Proof. It is enough to construct an effective relative Cartier divisor D on X whose support meets only a single given component X_j of X_k. In order to do this, choose a closed point

$$x \in X_j - \bigcup_{i \neq j} X_i$$

which is regular on X; such a point exists since there are at most finitely many points where X is not regular. Using the fact that $\operatorname{prof} \mathcal{O}_{X,x} = 2$, one can find an affine open neighborhood $U = \operatorname{Spec} A$ of x such that there is a non-zero-divisor $\bar{f} \in A \otimes_R k$ which vanishes at x. Lifting $\bar{f}$ to $f \in A$, this element defines a closed subscheme $\Delta \subset U$ which we may interpret as an effective relative Cartier divisor on U. However, Δ might not be a closed subscheme of X; it can happen that its schematic closure $\bar{\Delta}$ cannot be interpreted as a relative Cartier divisor on X or that $\bar{\Delta}$ meets components C_i with $i \neq j$. So we cannot, in general expect, that Δ extends to a relative Cartier divisor on X satisfying the required properties.

But we know that $\Delta \longrightarrow \operatorname{Spec} R$ is quasi-finite. So, R being *henselian*, we can use 2.3/4 in order to obtain an open neighborhood $V \subset U$ of x such that $\Delta \cap V \longrightarrow \operatorname{Spec} R$ is finite. Then the immersion $\Delta \cap V \hookrightarrow X$ is finite, and its image is closed in X so that we may regard $\Delta \cap V$ as a relative Cartier divisor on X. The latter is of the required type. □

For the remainder of this section, we want to look at smooth and proper elliptic curves $E \subset \mathbb{P}^2_S$ (having a section) over a base scheme $S = \operatorname{Spec} A$ where $A = \mathbb{C}[\tau, \tau^{-1}]$ or $A = \mathbb{Q}[\tau, \tau^{-1}]$ and where τ is an indeterminate. So S is a Dedekind scheme; let K be its field of fractions. For $t \in \mathbb{C}^*$ (resp. $t \in \mathbb{Q}^*$), we will write t also for the closed point in S which corresponds to the ideal $(\tau - t) \subset A$. As usual, for closed points $t \in S$, the fibre of E over t is denoted by E_t.

Proposition 5. *Consider the following property of E at closed points $t \in S$:*

(P) *There exists a rational point $a_t \in E_t$ such that none of its multiples na_t, $n > 0$, (in the sense of the group law on E) lifts to an $\mathcal{O}_{S,t}$-valued point of E or, equivalently, of $E \otimes_A \mathcal{O}_{S,t}$.*

Then, if $A = \mathbb{C}[\tau, \tau^{-1}]$, and if E is a smooth and proper elliptic curve over $S = \operatorname{Spec} A$ with non-constant j-invariant, the property (P) *is true for all $t \in \mathbb{C}^*$. Furthermore, if $A = \mathbb{Q}[\tau, \tau^{-1}]$ and if $E \subset \mathbb{P}^2_S$ is given by the equation*

$$y^2 z = x^3 + \tau x z^2 ,$$

(P) *is true for some $t \in \mathbb{Q}^*$; for example, it holds for all primes $p \equiv 5 (\operatorname{mod} 8)$, where $p < 1000$.*

Proof. Let us start with the case $A = \mathbb{C}[\tau, \tau^{-1}]$. Fix a closed point $t \in S$ and set $R = \mathcal{O}_{S,t}$. Then, using the relative version of the Mordell-Weil theorem for function fields as contained in Lang and Néron [1], we see that the group $E(K)$ is finitely generated. By the valuative criteria of separatedness and of properness, the latter group is isomorphic to $E(R)$. Now let Γ be the image of $E(R)$ in $E_t(\mathbb{C})$ and let $\bar{\Gamma}$ be the subgroup of $E_t(\mathbb{C})$ consisting of all points b_t such that a multiple nb_t is contained in Γ. Then, since $E(R)$ is countable, the group $\bar{\Gamma}$ is countable. But $E_t(\mathbb{C})$ is not countable. So $E_t(\mathbb{C}) - \bar{\Gamma}$ contains a point a_t as required.

Next let us consider the case where $A = \mathbb{Q}[\tau, \tau^{-1}]$. We claim that $E(K)$ is finite. In order to justify this, we look for $t \in \mathbb{Q}^*$ at the specialization map

$$E(K) \xrightarrow{\sim} E(\mathcal{O}_{S,t}) \longrightarrow E_t(\mathbb{Q})$$

and use the following facts which we cite without proof:

(a) $E_t(\mathbb{Q})$ is finite for infinitely many $t \in \mathbb{Q}^*$; for example for all primes p with $p \equiv 7$ or $p \equiv 11 (\operatorname{mod} 16)$; cf. Silverman [1], Chap. X, 6.2 and 6.2.1.

(b) The specialization map $E(K) \longrightarrow E_t(\mathbb{Q})$ is injective for allmost all $t \in \mathbb{Q}^*$; cf. Silverman [1], Appendix C, 20.3.

(c) There exist elements $t \in \mathbb{Q}^*$ such that $E_t(\mathbb{Q})$ is of rank ≥ 1, for example for all primes $p \equiv 5 (\operatorname{mod} 8)$ less than 1000; cf. Silverman [1], Chap. X, 6.3.

It follows from (a) and from (b) that $E(K) \simeq E(\mathcal{O}_{S,t})$ is finite for all $t \in \mathbb{Q}^*$. Choosing t as in (c), one can find a rational point $a_t \in E_t(\mathbb{Q})$ which has infinite order. But then none of its multiples can admit a lifting to a point of $E(\mathcal{O}_{S,t})$. □

Now let E be a smooth and proper elliptic curve over a discrete valuation ring R such that the special fibre E_k contains a rational point a_k whose multiples na_k, $n > 0$, (in the sense of the group law on E) do not admit liftings to R-valued points of E. As we have just seen, examples of such curves do exist. By blowing up a_k in E, one obtains a proper curve X over R which is regular. Its special fibre X_k consists of the strict transform $\tilde{E}_k$ of E_k and of the inverse image of a_k which is a projective line P_k; both intersect transversally at a single point.

Lemma 6. *The strict transform $\tilde{E}_k$ of E_k under the blowing-up $X \longrightarrow E$ cannot be contracted in X. More precisely, there is no R-morphism $u : X \longrightarrow Y$ onto a proper normal R-curve Y which maps $\tilde{E}_k$ onto a point $y \in Y$ and which is an isomorphism over $Y - \{y\}$.*

Proof. Assume that such a contraction $u : X \longrightarrow Y$ exists. Then Y is regular at all its points except possibly for y, and the complement of any affine open neighborhood of y yields an effective relative Cartier divisor D on X, whose support meets P_k and is disjoint from $\tilde{E}_k$; cf. Corollary 3. Let D_k be the generic fibre of D and D' its schematic closure in E. Then D' is an effective relative Cartier divisor on E; let $d > 0$ be its degree. The support of D' is the projection of D on E; so the closed fibre D'_k is da_k. If e is the unit section of E, the invertible sheaf $\mathscr{L} = \mathcal{O}_E(D' - de)$ has degree 0 and, thus, corresponds to an element of $\operatorname{Pic}^0_{E/R}(R)$; cf. Section 9.2. Now, using the canonical isomorphism

$$E \longrightarrow \operatorname{Pic}^0_{E/R} , \qquad x \longmapsto \mathcal{O}_E(x - e) ,$$

it follows that $\mathscr{L}$ corresponds to a point $b \in E(R)$. Restricting ourselves to special fibres, we see that $b_k = da_k$. However, this contradicts the choice of $a_k \in E_k$. □

Chapter 7. Properties of Néron Models

Although the notion of a Néron model is functorial, it cannot be said that Néron models satisfy the properties, one would expect from a good functor. For example, Néron models do not, in general, commute with (ramified) based change; also, in the group scheme case, the behavior with respect to exact sequences can be very capricious. The situation stabilizes somewhat if one considers Néron models with semi-abelian reduction.

The purpose of the present chapter is to collect several properties of Néron models, and to give a number of examples which show that certain other, perhaps desirable, properties are in general not true. We prove a criterion for a smooth group scheme to be a Néron model and discuss the behavior of Néron models with respect to the formation of subgroups as well as with respect to base change and descent. Then we look at isogenies and Néron models with semi-abelian reduction. For example, we prove the criterion of Néron-Ogg-Shafarevich for good reduction. There is also a section dealing with various aspects of exactness properties. The chapter ends with a supplementary section where we explain the Weil restriction functor. If one works with respect to a finite and faithfully flat extension of Dedekind schemes $S' \longrightarrow S$, this functor respects Néron models. Furthermore, if K and K' are the rings of rational functions on S and S', the Weil restriction is used to describe the behavior of associated Néron models if one descends from a K'-group scheme $X_{K'}$ to a K-group scheme X_K.

