'. One easily checks that the canonical map

$$\mathrm{Pic}_{X'/K} = (\mathrm{Pic}_{X'/K}, Y') \longrightarrow (\mathrm{Pic}_{X/K}, Y)$$

is an isomorphism. Thus, the general case is reduced to the case discussed above.

(b) Let Y_1 and Y_2 be finite subschemes of X. Then the diagram

$$\begin{tikzcd} (\mathrm{Pic}_{X/K}, Y_1 \cup Y_2) \arrow[r] \arrow[d] & (\mathrm{Pic}_{X/K}, Y_2) \arrow[d] \\ (\mathrm{Pic}_{X/K}, Y_1) \arrow[r] & (\mathrm{Pic}_{X/K}, Y_1 \cap Y_2) \end{tikzcd}$$

is co-cartesian. Thus, by using the characterization given in (a), we see that the intersection of two conductors is a conductor again. So the existence of a conductor implies the existence of a unique smallest one. Furthermore, one can see by the same argument that the smallest conductor of φ is compatible with finite Galois extensions of the base field; thus assertion (e) is clear. So it remains to show that there is at least one conductor for φ which satisfies assertion (c); hence the smallest one will satisfy (c), too. By what we have said above, we may assume that K is separably closed. Denote by $\pi : \tilde{X} \longrightarrow X$ the normalization of X. Assume for a moment that the base field is algebraically closed. Then, due to Rosenlicht and Serre, there exists a conductor $\tilde{Y}$ for $\varphi \circ \pi$ whose support is disjoint from the domain of definition of $\varphi \circ \pi$. Now let Y be the schematic image of $\tilde{Y}$ in X. Then one shows easily by using the very definition of conductors that Y is a conductor for φ satisfying the assertion (c). When K is not necessarily algebraically closed, we can first work over an algebraic closure $\bar{K}$ of K. So there is a conductor $\bar{Y}$ of $\varphi \otimes_K \bar{K}$. We can replace $\bar{Y}$ by a larger conductor, say Y, without changing its support. Furthermore, we can assume that Y is defined over K, since $\bar{K}$ is radicial over K. So Y fulfills assertion (c).

(d) follows from (a). □

Corollary 3. *Let X be a proper curve over a field K and assume that X is normal and geometrically reduced. Let G be a smooth commutative algebraic K-group. Let*

$\varphi : X \dashrightarrow G$ be a rational map and let Y be the conductor of φ.

(a) *If G does not contain a subgroup of type $\mathbb{G}_a$, then Y is reduced.*

(b) *If $\mathrm{uni}(G) = 0$, the conductor of φ is empty and φ decomposes into a composition $\varphi = \Phi \circ \iota$ where $\Phi : \mathrm{Pic}_{X/K} \longrightarrow G$ is a morphism of algebraic groups. In particular, φ is defined on the smooth locus of X.*

Proof. Denote by $\bar{Y}$ the largest reduced subscheme of Y. Then, we get an exact sequence

$$1 \longrightarrow U \longrightarrow V_Y^* \longrightarrow V_{\bar{Y}}^* \longrightarrow 1$$

of algebraic groups where V_Y^* and $V_{\bar{Y}}^*$ are the algebraic groups representing the functor of global units on Y and on $\bar{Y}$; cf. 8.1/10. The kernel U is a unipotent group which is a successive extension of groups of type $\mathbb{G}_a$. Now look at the exact sequence of 8.1/11

$$0 \longrightarrow V_X^* \longrightarrow V_Y^* \longrightarrow (\mathrm{Pic}_{X/K}, Y) \longrightarrow \mathrm{Pic}_{X/K} \longrightarrow 0$$