7.1 A Criterion

Throughout this section we will denote by R a discrete valuation ring, by R^{sh} its strict henselization, and by K and K^{sh} the corresponding fields of fractions. Furthermore, k is the residue field of R, and k_s its separable algebraic closure. In the following we will consider R-group schemes G of finite type with a smooth generic fibre and with the property that each K^{sh}-valued point of G extends to an R^{sh}-valued point of G. We are interested in conditions under which G is a Néron model of its generic fibre G_K or, more generally, in the way of deriving a Néron model of G_K from G.

Theorem 1. *Let G be a smooth R-group scheme of finite type or a torsor under a smooth R-group scheme of finite type. Then the following conditions are equivalent:*

(i) *G is a Néron model of its generic fibre G_K.*

(ii) *G is separated and the canonical map $G(R^{sh}) \longrightarrow G(K^{sh})$ is surjective.*

(iii) *The canonical map $G(R^{sh}) \longrightarrow G(K^{sh})$ is bijective.*

Proof. It is enough to consider the case where G is a group scheme. Indeed, if G is a torsor we may assume by 6.5/3 that R is strictly henselian and, furthermore, that G is unramified. Then G admits a section over R and we can view G as a group scheme.

In the following, let us assume that G is a group scheme. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are trivial, the second one by the valuative criterion of separatedness. Moreover, it is easy to see that condition (ii) implies condition (i). Namely, if G satisfies (ii), it is a weak Néron model of its generic fibre G_K. Hence the weak Néron property 3.5/3 and the extension theorem 4.4/1 show that G satisfies the definition of Néron models.

Turning to the remaining implication (iii) $\Longrightarrow$ (ii), we have to verify that (iii) implies the separatedness of G. Using Lemma 2 below, it is only to show that the unit section $\varepsilon : \operatorname{Spec} R \longrightarrow G$ is a closed immersion or, what amounts to the same, that $\operatorname{im} \varepsilon$ is closed in G. Restricting ε to generic fibres, we know that $\varepsilon_K : \operatorname{Spec} K \longrightarrow G_K$ is a closed immersion. Let F be the schematic image of ε_K in G. Then, pointwise, $\operatorname{im} \varepsilon$ and F coincide on G_K, and we have to show the same for the special fibre G_k of G. So consider a point $e_k \in F \cap G_k$. Working in an affine open neighborhood $U \subset G$ of e_k, let A be the ring of global sections on $F \cap U$. Then $R \subset A \subset K$ and, thus, $R = A$ since R is a discrete valuation ring. Hence the inclusion of $F \cap U$ into G gives rise to a point $e \in G(R)$ extending $\varepsilon_K \in G(K)$. However, condition (iii) implies $e = \varepsilon$. So F consists of only two points, namely, the points of $\operatorname{im} \varepsilon$, and it follows that $\operatorname{im} \varepsilon$ is closed in G. □

Lemma 2. *A group scheme G is separated over a base scheme S if and only if the unit section ε is a closed immersion.*

Proof. If G is separated, the diagonal morphism $\delta : G \longrightarrow G \times_S G$ is a closed immersion. Then the same is true for the unit section $\varepsilon : S \longrightarrow G = S \times_S G$, since ε is obtained from δ by means of the base change $\varepsilon : S \longrightarrow G$.

Conversely, viewing the diagonal in $G \times_S G$ as the inverse image of $\operatorname{im} \varepsilon$ with respect to the morphism

$$G \times_S G \longrightarrow G\,, \qquad (g,h) \longmapsto g \cdot h^{-1}\,,$$

it follows that G is separated if ε is a closed immersion. □

In order to demonstrate how Theorem 1 can be applied, let us give an example of an algebraic K-group which, although it is affine, admits a Néron model.

Example 3. Let R be a discrete valuation ring of equal characteristic $p > 0$, and let π be a uniformizing element of R. Consider the subgroup G of $\mathbb{G}_{a,R} \times_R \mathbb{G}_{a,R}$ which is given by the equation

$$x + x^p + \pi y^p = 0\,.$$

Then G is a smooth R-group scheme of finite type. Furthermore, looking at values of solutions of the above equation, one shows easily that the map $G(R^{sh}) \longrightarrow G(K^{sh})$ is surjective. Thus G is a Néron model of its generic fibre G_K. The group G_K is an example of a so-called K-wound unipotent group; i.e., of a connected unipotent algebraic K-group which does not contain $\mathbb{G}_{a,K}$ as a subgroup. Smooth commutative groups of this kind admit Néron models of finite type, at least in the case where R is excellent; cf. 10.2/1.

Next consider an R-group scheme G of finite type such that the generic fibre G_K is smooth. If the residue characteristic of R is zero, the special fibre G_k is smooth by Cartier's theorem, [SGA 3_{I}], Exp. $\mathrm{VI_B}$, 1.6.1, so that, if G is flat, it will be smooth over R. However, since the latter result does not extend to the general case, we want to describe a procedure which, by means of the smoothening process, associates a smooth R-group scheme G' to G such that the canonical map $G'(R^{sh}) \longrightarrow G(R^{sh})$ is bijective. Let us call a morphism of R-group schemes $G' \longrightarrow G$, where G' is smooth and of finite type over R, a *group smoothening* of G if each R-morphism $Z \longrightarrow G$ from a smooth R-scheme Z admits a unique factorization through G'. Then, by the defining universal property, $G' \longrightarrow G$ is an isomorphism on generic fibres since G_K is smooth. In particular, if $G(R^{sh}) \longrightarrow G(K^{sh})$ is bijective, G' will be a Néron model of G_K by Theorem 1. Group smoothenings can be defined in the same way using a global Dedekind scheme as base. However, their existence can only be guaranteed in the local case; cf. Theorem 5 below.

Lemma 4. *Let G be an R-group scheme of finite type which has a smooth generic fibre. Denote by F_k the Zariski closure in G_k of the set of k_s-valued points in G_k which lift to R^{sh}-valued points of G. Then F_k, provided with its canonical reduced structure, is a closed subgroup scheme of G_k. Furthermore, let $u: Y \longrightarrow G$ be the dilatation of F_k in G. Using the notation δ for the defect of smoothness as in 3.3, we have*

$$\delta(a') \le \max\{0, \delta(a) - 1\}$$

for each R^{sh}-valued point a of G and its lifting a' to Y.

Proof. Since the set of R^{sh}-valued points of G forms a group, it is clear that F_k is a subgroup scheme of G_k. In order to justify the second assertion, we use Lemma 3.4/1; it is only to show that $F_k \subset G_k$ is E-permissible, where $E = G(R^{sh})$. However this is clear. By construction, F_k is geometrically reduced and, hence, smooth over k, being a group scheme of finite type over a field. Furthermore, using 4.2/2, we see that the restriction of the sheaf of differentials $\Omega^1_{G/R}$ to G_k is free and, hence, that the restriction of $\Omega^1_{G/R}$ to F_k is free. Thus the two conditions characterizing E-permissibility are satisfied. □

It follows from 3.2/2(d) that the scheme Y of Lemma 4 is an R-group scheme again and that $u: Y \longrightarrow G$ is a group homomorphism. So a finite repetition of the construction leads to an R-group scheme G' which has generic fibre G_K and defect of smoothness 0, and thus is smooth at all its R^{sh}-valued points. In particular, G' is smooth at the unit section and therefore smooth everywhere since it is flat. We claim that the morphism $G' \longrightarrow G$ is a group smoothening of G. To justify this, consider

an R-morphism $Z \longrightarrow G$ where Z is a smooth R-scheme. Writing k_s for the separable algebraic closure of k, the set of k_s-valued points of Z_k which lift to R^{sh}-valued points of Z is schematically dense in Z_k; cf. 2.3/5. Thus, we see that, in the situation of Lemma 4, the special fibre of Z is mapped into F_k. Then the desired factorization of $Z \longrightarrow G$ follows from 3.2/1(b), again. So we have derived the following facts on group smoothenings.

Theorem 5. *Let G be an R-group scheme of finite type with a smooth generic fibre G_K. Then there exists a group smoothening $G' \longrightarrow G$ of G. Due to its definition, G' is smooth and of finite type; it is characterized by the property that each R-morphism $Z \longrightarrow G$, where Z is smooth over R, factors uniquely through G'.*

Furthermore, if the map $G(R^{sh}) \longrightarrow G(K^{sh})$ is surjective and if G is separated, G' is a Néron model of G_K.

Proof. Only the assertion concerning the Néron model remains to be verified. If $G(R^{sh}) \longrightarrow G(K^{sh})$ is surjective and if G is separated, the same is true for $G'(R^{sh}) \longrightarrow G'(K^{sh})$ and G'. Thus G' is a Néron model of G_K by the criterion given in Theorem 1. □

As an application we want to examine how the Néron model G of a K-group scheme G_K behaves if we pass from G_K to a subgroup $H_K \subset G_K$.