In the case of assertion (a), the canonical map

$$\Phi : (\mathrm{Pic}_{X/K}, Y) \longrightarrow G$$

induced by φ sends the image of U in $(\mathrm{Pic}_{X/K}, Y)$ to zero. Hence, Φ factors through

$$(\mathrm{Pic}_{X/K}, Y) \longrightarrow (\mathrm{Pic}_{X/K}, \bar{Y}) .$$

Thus, due to Theorem 2, $\bar{Y}$ is also a conductor for φ, hence $Y = \bar{Y}$ is reduced. In the case of assertion (b), the kernel of the map

$$(\mathrm{Pic}_{X/K}, Y) \longrightarrow \mathrm{Pic}_{X/K}$$

is the group of global units on Y modulo K^* which is unirational. Thus, we see that Φ factors through $\mathrm{Pic}_{X/K}$ and that the conductor of φ is empty. Then the assertion follows by Theorem 2. □

Corollary 3 yields the proof of the implication (a) $\Longrightarrow$ (c) of Theorem 1 and thus completes the proof of Theorem 1.

Remark 4. Using the characterization (c) of Theorem 2, one sees immediately that the condition $\mathrm{uni}(G_K) = 0$ is stable under finite separable field extensions.

Conjecture II. *Let S be an excellent Dedekind scheme with ring of rational functions K and let G_K be a smooth commutative algebraic K-group. If $\mathrm{uni}(G_K) = 0$ then G_K admits a Néron model over S.*

If one admits Conjecture II, Conjecture I is mainly a problem of unirational groups; use the technique of 7.5/1 (b). Conjecture II is true if K has characteristic zero. Indeed, if $\bar{K}$ is an algebraic closure of K, one has $\mathrm{uni}(G_K \otimes_K \bar{K}) = 0$ due to Remark 4. Then $G_K \otimes_K \bar{K}$ cannot contain a subgroup of type $\mathbb{G}_a$ or $\mathbb{G}_m$ and, hence, G_K is an abelian variety. In the case of positive characteristic, some parts of the conjecture can be proved, provided it is known that G_K admits a regular compactification.

Theorem 5. *Let S be an excellent Dedekind scheme with ring of rational functions K and let G_K be a smooth commutative algebraic K-group.*

(a) *Assume that G_K admits a regular compactification $\bar{G}_K$. If $\mathrm{uni}(G_K) = 0$, then G_K admits a Néron model over S.*

(b) *If S is a normal algebraic curve over a field and if G_K admits a Néron model over S, then $\mathrm{uni}(G_K) = 0$.*

Proof. (a) Let us first show that the local Néron models exist. So, we may assume for a moment that S is the affine scheme of a local ring R. Since $\mathrm{uni}(G_K) = 0$, it follows by Remark 4 that $\mathrm{uni}(G_K \otimes_K K^{sh}) = 0$ where K^{sh} is the field of fractions of a strict henselization of R. Then $G_K \otimes_K K^{sh}$ cannot contain a subgroup of type $\mathbb{G}_a$ or of type $\mathbb{G}_m$. Since S is excellent, it follows from 10.2/1 that a Néron model of G_K exists over S. Now let us return to the general situation. It remains to see that there exists a dense open subscheme U of S such that a Néron model of G_K exists over U; cf. 1.4/1. There exists a dense open subscheme U of S such that $\bar{G}_K$ extends to a proper flat U-scheme $\bar{G}_U$. Since S is excellent, the regular locus of $\bar{G}_U$ is open by [EGA IV_2], 7.8.6. So we may assume that $\bar{G}_U$ is regular. Let G_U be the smooth locus of $\bar{G}_U$. Since $\mathrm{uni}(G_K) = 0$, we see by Theorem 1 that the generic fibre of G_U coincides with G_K. After replacing U by a dense open subset, we may assume that G_U is a group scheme over U. Now we claim that G_U is the Néron model of G_K over U. Let $U(s)$ be the spectrum of the strict henselization of the local ring of U at a closed point s of U. Since $\bar{G}_U \times_U U(s)$ is regular, the $U(s)$-valued points of $\bar{G}_U$ factor through the smooth locus G_U by 3.1/2. Then it follows from 7.1/1 that $G_U \times_U \mathrm{Spec}\, \mathcal{O}_{S,s}$ is the local Néron model of G_K over $\mathcal{O}_{S,s}$ and the assertion follows from 1.2/4.

(b) Let us assume that $\mathrm{uni}(G_K)$ is non-trivial. Due to Theorem 1, there exists an affine smooth curve C_K with a closed point x_K and a morphism

$$\varphi_K : C_K - \{x_K\} \longrightarrow G_K$$

such that φ_K does not extend to C_K. Since we are free to replace S by an étale extension (cf. 1.2/2), we may assume that the residue field $k(x_K)$ is radicial over K. Since C_K is smooth over K, the extension $k(x_K)$ can be generated by one element over K. So, after shrinking S, there exist an element $f \in \Gamma(S, \mathcal{O}_S)$ and a p-power p^n such that $k(x_K)$ is generated by the p^n-th root of f. Now $C_K \longrightarrow \mathrm{Spec}(K)$ is induced by a smooth relative curve $C \longrightarrow S$. Denote by Z the schematic closure of the point x_K in C. We may assume, after shrinking S, that Z is a subscheme of $\mathbb{A}_S^1$ defined by $(T^{p^n} - f)$. It is a general fact that there exist infinitely many closed points s of S such that the polynomial $(T^{p^n} - f)$ has a solution over the residue field $k(s)$; cf. Lemma 6 below. If G_K admits a Néron model G of finite type over S, the morphism φ_K extends to a morphism

$$\varphi : (C - Z) \longrightarrow G .$$

Now look at the graph $\Gamma_\varphi \subset C \times_S G$ of φ viewed as a rational map $C \dashrightarrow G$. So Γ_φ

is closed in $C \times_S G$. Let Q be the image of Γ_φ under the first projection $p_1 : C \times_S G \longrightarrow C$. Since G is of finite type over S, the subset Q is constructible. The point x_K is not contained in Q, because φ_K is not defined at x_K. As x_K is the generic point of Z, we may assume, after shrinking S, that Q is disjoint from Z. Now let z be a point of Z such that the field extension $k(z)/k(s)$ is trivial where s is the image of z in S. So there exist an étale extension $S' \longrightarrow S$ and an S'-valued point x' of C such that z is the image of a point s' of S' under x' and such that x_K does not belong to the image of x'. Due to the Néron mapping property, $x'_K \circ \varphi_K$ extends to an S'-valued point of G. By continuity, x' factors through the graph Γ_φ. Thus, we see that the point z must belong to Q and we get a contradiction. □

In the last proof we have used the following fact.

Lemma 6. *Let k be a field of positive characteristic p and let A be an integral k-algebra of finite type and of dimension $d \geq 1$. Let n be a positive integer and let f be an element of A. Then, for any $n \geq 1$, there exist infinitely many prime ideals $\mathfrak{p}$ of A of codimension 1 such that the equation $T^{p^n} - f = 0$ has a solution modulo $\mathfrak{p}$.*

Proof. It suffices to show that there is at least one such prime ideal. By standard limit arguments, we may assume that k is of finite type over its prime field k_0. Then there exists a smooth and irreducible k_0-scheme R_0 such that k is the field of rational functions of R_0, and there exists an R_0-scheme S_0 of finite type such that the generic fibre of S_0 is isomorphic to S, where S is the affine scheme of A. We may assume that S_0 is affine, irreducible, and reduced. Moreover we may assume that f extends to a global section of $\mathcal{O}_{S_0}$. Now let x be a closed point of S_0. Then $k(x)$ is a finite field and, hence, perfect. So we can write

$$f = g^{p^n} + h$$

where g and h are global sections of $\mathcal{O}_{S_0}$ and where $h(x) = 0$. Since the relative dimension of S over R_0 is $d \geq 1$, we can choose g and h in such a way that the subscheme $V(h)$ defined by h is dominant over R_0. So there is a generic point s of $V(h)$ lying above the generic point of R_0. Let $\mathfrak{p} \subset \Gamma(S_0, \mathcal{O}_{S_0})$ be the prime ideal corresponding to s. Then g is a solution of the equation $T^{p^n} - f = 0$ modulo $\mathfrak{p}$, and $\mathfrak{p}$ gives rise to a prime ideal of A as required. □