Corollary 6. *Let S be a Dedekind scheme with ring of rational functions K. Furthermore, let G be an S-group scheme which is a Néron model of its scheme of generic fibres G_K, and let H_K be a smooth subgroup of G_K. Then H_K admits a Néron model H over S; more precisely, one can define H as a group smoothening of the schematic closure $\bar{H}$ of H_K in G. The schematic closure $\bar{H}$ itself is a Néron model of H_K if and only if it is smooth. In particular, the latter is the case if $\operatorname{char} k(s) = 0$ for all closed points $s \in S$.*

Proof. First, let us show that there exists a group smoothening of $\bar{H}$ over S. Since H_K is smooth, its schematic closure $\bar{H}$ is smooth over a dense open part S' of S. On the other hand, we know from Theorem 5 that, for each of the finitely many points $s \in S' - S$, the group scheme $\bar{H} \otimes_S \mathcal{O}_{S,s}$ admits a group smoothening. Then, similarly as explained in the proof of 1.4/1, we can glue $\bar{H} \otimes_S \mathcal{O}_{S,s}$ for $s \in S - S'$ to $\bar{H} \times_S S'$, thereby obtaining a global group smoothening H of $\bar{H}$ over S.

It remains to show that H is a Néron model of H_K. To do so, we may assume that S is local. Consider a smooth S-scheme Z and a K-morphism $Z_K \longrightarrow H_K$. Then, since $H_K \subset G_K$ and since G is a Néron model of G_K, this morphism extends uniquely to an S-morphism $Z \longrightarrow G$ which, by the definition of $\bar{H}$, must factor through $\bar{H}$. Furthermore, we conclude from Theorem 5 that $Z \longrightarrow \bar{H}$ extends uniquely to an R-morphism $Z \longrightarrow H$. The latter is unique as an extension of $Z_K \longrightarrow H_K$. So H is a Néron model of G_K and the remaining assertions are clear since $\bar{H}$ is flat over S. □

7.2 Base Change and Descent

One cannot expect that, for a faithfully flat extension of discrete valuation rings $R \subset R'$, the base change $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$ transforms Néron models over R into Néron models over R'. In Example 7.1/3 of the preceding section we can see that, after adjoining a p-th root of the uniformizing element π of R to K, the boundedness of $G_K(K^{sh})$ and, hence, the existence of a Néron model of G_K is lost, since G_K becomes isomorphic to the additive group $\mathbb{G}_{a,K}$. On the other hand, it follows from 1.2/2 and 6.5/3 that Néron models behave well with respect to étale base change. The latter is true for a more general class of morphisms as we will see in this section (cf. 6.5/5 for a partial result of this type).

Consider a faithfully flat extension $R \subset R'$ of discrete valuation rings with fields of fractions K and K'. As usual we indicate strict henselizations by an exponent "sh" and we may assume that R^{sh} is a subring of R'^{sh}. Recall that R' is said to have ramification index 1 over R if a uniformizing element of R gives rise to a uniformizing element of R' and if the residue extension of R'/R is separable (cf. 3.6/1).

Theorem 1. *Let $R \subset R'$ and $K \subset K'$ be as above and consider a torsor X_K under a smooth K-group scheme G_K of finite type. Denote by $X_{K'}$ the torsor under $G_{K'}$ obtained by base change with K'.*

(i) *Assume that $X_{K'}$ admits a Néron model X' over R'. Then X_K admits a Néron model X over R, and there is a canonical R'-morphism $X \otimes_R R' \longrightarrow X'$, called morphism of base change.*

(ii) *Let R'/R be of ramification index 1. Then X_K admits a Néron model X over R if and only if $X_{K'}$ admits a Néron model X' over R'. If the latter is the case, the morphism of base change $X \otimes_R R' \longrightarrow X'$ is an isomorphism.*

Proof. If $X_{K'}$ admits a Néron model, $X_{K'}(K'^{sh})$ is bounded in $X_{K'}$. Using 1.1/5, we see that $X_K(K'^{sh})$ is bounded in X_K. But then $X_K(K^{sh})$ is bounded in X_K and a Néron model X of X_K exists by 6.5/4. Since $X \otimes_R R'$ is a smooth R'-model of $X_{K'}$, the identity on $X_{K'}$ extends to an R'-morphism $X \otimes_R R' \longrightarrow X'$ as required in assertion (i).

In the situation of assertion (ii) we have only to consider the case where X_K has a Néron model X. Furthermore, since Néron models are compatible with étale base change, we may assume that R and R' are strictly henselian. It has to be shown that $X \otimes_R R'$ is a Néron model of $X_{K'}$. To do this, it is enough to look at the case where the torsor $X_{K'}$ is unramified. So consider a K'-valued point of $X_{K'}$. Interpreting it as a point $a_K \in X_K(K')$ and working in an affine open neighborhood of its image in X_K, we can find an R-model $\tilde{X}$ of X_K of finite type such that a_K extends to a point $a \in \tilde{X}(R')$. Due to 3.6/4, we may assume that $\tilde{X}$ is smooth. But then, since X is a Néron model of X_K, we have a morphism $\tilde{X} \longrightarrow X$. Thus each $a_K \in X_K(K')$ extends to a point $a \in X(R')$ and, consequently, the canonical map $(X \otimes_R R')(R') \longrightarrow (X \otimes_R R')(K')$ is surjective. So $X \otimes_R R'$ is a Néron model of $X_{K'}$ by 7.1/1. □

It will be of interest in 10.1/3 that the argument for showing that $X \otimes_R R'$ is a Néron model of $X_{K'}$ can be changed slightly so that the use of 7.1/1 can be avoided. Namely, look at a discrete valuation ring R'' which is of ramification index 1 over R'. Then R'' has ramification index 1 also over R and, if K'' is the field of fractions of R'', the above given argument shows that the map $X(R'') \longrightarrow X(K'')$ is surjective. In particular, taking for R'' the local ring of a smooth R'-scheme Z' at a generic point of the special fibre Z'_k, we see that $X \otimes_R R'$ satisfies the weak Néron property. So if $X_{K'}$ is unramified, we may view $X \otimes_R R'$ as an R'-group scheme, which satisfies the Néron mapping property by the extension argument 4.4/1 for morphisms into group schemes. Thus $X \otimes_R R'$ is a Néron model of $X_{K'}$ in this case.

Corollary 2. *Over discrete valuation rings, the formation of Néron models (of torsors or group schemes) is compatible with extensions R'/R of ramification index 1. For example, R' can be the completion of R.*

Giving another application of Theorem 1, we show that the Néron mapping property can be strengthened.

Proposition 3. *Let X_K be a K-torsor under a smooth K-group scheme G_K of finite type, and assume that a Néron model X of X_K exists. Let A be an R-algebra of type $R\{t\}$ or $R[[t]]$ (strictly convergent or formal power series in a system of variables $t = (t_1, \ldots, t_n)$) where R is complete. Then each K-morphism*

$$u_K : \operatorname{Spec}(A \otimes_R K) \longrightarrow X_K$$

extends uniquely to an R-morphism $u : \operatorname{Spec} A \longrightarrow X$.

Proof. Let η be the generic point of the special fibre $\operatorname{Spec}(A \otimes_R k)$ of $\operatorname{Spec} A$. Then A_η is a discrete valuation ring which is of ramification index 1 over R. Writing F for the field of fractions of A_η, we see that u_K gives rise to an F-morphism $\operatorname{Spec} F \longrightarrow X_K \otimes_K F$. Applying Theorem 1, this morphism extends to an A_η-morphism $\operatorname{Spec} A_\eta \longrightarrow X \otimes_R A_\eta$ and, hence, to an R-rational map $u : \operatorname{Spec} A \dashrightarrow X$. In particular, the special fibre X_k is not empty and, thus, X_K cannot be a ramified torsor. We claim that u is a morphism. Then u extends u_K, and it is unique since X is separated.

If $X(R) \neq \emptyset$, we may view X as an R-group scheme, and one can conclude from Remark 4.4/3 that the R-rational map u is a morphism. In the general case, we choose a discrete valuation ring R' which is finite and étale over R and which satisfies the property that $X(R') \neq \emptyset$. The latter is possible since the torsor X_K is unramified. Set $A' = R'\{t\}$ or $A' = R'[[t]]$ depending on the type of power series we consider for A; note that R' is complete. Then it follows from the above special case that the composition of morphisms

$$\operatorname{Spec}(A' \otimes_R K) \xrightarrow{\text{pr}} \operatorname{Spec}(A \otimes_R K) \xrightarrow{u_K} X_K ,$$

where pr is the canonical projection, extends to an R-morphism $u' : \operatorname{Spec} A' \longrightarrow X$. In other words, the composition of the projection $\operatorname{Spec} A' \longrightarrow \operatorname{Spec} A$ with the R-rational map $u : \operatorname{Spec} A \dashrightarrow X$ is a morphism. But then, by 2.5/5, u is defined everywhere and, thus, is a morphism. □

Using the technique of Weil restriction to be explained in Section 7.6, one can describe in a precise way how, in the situation of Theorem 1 (i) and under the assumption that the extension of discrete valuation rings $R \subset R'$ is *finite*, a Néron model X of X_K can be constructed from a Néron model X' of $X_{K'}$, at least in the case of group schemes.

Proposition 4. *Let $S' \longrightarrow S$ be a flat and finite morphism of Dedekind schemes with rings of rational functions K and K'. Let G_K be a smooth K-group scheme of finite type and denote by $G_{K'}$ the K'-group scheme obtained from G_K by base change. Assume that the Néron model G' of $G_{K'}$ exists over S'. Then the Néron model G of G_K exists over S and can be constructed as a group smoothening of the schematic closure of G_K in the Weil restriction $\mathfrak{R}_{S'/S}(G')$.*

Proof. Using 7.6/6, we see that the Weil restriction $\mathfrak{R}_{S'/S}(G')$ exists as a scheme and that it is a Néron model of its scheme of generic fibres, i.e. of $\mathfrak{R}_{K'/K}(G'_{K'})$. Thus, considering the canonical closed immersion

$$\imath : G_K \longrightarrow \mathfrak{R}_{K'/K}(G_{K'}),$$

the assertion follows from 7.1/6. $\square$

7.3 Isogenies

We want to investigate under what conditions an isogeny $G_K \longrightarrow G'_K$ between smooth and connected K-group schemes extends to an isogeny between associated Néron models. In order to attack this problem, we begin by recalling some well-known facts about homomorphisms between group schemes over a field k.

Lemma 1. *Let $f : G \longrightarrow G'$ be a homomorphism of group schemes which are smooth and of finite type over a field k. Assume that $\dim G = \dim G'$. Then the following conditions are equivalent:*
 (a) *f is flat.*
 (b) *$f(G^0) = G'^0$ where G^0 and G'^0 denote identity components of G and G'.*
 (c) *$\ker f$ is finite.*
 (d) *f is quasi-finite.*
 (e) *f is finite.*

A commutative group scheme G which is smooth and of finite type over a field k is called *semi-abelian* if its identity component G^0 is an extension of an abelian variety by a (not necessarily deployed) affine torus. The latter fact can be checked over the algebraic closure $\bar{k}$ of k. Indeed, one knows from Chevalley's theorem 9.2/1 that $G^0_{\bar{k}}$ is uniquely an extension of an abelian variety by a connected affine group $H_{\bar{k}}$. Then $H_{\bar{k}}$ decomposes into the product of a torus part and a unipotent part, where the torus part is already defined over k; cf. [SGA 3_{II}], Exp. XIV, 1.1. So we see that G is semi-abelian if and only if the unipotent part of $H_{\bar{k}}$ is trivial. Over a general base scheme S, an S-group scheme G is called *semi-abelian* if it is smooth over S and if all its fibres are semi-abelian in the sense explained above.

Lemma 2. *Let G be a commutative S-group scheme which is smooth and of finite type over an arbitrary base scheme S. Let l be a positive integer.*
 (a) *Suppose that G is semi-abelian. Then the l-multiplication $l_G : G \longrightarrow G$ is quasi-finite and flat.*
 (b) *Suppose that* char $k(s)$ *does not divide l for all $s \in S$. Then the l-multiplication $l_G : G \longrightarrow G$ is étale.*

Proof. In order to verify the flatness of l_G in the situation (a) or (b), we can use the characterization of flatness in terms of fibres 2.4/2. So we may assume that S consists of a field k. Then, since l_G is surjective on abelian varieties and on tori, and in the situation (b), also on unipotent groups, it follows from the structure of commutative smooth and connected group schemes over k that $G^0 \subset \operatorname{im} l_G$. By Lemma 1 we see that l_G is quasi-finite and flat.

In the situation of assertion (b) we have just seen that l_G is flat. So we may use the criterion 2.4/8. Thus, just as before, we can assume that S consists of a field k. Then we can consider the Lie algebra $\mathrm{Lie}(G)$ and the endomorphism $\mathrm{Lie}(l_G) : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G)$ induced on it by l_G. Since $\mathrm{Lie}(l_G)$ is just the multiplication by l and since l is not divisible by char k, we see that it is bijective. So $l_G : G \longrightarrow G$ is étale by 2.2/10. $\square$

For an S-group scheme G as in Lemma 2, we write ${}_lG$ for the kernel of the l-multiplication $l_G : G \longrightarrow G$. If char $k(s)$ does not divide l for all $s \in S$, we deduce from Lemma 2 that ${}_lG$, being the fibre of l_G over the unit section, is étale over S, whereas in the situation of Lemma 2 (a) we only know that ${}_lG$ is quasi-finite and flat over S.

In general, an S-group scheme H of finite type which is quasi-finite over S is not finite over S unless S consists of a field. However, if S is the spectrum of a *henselian* discrete valuation ring R and if H is quasi-finite and separated, one can consider its *finite part* H'. The latter is the open and closed subscheme of H consisting of the special fibre H_k and of all points of the generic fibre H_K which specialize into points of H_k. Namely, applying 2.3/4, one shows that H is the disjoint sum of two open and closed subschemes H' and H'', where H' is finite over S and where the special fibre of H'' is empty. The finite part H' of H is an open subgroup scheme of H.

Proposition 3. *Let R be a discrete valuation ring and let l be a positive integer such that the residue characteristic of R does not divide l. Then, for any smooth commutative R-group scheme G of finite type, the canonical map ${}_lG(R^{sh}) \longrightarrow {}_lG(k_s)$ is bijective, where R^{sh} is a strict henselization of R and where k_s is the residue field of R^{sh}.*

Proof. We may assume that R is strictly henselian. Since ${}_lG$ is étale over R by Lemma 2, its finite part is a disjoint union of copies of $S = \operatorname{Spec} R$; cf. 2.3/1. $\square$

Definition 4. *Let $f: G \longrightarrow G'$ be a homomorphism of commutative group schemes of finite type over an arbitrary base scheme S. Then f is called an isogeny if, for each $s \in S$, the homomorphism $f_s: G_s \longrightarrow G'_s$ is an isogeny in the classical sense; i.e., if f_s is finite and surjective on identity components.*

Examples of isogenies are provided by l-multiplications on commutative group schemes G where l and G have to be chosen as required in Lemma 2 (a) or (b). In the situation of the definition, each f_s has a degree $\deg f_s$, which can be defined as the rank of the finite $k(s)$-group scheme $\ker f_s$. Recalling some facts on commutative finite group schemes H over a field k, we mention that H is étale if $\operatorname{char} k = 0$ (by Cartier's theorem) or, more generally, if $\operatorname{char} k$ does not divide the rank of H. If H is connected, its rank is a power of char k. Furthermore, the l-multiplication $l_H : H \longrightarrow H$ is the zero-homomorphism if l is a multiple of the rank of H.

We need a well-known result relating isogenies over fields to l-multiplications.

Lemma 5. *Let $f: G \longrightarrow G'$ be an isogeny between smooth and connected commutative group schemes of finite type over a field k. Assume either that* char k *does not divide* $\deg f$ *or that G is semi-abelian. Then there is an isogeny $g : G' \longrightarrow G$ such that $g \circ f = l_G$ where $l = \deg f$.*

Proof. Setting $l = \deg f$, we see that $\ker f \subset \ker l_G$. Then, f being flat and surjective, we have $G' = G/\ker f$ and, thus, homomorphisms

$$G \xrightarrow{f} G' \longrightarrow G/\ker l_G \, .$$

Since the l-multiplication $l_G : G \longrightarrow G$ is finite by Lemma 2, and since l_G factors through $G/\ker l_G$, the existence of g is clear. □

Now, working over a discrete valuation ring R and its field of fractions K, we can deal with the question of whether a homomorphism between R-group schemes is an isogeny as soon as it is an isogeny on generic fibres.

Proposition 6. *Let G_K and G'_K be smooth commutative and connected K-group schemes of finite type admitting Néron models G and G' over R. Consider an isogeny $f_K : G_K \longrightarrow G'_K$ and assume either that the residue characteristic of R does not divide* $\deg f_K$ *or that G is semi-abelian. Then f_K extends to an isogeny $f: G \longrightarrow G'$, and there is an isogeny $g : G' \longrightarrow G$ such that $g \circ f = l_G$ for $l = \deg f_K$.*

Proof. Using Lemma 5, there is an isogeny $g_K : G'_K \longrightarrow G_K$ satisfying $g_K \circ f_K = l_{G_K}$ for $l = \deg f_K$. Due to the Néron mapping property, f_K and g_K extend to homomorphisms $f: G \longrightarrow G'$ and $g : G' \longrightarrow G$ such that $g \circ f = l_G$. Then, by our assumptions on $l = \deg f_K$ or on G, we see from Lemma 2 that l_G is an isogeny, and it follows easily that f and g are isogenies. □

Corollary 7. *Let $f_K : G_K \longrightarrow G'_K$ be an isogeny of abelian varieties with Néron models G and G'. Then G is semi-abelian if and only if G' is semi-abelian.*

Proof. By the Néron mapping property, the isogeny f_K extends to a homomorphism $f: G \longrightarrow G'$. If G is semi-abelian, f is an isogeny by Proposition 6 and, consequently, G' is semi-abelian. Using an isogeny $g_K : G'_K \longrightarrow G_K$, one shows in the same way that G is semi-abelian if G' is semi-abelian. □

7.4 Semi-Abelian Reduction

Let G be a smooth group scheme of finite type over a Dedekind scheme S which, for simplicity, we will assume to be connected. We say that G has *abelian reduction* (resp. *semi-abelian reduction*) at a closed point $s \in S$ if the identity component G_s^0 is an abelian variety (resp. an extension of an abelian variety by an affine torus). In particular, if G is a Néron model of its generic fibre G_K, where K is the field of fractions of S, we will say that G_K has abelian (resp. semi-abelian) reduction at $s \in S$ if the corresponding fact is true for G. The latter amounts to the same as saying that the local Néron model $G \times_S \operatorname{Spec} \mathcal{O}_{S,s}$ of G_K at $s \in S$ has abelian (resp. semi-abelian) reduction.