If we want to apply Theorem 5(a) to an algebraic K-group G_K, it has to be known that G_K admits a regular compactification $\bar{G}_K$, a question which is related to the resolution of singularities in characteristic >0. Since it is widely accepted that the latter problem should admit a positive answer, we get strong indications for Conjecture II being true. Also note that, for a K-wound unipotent group G_K, Thm. VI.3.1 of Oesterlé [1] implies $\mathrm{uni}(G_K) = 0$ if K is of characteristic p and if $\dim G_K < p - 1$.

Bibliography

Grothendieck, A.

[FGA] *Fondements de la Géométrie Algébrique*
Sém. Bourbaki, exp. n° *149* (1956/57), *182* (1958/59), *190* (1959/60), *195* (1959/60), *212* (1960/61), *221* (1960/61), *232* (1961/62), *236* (1961/62), Benjamin, New York (1966)

Grothendieck, A., and Dieudonné, J.

Eléments de Géométrie Algébrique

[EGA I] Le langage des schémas. Publ. Math. IHES *4* (1960)

[EGA II] Etude globale élémentaire de quelques classes de morphismes. Publ. Math. IHES *8* (1961)

[EGA III] Etude cohomologique des faisceaux cohérents. Publ. Math. IHES *11* (1961), *17* (1963)

[EGA IV] Etude locale des schémas et des morphismes de schémas. Publ. Math. IHES *20* (1964), *24* (1965), *28* (1966), *32* (1967)

Grothendieck, A., et al.

Séminaire de Géométrie Algébrique

[SGA 1] Revêtements Etales et Groupe Fondamental. Lect. Notes Math. *224*, Springer, Berlin-Heidelberg-New York (1971)

[SGA 3] Schémas en Groupes I,II,III. Lect. Notes Math. *151*, *152*, *153*, Springer, Berlin-Heidelberg-New York (1970)

[SGA 4] Théorie des Topos et Cohomologie Etale. Lect. Notes Math. *269*, *270*, *305*, Springer, Berlin-Heidelberg-New York (1972–73)

[SGA 6] Théorie des Intersections et Théorème de Riemann-Roch. Lect. Notes Math. *225*, Springer, Berlin-Heidelberg-New York (1971)

[SGA 7] Groupes de Monodromie en Géométrie Algébrique. Lect. Notes Math. *288*, *340*, Springer, Berlin-Heidelberg-New York (1972–73)

Abhyankar, S. S.

1. Resolution of singularities of arithmetical surfaces. Proc. Conf. on Arithm. Alg. Geometry, Purdue (1963)

Altman, A., and Kleiman, S.

1. Compactifying the Picard Scheme. Adv. Math. *35*, 50–112 (1980)
2. Compactifying the Picard Scheme II. Am. J. Math. *101*, 10–41 (1979)

Artin, M.

1. Some numerical criteria for contractibility of curves on algebraic surfaces. Am. J. Math. *84*, 485–496 (1962)
2. On isolated rational singularities of surfaces. Am. J. Math. *88*, 129–136 (1966)
3. The implicit function theorem in algebraic geometry. Algebraic Geometry, Bombay Colloquium 1968. Oxford University Press, Oxford, 13–34 (1969)
4. Algebraic approximation of structures over complete local rings. Publ. Math. IHES *36*, 23–58 (1969)
5. Algebraization of formal moduli I. Global Analysis, Papers in Honor of K. Kodaira. University of Tokyo Press, Princeton University Press, 21–71 (1969)
6. Algebraization of formal moduli II. Ann. Math. *91*, 88–135 (1970)
7. Versal Deformations and Algebraic Stacks. Invent. Math. *27*, 165–189 (1974)