If A_K is an abelian variety over K, then A_K is said to have *potential abelian reduction* (resp. *potential semi-abelian reduction*) at a closed point $s \in S$ if there is a finite Galois extension L of K such that A_L has abelian (resp. semi-abelian) reduction at all points over s. To be precise, we thereby mean that the Néron model A' of A_L over the normalization S' of S in L has abelian (resp. semi-abelian) reduction at all closed points $s' \in S'$ lying over s. Instead of abelian reduction, we will also talk about *good reduction.* Let us begin by mentioning the fundamental theorem on the potential semi-abelian reduction of abelian varieties.

Theorem 1. *Each abelian variety A_K over K has potential semi-abelian reduction at all closed points of S.*

The easiest way to obtain this result is via the potential semi-stable reduction of curves, as proved by Artin and Winters [1], a topic which is beyond the scope of the present book. So we will restrict ourselves to briefly indicating how the assertion of the theorem can be deduced from the corresponding results on curves.

Since abelian varieties have good reduction almost everywhere, see 1.4/3, the problem is a local one, and we may assume that S consists of a discrete valuation ring R. One starts with the case where A_K is the Jacobian $J_K = \operatorname{Pic}^0_{C_K/K}$ of a smooth and proper K-curve C_K. Then the theorem on the potential semi-stable reduction of curves asserts that, replacing K by a finite separable extension if necessary, we can extend C_K into a proper flat R-curve C whose geometric fibres have at most ordinary double points as singularities; cf. 9.2/7. For such a curve it is shown in 9.4/1 that the relative Jacobian $\operatorname{Pic}^0_{C/S}$ is a smooth and separated R-group scheme having semi-abelian reduction. Since $\operatorname{Pic}^0_{C/S}$ is an S-model of J_K, it follows from Proposition 3 below or from the more general discussion of the relationship between

Néron models and the relative Picard functor in 9.5/4 or 9.7/2 that $\mathrm{Pic}^0_{C/S}$ is the identity component of the Néron model of J_K. Thus J_K has semi-abelian reduction.

If A_K is a general abelian variety, one knows, see Serre [1], Chap. VII, § 2, n°13, that there is an exact sequence of abelian varieties

$$0 \longrightarrow A'_K \longrightarrow J_K \longrightarrow A_K \longrightarrow 0$$

where J_K is a product of Jacobians. Using the fact that J_K has potential semi-abelian reduction, it follows from the lemma below that A_K has potential semi-abelian reduction also. □

Lemma 2. *Let $0 \longrightarrow A'_K \longrightarrow A_K \longrightarrow A''_K \longrightarrow 0$ be an exact sequence of abelian varieties over K. Then A_K has semi-abelian (resp. abelian) reduction if and only if A'_K and A''_K have semi-abelian (resp. abelian) reduction.*

Proof. Due to Poincaré's complete reducibility theorem, see Mumford [3], Chap. IV, § 19, Thm. 1, there is an abelian subvariety $\tilde{A}''_K$ in A_K such that the canonical map $\tilde{A}''_K \times A'_K \longrightarrow A_K$ and, thus, also the composition $\tilde{A}''_K \longrightarrow A_K \longrightarrow A''_K$ are isogenies. So we see that A_K is isogenous to $A'_K \times A''_K$ and it follows from 7.3/7 that A_K has semi-abelian reduction if and only if the same is true for A'_K and A''_K. An application of 7.3/6 settles the case of abelian reduction. □

For the remainder of this section, let us assume that the base scheme S consists of a *discrete valuation ring* R with field of fractions K. We want to discuss properties of Néron models with abelian or semi-abelian reduction and to give criteria for the existence of Néron models with abelian or semi-abelian reduction over the given field K.

Proposition 3. *Let A_K be an abelian variety with Néron model A and let G be a smooth and separated R-group scheme which is an R-model of A_K. Assume that G has semi-abelian reduction. Then the canonical morphism $G \longrightarrow A$ is an open immersion; it is an isomorphism on identity components.*

Proof. We can assume that R is strictly henselian. Furthermore, it is enough to show that $G^0 \longrightarrow A^0$ is an isomorphism. So assume that $G = G^0$. Let l be a positive integer which is not divisible by the characteristic of the residue field k of R. Considering the kernels ${}_lG$ and ${}_lA$ of l-multiplications on G and A, we have a canonical commutative diagram

$$\begin{tikzcd}
{}_lG(K) \arrow[r, "\sim"] & {}_lA(K) \\
{}_lG(R) \arrow[r] \arrow[u, hook] \arrow[d, "\wr"] & {}_lA(R) \arrow[u, "\wr"'] \arrow[d, "\wr"'] \\
{}_lG(k) \arrow[r] & {}_lA(k)
\end{tikzcd}$$

where ${}_lG(R) \longrightarrow {}_lG(K)$ is injective since G is separated and where all other vertical maps are bijective; the upper one on the right-hand side because A is a Néron model of A_K and the lower ones by 7.3/3. So the middle horizontal map is injective, and the same is true for the lower horizontal one. Now, using the facts that G has semi-abelian reduction and that k is separably closed, it follows that the points in $G(k)$ which have finite order not divisible by char k are topologically dense in each connected subgroup of G_k. Therefore $G_k \longrightarrow A^0_k$ has a finite kernel. In particular, $G \longrightarrow A^0$ is quasi-finite and, thus, surjective by reasons of dimension. But then Zariski's Main Theorem 2.3/2′ shows that $G \longrightarrow A^0$ is an isomorphism. □

Corollary 4. *If an abelian variety A_K has semi-abelian reduction, then the formation of the identity component of the Néron model of A_K is compatible with faithfully flat extensions of discrete valuation rings R'/R.*

We have seen above that points of finite order play an important role when dealing with Néron models of abelian varieties. We want to use them in order to give a criterion for the existence of abelian or semi-abelian reductions over the given field K. As before, R will be a discrete valuation ring with field of fractions K and with residue field k. Let K_s be a separable algebraic closure of K and consider rings $R \subset R^h \subset R^{sh} \subset R_s \subset K_s$ where R^h is a henselization of R, where R^{sh} is a strict henselization of R, and where R_s is the localization of the integral closure of R in K_s at a maximal ideal lying over the maximal ideal of R^{sh}. As usual K^h and K^{sh} denote the fields of fractions of R^h and of R^{sh}. Then the inertia group of the maximal ideal of R_s coincides with the Galois group $\mathrm{Gal}(K_s/K^{sh})$; cf. 2.3/11. Fixing the above situation, we will call $I := \mathrm{Gal}(K_s/K^{sh})$ "the" inertia group of $\mathrm{Gal}(K_s/K)$.

Theorem 5. *Let A_K be an abelian variety over K with Néron model A over R, and let l be a prime different from char k. Then the following conditions are equivalent:*

(a) *A_K has abelian reduction; i.e., the identity component A^0_k is an abelian variety over k.*

(b) *A is an abelian scheme over R.*

(c) *For each $\nu \geq 0$ the inertia group I of* $\mathrm{Gal}(K_s/K)$ *acts trivially on ${}_{l^\nu}A_K(K_s)$, the set of K_s-valued points of the kernel of the l^ν-multiplication $l^\nu_{A_K}: A_K \longrightarrow A_K$. In other words, the canonical map ${}_{l^\nu}A_K(K^{sh}) \longrightarrow {}_{l^\nu}A_K(K_s)$ is bijective.*

(d) *The Tate module $T_l(A_K) = \varprojlim {}_{l^\nu}A_K(K_s)$ is unramified over R; i.e., the inertia group I of* $\mathrm{Gal}(K_s/K)$ *operates trivially on $T_l(A_K)$.*

Proof. We begin by showing that conditions (a) and (b) are equivalent. If A^0_k is an abelian variety, we can conclude from [EGA IV$_3$], 15.7.10, that A^0 is proper over R and, thus, is an abelian scheme over R. But then A^0 is a Néron model of its generic fibre by 1.2/8; thus, $A = A^0$. This verifies the implication (a) $\Longrightarrow$ (b); the converse is trivial.