8. Algebraic structure of power series rings. Algebraist's Homage: Papers in Ring Theory and Related Topics. Proc. Conf. on Algebra in Honor of N. Jacobson, New Haven (1981), Contemp. Math. *13*, AMS, Providence, 223–227 (1982)
9. Néron models. Arithmetic Geometry, edited by G. Cornell and J.H. Silverman. Proc. Storrs 1984, Springer, Berlin-Heidelberg-New York (1986)

Artin, M., and Rotthaus, C.
1. A structure theorem for power series rings. Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, vol. 1, pp. 35–44. Kinokuniya Tokyo (1987)

Artin, M., and Winters, G.
1. Degenerate fibres and stable reduction of curves. Topology *10*, 373–383 (1971)

Becker, J., Denef, J., Lipshitz, L., van den Dries, L.
1. Ultraproducts and Approximation in Local Rings I. Invent. Math. *51*, 189–203 (1979)

Bégueri, L.
1. Dualité sur un corps local à corps résiduel algébriquement clos. Bull. Soc. Math. Fr., Suppl. Mém. (N.S.) *4* (1980)

Borel, A., and Tits, J.
1. Eléments unipotents et sous-groupes paraboliques de groupes réductifs I. Invent. Math. *12*, 95–104 (1971)

Bosch, S. and Lütkebohmert, W.
1. Stable Reduction and Uniformization of Abelian Varieties I. Math. Ann. *270*, 349–379 (1985)
2. Stable Reduction and Uniformization of Abelian Varieties II. Invent. Math. *78*, 257–297 (1984)
3. Néron models from the rigid analytic viewpoint. J. Reine Angew. Math. *364*, 69–84 (1986)

Bourbaki, N.
1. Algèbre. Chap. 1–7, 10, Hermann, Paris (1947–59), Masson, Paris (1980–81)
2. Algèbre commutative. Chap. 1–9, Hermann, Paris (1961–65), Masson, Paris (1980–85)

Chai, C.-L.
1. Compactification of Siegel moduli schemes. London Mathematical Society Lecture Note Series *107*, Cambridge University Press, Cambridge (1985)

Chevalley, C.
1. La théorie des groupes algébriques. Proc. Int. Cong. Math. 1958, Cambridge University Press, Cambridge, 53–68 (1960)

Deligne, P.
1. Le Lemme de Gabber, Séminaire sur les pinceaux arithmétiques: La conjecture de Mordell, edited by L. Szpiro. Astérisque *127*, 131–150 (1985)

Deligne, P., and Mumford, D.
1. The irreducibility of the space of curves of given genus. Publ. Math. IHES *36*, 75–110 (1969)

Deligne, P., and Rapoport, M.
1. Les schémas de modules de courbes elliptiques. Modular Functions of One Variable II. Lect. Notes Math. *349*, Springer, Berlin-Heidelberg-New York (1973)

Faltings, G.
1. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. Invent. Math. *73*, 349–366 (1983)

Greenberg, M. J.
1. Schemata over local rings. Ann. Math. *73*, 624–648 (1961)
2. Rational points in henselian discrete valuation rings. Publ. Math. IHES *31*, 59–64 (1966)

Grothendieck, A.
1. Techniques de construction en géométrie analytique. IV: Formalisme général des foncteurs représentables. Sém. Cartan 1960/61, n° *11* (1961)
2. Techniques de construction en géométrie analytique. V: Fibrés vectoriels, fibrés projectifs, fibrés en drapeaux. Sém. Cartan 1960/61, n° *12* (1961)

3. Le groupe de Brauer III. Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 88–188 (1968)

Hironaka, H.
1. An example of a non-Kählerian complex-analytic deformation of Kählerian complex structures. Ann. Math. *75*, 190–208 (1962)
2. Resolution of singularities of an algebraic variety over a field of characteristic zero (I, II). Ann. Math. *79*, 109–203, 205–326 (1964)

Igusa, J.
1. Fibre systems of Jacobian varieties I, II. Am. J. Math. *78*, 171–199 and 745–760 (1956)

Katz, N., and Mazur, B.
1. Arithmetic Moduli of Elliptic Curves. Annals of Mathematics Studies *108*, Princeton University Press, Princeton (1985)

Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.
1. Toroidal embeddings I. Lect. Notes Math. *339*, Springer, Berlin-Heidelberg-New York (1973)

Knutson, D.
1. Algebraic spaces. Lect. Notes Math. *203*, Springer, Berlin-Heidelberg-New York (1971)

Kodaira, K.
1. On compact analytic surfaces. Princeton Math. Series *24*, 121–135
2. On compact analytic surfaces II. Ann. Math. *77*, 563–626 (1963)

Koizumi, S.
1. On specialization of the Albanese and Picard varieties. Mem. College Sci. Univ. Kyoto *32*, 371–382 (1960)

Koizumi, S., and Shimura, G.
1. On specialization of abelian varieties. Sci. Papers Coll. Gen. Educ. Univ. Tokyo *9*, 187–211 (1959)

Lang, S.
1. Abelian Varieties. Interscience Publishers, New York (1959)

Lang, S., and Néron, A.
1. Rational points of abelian varieties over function fields. Am. J. Math. *81*, 95–118 (1959)

Lazard, D.
1. Autour de la platitude. Bull. Soc. Math. Fr. *97*, 81–128 (1968)

Lichtenbaum, S.
1. Curves over discrete valuation rings. Am. J. Math. *90*, 380–405 (1968)

Lipman, J.
1. Desingularization of two-dimensional schemes. Ann. Math. *107*, 151–207 (1978)

Lorenzini, D. J.
1. Degenerating curves and their Jacobians. Preprint, Berkeley (1988)

Mazur, B.
1. Modular curves and the Eisenstein ideal. Publ. Math. IHES *47*, 173–186 (1977)

Mazur, B., and Swinnerton-Dyer, P.
1. Arithmetic of Weil curves. Invent. Math. *25*, 1–61 (1974)

McCallum, W. G.
1. The component group of a Néron model. Preprint, Berkeley (1988)

Milne, J. S.
1. Etale Cohomology. Princeton Math. Series 33, Princeton University Press, Princeton (1980)
2. Arithmetic Duality Theorems. Perspectives in Math., Academic Press, London (1986)

Moret-Bailly, L.
1. Groupes de Picard et problèmes de Skolem I. Ann. Sci. Ec. Norm. Super., *22*, 161–179 (1989)

Mumford, D.
1. Geometric Invariant Theory. Ergebnisse *34*, Springer, Berlin-Heidelberg-New York (1965)
2. Lectures on Curves on an Algebraic Surface. Annals of Math. Studies *59*, Princeton University Press, Princeton (1966)
3. Abelian Varieties. Oxford University Press, Oxford (1970)

Murre, J. P.
1. On contravariant functors from the category of preschemes over a field into the category of abelian groups. Publ. Math. IHES *23*, 5–43 (1964)
2. Representation of unramified functors. Applications. (according to unpublished results of A. Grothendieck). Sém. Bourbaki 1964/65, exp. n° *294*, Benjamin, New York (1966)

Nagata, M.
1. Imbedding of an abstract variety in a complete variety. J. Math. Kyoto Univ. *2*, 1–10 (1962)
2. A generalization of the imbedding problem of an abstract variety in a complete variety. J. Math. Kyoto Univ. *3*, 89–102 (1963)

Néron, A.
1. Modèles p-minimaux des variétés abéliennes. Sém. Bourbaki 1961/62, exp. n° *227*, Benjamin, New York (1966)
2. Modèles minimaux des variétés abéliennes. Publ. Math. IHES *21* (1964)
3. Quasi-fonctions et hauteurs sur les variétés abéliennes. Ann. Math. *82*, 249–331 (1965)