The equivalence of (c) and (d) is clear. In order to verify the remaining implications, consider the canonical maps

$$(*) \qquad {}_{l^\nu}A(K_s) \hookleftarrow {}_{l^\nu}A(K^{sh}) \xleftarrow{\sim} {}_{l^\nu}A(R^{sh}) \xrightarrow{\sim} {}_{l^\nu}A(k_s)$$

where k_s is the residue field of R^{sh} and where the map on the right-hand side is bijective by 7.3/3. If A is an abelian scheme over R, the cardinality of both sets ${}_{l^\nu}A(K_s)$ and ${}_{l^\nu}A(k_s)$ is $l^{\nu \cdot 2n}$ where n is the dimension of A; cf. Mumford [3], p. 64. Therefore, all maps in $(*)$ are bijective and we see that (b) implies (c).

Conversely, assume that all maps in $(*)$ are bijective. Then the cardinality of ${}_{l^\nu}A(k_s)$ is $l^{\nu \cdot 2n}$ for each $\nu \geq 0$, and it follows from the structure of commutative group schemes of finite type (over an algebraically closed or perfect field k) that the identity component A_k^0 is an abelian variety. So we see that condition (c) implies condition (a). □

The equivalence of (a) and (d) in the above theorem is called the criterion of Néron-Ogg-Shafarevich for good reduction. To apply it, one may work over a strictly henselian base ring R. Then A_K has abelian reduction if and only if all l^ν-torsion points of A_K are rational over K. The criterion can be generalized to the semi-abelian reduction case; see [SGA 7_I], Exp. IX, 3.5. We include this generalization here without proof.

Theorem 6. *Let A_K be an abelian variety over K, and let l be a prime different from* char k. *Then the following conditions are equivalent*:

(a) *A_K has semi-abelian reduction over R.*

(b) *There is a submodule $T' \subset T := T_l(A_K(K_s))$ which is stable under the action of the inertia group I of* $\mathrm{Gal}(K_s/K)$ *such that I acts trivially on T' and on T/T'.*

7.5 Exactness Properties

In the following let S be a Dedekind scheme with ring of rational functions K. Except for the purposes of Proposition 1 below, we will only be concerned with the case where S consists of a discrete valuation ring R. Let G_K be a smooth K-group scheme of finite type, and let X_K be a torsor under G_K. Then the Néron model X of X_K, if it exists, may be viewed as a direct image $\iota_* X_K$ with respect to the canonical inclusion $\iota : \mathrm{Spec}\, K \longrightarrow S$. More precisely, X represents this direct image if one restricts to *smooth* schemes over S. This consideration suggests that the Néron model might behave reasonably well with respect to left exactness. However we will see that, except for quite special cases, there will be a defect of exactness, the defect of right exactness being much more serious than the one of left exactness. We will give some examples at the end of this section, after we have presented the general results. Let us begin with an assertion concerning the existence of Néron models.

Proposition 1. *Let S be a Dedekind scheme with ring of rational functions K and let*

$$(*) \qquad 0 \longrightarrow G'_K \longrightarrow G_K \longrightarrow G''_K \longrightarrow 0$$

be an exact sequence of smooth K-group schemes of finite type (*not necessarily commutative*).

(a) *If G_K admits a Néron model over S, the same is true for G'_K, but not necessarily for G''_K.*

(b) *If G'_K and G''_K admit Néron models over S, the same is true for G_K.*

Proof. If G_K admits a Néron model, then G'_K admits a Néron model by 7.1/6. To justify the second part of assertion (a), we give an example showing that the existence of a Néron model for G_K does not imply the same for G''_K. Assume that S consists of a discrete valuation ring of equal characteristic $p > 0$ and, as in Example 7.1/3, let G_K be the subgroup of $\mathbb{G}_{a,K} \times_K \mathbb{G}_{a,K}$ given by the equation $x + x^p + \pi y^p = 0$, where π is a uniformizing element of R. Then G_K admits a Néron model over S and the projection of $\mathbb{G}_{a,K} \times_K \mathbb{G}_{a,K}$ onto its second factor gives rise to a smooth group epimorphism $G_K \longrightarrow \mathbb{G}_{a,K}$. Writing G'_K for its kernel, we have a short exact sequence

$$0 \longrightarrow G'_K \longrightarrow G_K \longrightarrow \mathbb{G}_{a,K} \longrightarrow 0$$

of smooth K-group schemes of finite type. The middle term admits a Néron model whereas the group $\mathbb{G}_{a,K}$ on right-hand side does not. The example is quite typical; the reason that a Néron model for G_K does not imply the existence of a Néron model for G''_K, comes mainly from the fact that the quotient of a K-wound unipotent group is not necessarily K-wound again.

Next, to prove assertion (b), assume that G'_K and G''_K admit Néron models G' and G'' over S, where S is an arbitrary Dedekind scheme again. First, if the given exact sequence $(*)$ extends to an exact sequence of smooth S-group schemes of finite type

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0\,,$$

we claim that G is automatically a Néron model of G_K by the criterion given in 7.1/1. Namely, in order to verify this, we may assume that S consists of a strictly henselian discrete valuation ring R. Then it is enough to show that the canonical map $G(R) \longrightarrow G(K)$ is bijective. However, this follows easily from the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G'(R) & \longrightarrow & G(R) & \longrightarrow & G''(R) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \\
0 & \longrightarrow & G'(K) & \longrightarrow & G(K) & \longrightarrow & G''(K) & &
\end{array}$$

by realizing that the first row is exact, due to the fact that the smoothness of $G \longrightarrow G''$ implies the surjectivity of $G(R) \longrightarrow G''(R)$; cf. 2.2/14.

In the general case we can apply a limit argument ([EGA IV_3], 8.8.2), and thereby extend $(*)$ to an exact sequence of smooth group schemes of finite type over a dense open subscheme S' of S. Consequently, there is a Néron model of G_K over S'. Then, using 1.4/1, it is enough to construct the local Néron models of G_K at the finitely many remaining points of $S - S'$. So, in the proof of assertion (b), we are reduced to the case where S consists of a discrete valuation ring R. Since this problem does not seem to be accessible by elementary methods, we have to make use of a later criterion characterizing the existence of Néron models in terms of the structure of algebraic groups; cf. 10.2/1. It says that a smooth K-group scheme of finite type like G_K admits a Néron model if and only if, after the base change

$K \longrightarrow \hat{K}^{sh}$, the group G_K does not contain subgroups of type $\mathbb{G}_a$ or $\mathbb{G}_m$; here $\hat{K}^{sh}$ is the field of fractions of $\hat{R}^{sh}$, the strict henselization of the completion of R. Using this criterion, it is easily verified that G_K admits a Néron model over R if the same is true for G'_K and G''_K. □

Next, consider an exact sequence

$$0 \longrightarrow G'_K \longrightarrow G_K \longrightarrow G''_K \longrightarrow 0$$

and assume that the corresponding Néron models G', G, and G'' exist so that, due to the universal mapping property, there is an associated complex

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0 .$$

We want to examine under what conditions parts of the latter sequence are exact. To do this, it is enough to look at the local case. So, in the following, the base S will consist of a discrete valuation ring R with field of fractions K and with residue field k.

Proposition 2. *If* $\operatorname{char} k = 0$, *the closed immersion* $G'_K \longrightarrow G_K$ *gives rise to a closed immersion* $G' \longrightarrow G$ *of associated Néron models.*

Proof. Denote by H the schematic closure of G'_K in G. Then $G' \longrightarrow G$ factors through $H \subset G$ and we know from 7.1/6 that the induced morphism $G' \longrightarrow H$ is an isomorphism. □

Next, let us look at abelian varieties.

Proposition 3. *Consider an exact sequence of abelian varieties*

$$0 \longrightarrow A'_K \longrightarrow A_K \longrightarrow A''_K \longrightarrow 0$$

and the corresponding complex of Néron models

$$(\dagger) \qquad 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 .$$

Let B_K be an abelian subvariety of A_K such that $A_K \longrightarrow A''_K$ induces an isogeny $u_K : B_K \longrightarrow A''_K$; let $n = \deg u_K$.

(a) *If* $\operatorname{char} k$ *does not divide* n, *then* $A' \longrightarrow A$ *is a closed immersion,* $A \longrightarrow A''$ *is smooth with kernel* A', *and the cokernel of* $A_k \longrightarrow A''_k$ *is killed by multiplication with* n. *If, in addition,* A *has abelian reduction,* $(\dagger)$ *is exact.*

(b) *If A has semi-abelian reduction, the sequence $(\dagger)$ is exact up to isogeny; i.e., it is isogenous to an exact sequence of commutative S-group schemes.*

Proof. The isogeny $u_K : B_K \longrightarrow A''_K$ gives rise to an isogeny $v_K : A'_K \times_K B_K \longrightarrow A_K$ of degree n. So there is an isogeny $w_K : A_K \longrightarrow A'_K \times_K B_K$ such that $w_K \circ v_K$ is multiplication by n. Let B be the Néron model of B_K. Then u_K, v_K, and w_K extend to R-morphisms $u : B \longrightarrow A''$, $v : A' \times_R B \longrightarrow A$, and $w : A \longrightarrow A' \times_R B$ such that $w \circ v$ is multiplication by n on $A' \times_R B$. Assuming the condition of (a), the multiplication by n is an étale isogeny on $A' \times_R B$, and u, v, and w are easily checked to be étale isogenies, too. Then $H := w^{-1}(A')$ is a smooth closed subgroup scheme of A which

satisfies $H^0_K = A'_K$. It follows that the schematic closure of A'_K in H or A is an open subgroup scheme of H and, thus, is smooth over R. So, by 7.1/6, it coincides with the Néron model A' of A'_K and we see that $A' \longrightarrow A$ is a closed immersion. The remaining assertions of (a) follow by using the étale isogeny u. One shows that $A \longrightarrow A''$ is flat, has kernel A' and, hence, is smooth. Furthermore, if A has abelian reduction, the same is true for A'' by 7.4/2 so that $A \longrightarrow A''$ is surjective.

Assertion (b) follows from the fact that $v : A' \times_R B \longrightarrow A$ and $u : B \longrightarrow A''$ are isogenies; use 7.3/6 and 7.3/7. □

Theorem 4. *Let $0 \longrightarrow A'_K \longrightarrow A_K \longrightarrow A''_K \longrightarrow 0$ be an exact sequence of abelian varieties and consider the associated sequence of Néron models $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$. Assume that the following condition is satisfied:*

$(*)$ *R has mixed characteristic and the ramification index $e = v(p)$ satisfies $e < p - 1$, where p is the residue characteristic of R and where v is the valuation on R, which is normalized by the condition that v assumes the value 1 at uniformizing elements of R.*

Then the following assertions hold:

(i) *If A' has semi-abelian reduction, $A' \longrightarrow A$ is a closed immersion.*

(ii) *If A has semi-abelian reduction, the sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A''$ is exact.*

(iii) *If A has abelian reduction, the sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is exact and consists of abelian R-schemes.*

Proof. Let us first see how assertions (ii) and (iii) can be deduced from assertion (i). If A has semi-abelian or abelian reduction, the same is true for A' and A'' by 7.4/2. So $A' \longrightarrow A$ is a closed immersion by (i), and we can consider the quotient A/A'; it exists in the category of algebraic spaces, cf. 8.3/9. Furthermore, A/A' is smooth and separated and, thus, a scheme by 6.6/3. Now look at the canonical morphism $A/A' \longrightarrow A''$ which is an isomorphism on generic fibres. Since A has semi-abelian reduction, the same is true for A/A', and it follows from 7.4/3 that $A/A' \longrightarrow A''$ is an open immersion. So assertion (ii) is clear. Finally, if A has abelian reduction, the same is true for A/A'. So the latter is an abelian scheme by 7.4/5 and, thus, must coincide with the Néron model A'' of A''_K. Thereby we obtain assertion (iii).

It remains to verify assertion (i) under the assumption of condition $(*)$. As a key ingredient for the proof of this fact, we will need the following result on finite group schemes; cf. Raynaud [7], 3.3.6.

Lemma 5. *Let R be a discrete valuation ring satisfying condition $(*)$ of Theorem 4. Let $v : G' \longrightarrow G$ be a morphism of R-group schemes which are finite, flat, and commutative. Then, if $v_K : G'_K \longrightarrow G_K$ is an isomorphism, v is an isomorphism.*

The lemma implies a criterion for finite and flat R-group schemes to be étale. To state it in its simplest form, recall that a group scheme over a base scheme S is called constant if it is of the type H_S with an abstract group H.

Corollary 6. *Assume that R is as in condition $(*)$ of Theorem 4 and that, in addition, it is strictly henselian. Furthermore, consider a finite, flat, and commutative R-group scheme G whose generic fibre is constant. Then G is constant.*

Proof of Corollary 6. Let $G' \longrightarrow G$ be a group smoothening of G (see 7.1). Then G' coincides with its finite part and, thus, is finite over R since G is finite over R. Therefore $G' \longrightarrow G$ is an isomorphism by the lemma. Using the fact that G' is étale over R and that R is strictly henselian, G is constant. □

Now let us indicate how to obtain assertion (i) of Theorem 4 under the assumption of condition (∗). Since Néron models are preserved when R is replaced by its strict henselization or by its completion, we may assume that R is *strictly henselian* and *complete*.

We begin by showing that $u: A' \longrightarrow A$ is a monomorphism; i.e., that $N := \ker u$ is trivial. For this purpose it is enough to show that the special fibre N_k of N is trivial. If not, there is a prime l, not necessarily different from char k, such that ${}_lA'_k \cap N_k$ is non-trivial; as usual, ${}_lA'$ is the kernel of the l-multiplication on A'. Since A' has semi-abelian reduction, ${}_lA'$ is quasi-finite and flat over R; cf. 7.3/2. Now, R being henselian, we can consider the finite part G' of ${}_lA'$; see 7.3. It is enough to show that u is a monomorphism on G'. Let G be the schematic image of G' under u and consider the morphism $u': G' \longrightarrow G$ given by u. Then u' is an isomorphism on generic fibres and thus, by the lemma, an isomorphism on G'. In particular, u' is a monomorphism, and it follows that u is a monomorphism.

If A' has abelian reduction, it is an abelian scheme by 7.4/5 and, thus, proper over R. So it follows that u is proper. But then, being a monomorphism, it must be a closed immersion. This ends the proof in the special case where A' has abelian reduction.

In the general case, some work remains to be done since there exist monomorphisms which are not immersions; cf. [SGA 3_{II}], Exp. VIII, 7 and Exp. XVI, 1. Let B be the schematic image of $u: A' \longrightarrow A$; it is a closed subgroup scheme of A which is flat over R. We will show that B or, what is enough, that B^0 is smooth. Then, due to the Néron mapping property, the morphism $A' \longrightarrow B$ admits an inverse and u is a closed immersion. In order to do so, we denote by an index n reductions modulo π^n, where π is a uniformizing element of R. Since u is a monomorphism, it is a closed immersion modulo π^n for all $n > 0$; cf. [SGA 3_{I}], Exp. $\mathrm{VI_B}$, 1.4.2. So we can consider the exact sequence of R_n-schemes

$$0 \longrightarrow A_n'^0 \longrightarrow B_n^0 \longrightarrow Q_n \longrightarrow 0$$

where the quotient $Q_n = B_n^0/A_n'^0$ exists as an R-scheme by [SGA 3_{I}], Exp. $\mathrm{VI_A}$, Thm. 3.2, and is flat by [SGA 3_{I}], Exp. $\mathrm{VI_B}$, Thm. 9.2. Furthermore, Q_n is connected and, by reasons of dimension, finite over R_n. Taking inductive limits for n going to infinity, we obtain an exact sequence of formal group schemes over R

$$0 \longrightarrow \hat{A}' \longrightarrow \hat{B} \longrightarrow Q \longrightarrow 0$$

where Q is an R-scheme which is finite, flat, and connected. Let q be a power of p such that Q is annihilated by the q-multiplication on Q. Since $\hat{A}'$ is p-divisible, the above sequence restricts to an exact sequence

$$0 \longrightarrow {}_q\hat{A}' \longrightarrow {}_q\hat{B} \longrightarrow Q \longrightarrow 0$$

on the kernels of q-multiplications; the latter are finite flat R-group schemes by 7.3/2.

Furthermore, ${}_q\hat{A}'$ and ${}_q\hat{B}$ can be interpreted as the finite parts of the quasi-finite flat R-group schemes ${}_qA'^0$ and ${}_qB^0$.

Applying Grothendieck's orthogonality theorem [SGA 7_{I}], Exp. IX, Prop. 5.6, we see that the generic fibre of the quotient ${}_qA'/{}_q\hat{A}'$ is constant. Since A' and B coincide on generic fibres, it follows that the generic fibres of ${}_q\hat{B}/{}_q\hat{A}'$ and, thus, of Q are constant. But then Q is constant by Corollary 6 and, being connected, it must be trivial. So $\hat{A}'$ is isomorphic to $\hat{B}$ and, consequently, B^0 is smooth which remained to be shown. □

In the remainder of this section, we want to discuss the defect of exactness of Néron models by looking at some special examples.

Example 7. Let R be a complete discrete valuation ring with normalized valuation v. Let q be a non-zero element of R with $v(q) > 0$ and consider the Tate elliptic curves $E_K = \mathbb{G}_{m,K}/q^{\mathbb{Z}}$ and $E'_K = \mathbb{G}_{m,K}/(q^l)^{\mathbb{Z}}$ where l is a positive integer not divisible by char K. Since the l-multiplication on E_K factors through E'_K, it gives rise to an exact sequence

$$0 \longrightarrow G_K \longrightarrow E_K \longrightarrow E'_K \longrightarrow 0\,,$$

where G_K is a finite group scheme of order l, contained in the kernel of the l-multiplication on E_K; the latter is of order l^2. Let

$$0 \longrightarrow G \longrightarrow E \longrightarrow E' \longrightarrow 0$$

be the associated sequence of Néron models. We want to show that there can be a defect of exactness at G, at E, or at E', depending on l and on the residue characteristic of R.