Oesterlé, J.
1. Nombres de Tamagawa et groupes unipotents en caractéristique p. Invent. Math. *78*, 13–88 (1984)

Ogg, A. P.
1. Cohomology of abelian varieties over function fields. Ann. Math. *76*, 185–212 (1962)

Oort, F
1. Sur le schéma de Picard. Bull. Soc. Math. Fr. *90*, 1–14 (1962)

Peskine, Ch.
1. Une généralisation du "Main Theorem" de Zariski. Bull. Sci. Math. *90*, 119–127 (1966)

Raynaud, M.
1. Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variétés abéliennes. Sém. Bourbaki 1964/65, exp. n° *286*, Benjamin, New York (1966)
2. Modèles de Néron. C. R. Acad. Sci., Paris, Sér. A *262*, 345–347 (1966)
3. Un critère d'effectivité de descente. Sém. Algèbre commutative P. Samuel 1967/68, exp. *V* (1968)
4. Faisceaux amples sur les schémas en groupes et les espaces homogènes. Lect. Notes Math. *119*, Springer, Berlin-Heidelberg-New York (1970)
5. Anneaux locaux henséliens. Lect. Notes Math. *169*, Springer, Berlin-Heidelberg-New York (1970)
6. Spécialisation du foncteur de Picard. Publ. Math. IHES *38*, 27–76 (1970)
7. Schémas en groupes de type (p, . . . , p). Bull. Soc. Math. Fr. *102*, 241–280 (1974)

Raynaud, M., and Gruson, L.
1. Critères de platitude et de projectivité. Invent. Math. *13*, 1–89 (1971)

Rego, C. J.
1. The compactified Jacobian. Ann. Sci. Ec. Norm. Super., *13*, 211–223 (1980)

Rosenlicht, M.
1. Generalized Jacobian varieties. Ann. Math. *59*, 505–530 (1954)
2. Some basic theorems on algebraic groups. Am. J. Math. *78*, 401–443 (1956)

Rotthaus, C.
1. On the approximation property of excellent rings. Invent. Math. *88*, 39–63 (1987)

Serre, J -P.
1. Groupes Algébriques et Corps de Classes. Hermann, Paris (1959)
2. Groupes proalgébriques. Publ. Math. IHES *7* (1960)
3. Sur les corps locaux à corps résiduel algébriquement clos. Bull. Soc. Math. Fr. *89*, 105–154 (1961)
4. Corps locaux (2nd ed.). Hermann, Paris (1968)

Serre, J -P., and Tate, J.
1. Good reduction of abelian varieties. Ann. Math. *88*, 492–517 (1968)

Shafarevich, I. R.
1. Principal homogeneous spaces defined over a function field. Amer. Math. Soc. Transl. *37*, 85–114 (1964)
2. Lectures on minimal models and birational transformations of two-dimensional schemes. Tata Inst. Fund. Research *37*, Bombay (1966)

Shimura, G.
1. Reduction of algebraic varieties with respect to a discrete valuation of the basic field. Am. J. Math. *77*, 134–176 (1955)

Silverman, J. H.
1. The Arithmetic of Elliptic Curves. Grad. Texts in Math. *106*, Springer, Berlin-Heidelberg-New York (1986)

Tate, J.
1. WC-groups over p-adic fields. Sém. Bourbaki 1957/58, exp. n° *156*, Benjamin, New York (1966)
2. Algorithm for determining the Type of a Singular Fiber in an Elliptic Pencil. Modular Functions of One Variable IV. Lect. Notes Math. *476*, Springer, Berlin-Heidelberg-New York (1975)

Tits, J.
1. Lectures on algebraic groups. Yale University, New Haven (1967)

Weil, A.
1. Foundations of Algebraic Geometry. Amer. Math. Soc. Colloquium Publ. *29* (1946) (revised and enlarged edition 1962)
2. Variétés Abéliennes et Courbes Algébriques. Hermann, Paris (1948), republished in Courbes Algébriques et Variétés Abéliennes. Hermann, Paris (1971)

Subject Index