Defect of exactness at G. Assume that R is of mixed characteristic, that $l = p =$ char k, and that all p-torsion points of E_K are rational over K. The latter condition implies that the ramification index e is at least $p-1$; cf. Serre [4], Chap. IV, §4, Prop. 17. Then $G_K \simeq (\mathbb{Z}/p\mathbb{Z})_K$ and $G \simeq (\mathbb{Z}/p\mathbb{Z})_R$. Furthermore, the kernel of $E \longrightarrow E'$ is the group $\boldsymbol{\mu}_{p,R}$ of p-th roots of unity, and the morphism from G into the kernel of $E \longrightarrow E'$ coincides with a morphism $(\mathbb{Z}/p\mathbb{Z})_R \longrightarrow \boldsymbol{\mu}_{p,R}$ sending 1 to a primitive p-th root of unity of R. However, the latter is not a monomorphism since $p =$ char k. In particular, $G \longrightarrow E$ is not a monomorphism.

Defect of exactness at E. Keeping the situation we have developed above, we see that G cannot be mapped surjectively onto the kernel of $E \longrightarrow E'$ since the morphism $(\mathbb{Z}/p\mathbb{Z})_R \longrightarrow \boldsymbol{\mu}_{p,R}$ is not surjective.

Defect of exactness at E'. The group of connected components of the special fibre of E has order $v(q)$ whereas that of E' has order $l \cdot v(q)$. So, without restrictions on the residue characteristic of R, the morphism $E \longrightarrow E'$ cannot be surjective for arbitrary $l > 1$. □

Next we want to show that the assertion of Theorem 4 can be false if we do not require condition $(*)$ of this theorem.

Example 8 (Serre). We will construct a morphism $v : A' \longrightarrow A$ of abelian schemes over R which is not a monomorphism, but which has the property that v_K: $A'_K \longrightarrow A_K$ is a closed immersion. The valuation ring R is supposed to have mixed characteristic. So if $p = \operatorname{char} k$, we have to assume $e := v(p) \geq p - 1$ by Theorem 4. In the following we assume that R contains all p-th roots of unity so that e is a multiple of $p - 1$ by Serre [4], Chap. IV, § 4, Prop. 17. Now, similarly as in Example 7, consider a morphism $u : (\mathbb{Z}/p\mathbb{Z})_R \longrightarrow \boldsymbol{\mu}_p$ sending 1 to a primitive p-th root of unity. Let E be an elliptic curve over R (i.e., an abelian scheme with elliptic curves as fibres) which contains $\boldsymbol{\mu}_p$ as a subscheme. Then u extends to a morphism $u : (\mathbb{Z}/p\mathbb{Z})_R \longrightarrow E$, which is a closed immersion on generic fibres, but which is not a monomorphism. Let E' be a second elliptic curve over R which contains $(\mathbb{Z}/p\mathbb{Z})_R$ as a subscheme (for example, a Serre-Tate-lifting of an elliptic curve over k containing $(\mathbb{Z}/p\mathbb{Z})_k$ as a subscheme). Then consider the co-cartesian diagram

$$\begin{tikzcd}
(\mathbb{Z}/p\mathbb{Z})_R \arrow[r, "u"] \arrow[d] & E \arrow[d] \\
E' \arrow[r, "u'"] & F'
\end{tikzcd}$$

where F' is the quotient of $E \times E'$ with respect to the action of $(\mathbb{Z}/p\mathbb{Z})_R$. Since the action is free, F' is an abelian scheme over R. Furthermore, u'_K is a closed immersion, but u' itself cannot be a monomorphism since u is not a monomorphism. □

Finally, we want to show that the condition on the semi-abelian reduction of A' in Theorem 4 cannot be cancelled.

Example 9. Consider discrete valuation rings $R \subset R'$ where $R = \mathbb{Z}_{(p)}$ and $R' = \mathbb{Z}_{(p)}[a]$ with a being a primitive p-th root of unity; p is a prime different from 2. Let $u' : E' \longrightarrow F'$ be a morphism of abelian R'-schemes of the type constructed in Example 8; i.e., such that u' is not a monomorphism, but such that it is a closed immersion on generic fibres. Then apply the technique of Weil restriction of R' over R to u' (cf. Section 7.6) and consider the induced morphism $u^1 : E^1 \longrightarrow F^1$. It follows from 7.6/6 that E^1 and F^1 are Néron models of their generic fibres, and from 7.6/2 that u^1 is a closed immersion on generic fibres. We claim that u^1 is not a monomorphism. Indeed, the image of the map $\operatorname{Lie}(u') : \operatorname{Lie}(E') \longrightarrow \operatorname{Lie}(F')$ cannot be locally a direct factor in $\operatorname{Lie}(F')$. The same is true for the Weil restriction of $\operatorname{Lie}(u')$, and the latter is canonically identified with $\operatorname{Lie}(u^1) : \operatorname{Lie}(E^1) \longrightarrow \operatorname{Lie}(F^1)$. So $u^1 : E^1 \longrightarrow F^1$ cannot be a closed immersion and, thus, not a monomorphism. Since $v(p) = 1 < p - 1$, where v is the normalized valuation on R, we see from Theorem 4 that E^1 cannot have semi-abelian reduction. □

7.6 Weil Restriction

The main purpose of this section is to discuss a criterion for the existence of Weil restrictions and to study the behavior of Néron models with respect to Weil restrictions.

Let $h : S' \longrightarrow S$ be a morphism of schemes. Then, for any S'-scheme X', the contravariant functor

$$\mathfrak{R}_{S'/S}(X') : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}) , \qquad T \longmapsto \operatorname{Hom}_{S'}(T \times_S S', X') ,$$

is defined on the category (Sch/S) of S-schemes. If it is representable, the corresponding S-scheme, again denoted by $\mathfrak{R}_{S'/S}(X')$, is called the *Weil restriction* of X' with respect to h. Thus, the latter is characterized by a functorial isomorphism

$$\operatorname{Hom}_S(T, \mathfrak{R}_{S'/S}(X')) \xrightarrow{\sim} \operatorname{Hom}_{S'}(T \times_S S', X')$$

of functors in T where T varies over all S-schemes. There are several elementary properties of the functor $\mathfrak{R}_{S'/S}(X')$ and, hence, of Weil restrictions, which follow immediately from the definition. We will derive some of them once we have mentioned the adjunction formula in Lemma 1 below.

Imposing an appropriate condition on h such as being finite and locally free (which we mean as a synonym for being finite, flat, and of finite presentation), the existence of the Weil restriction of the affine n-space $\mathbb{A}^n_{S'}$ is trivial (cf. the beginning of the proof of Theorem 4). Then, in order to treat more general schemes, it is necessary to study the behavior of Weil restrictions with respect to open or closed immersions. In order not to worry about the representability of the functor $\mathfrak{R}_{S'/S}(X')$ too much, we will work entirely within the context of functors from schemes to sets. In particular, we will make no difference between an S-scheme X and its associated functor $\operatorname{Hom}_S(\cdot, X)$; in the same way we will proceed with S'-schemes.

It is convenient to define the functor $\mathfrak{R}_{S'/S}(X')$ not only for S'-schemes X', but, more generally, for arbitrary contravariant functors from the category (Sch/S') of S'-schemes to the category of sets. So consider a functor

$$F' : (\mathrm{Sch}/S')^0 \longrightarrow (\mathrm{Sets}) .$$

Then its direct image with respect to $h : S' \longrightarrow S$ consists of the functor

$$h_*F' : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}) , \qquad T \longmapsto F'(T \times_S S') .$$

Using 4.1/1, we see easily that the functor

$$(\mathrm{Sch}/S) \longrightarrow (\mathrm{Sch}/S') , \qquad T \longmapsto T \times_S S' ,$$

plays the role of an adjoint of h_*; namely, the so-called adjunction formula is valid.

Lemma 1. *For any S-scheme T and any functor $F' : (\mathrm{Sch}/S')^0 \longrightarrow (\mathrm{Sets})$, there is a canonical bijection*

$$\operatorname{Hom}_S(T, h_*F') \xrightarrow{\sim} \operatorname{Hom}_{S'}(T \times_S S', F')$$

which is functorial in T and in F'.

As an application of the above formula, we want to derive some elementary properties of Weil restrictions. Let X' be an S'-scheme. Then the identity on $\mathfrak{R}_{S'/S}(X')$ gives rise to a functorial morphism

$$\mathfrak{R}_{S'/S}(X') \times_S S' \longrightarrow X'$$

if $\mathfrak{R}_{S'/S}(X')$ exists as an S-scheme. Likewise, if X is an S-scheme, the identity on $X \times_S S'$ defines a functorial morphism

$$X \longrightarrow \mathfrak{R}_{S'/S}(X \times_S S') .$$

On the other hand, each functorial morphism $F' \longrightarrow G'$ between contravariant functors from (Sch/S') to (Sets) induces a functorial morphism $h_*F' \longrightarrow h_*G'$. Furthermore, h_* commutes with fibred products, and it follows that h_*F' is a group functor if the same is true for F'. In particular, the Weil restriction of a group scheme is, if it exists as a scheme, a group scheme again. Also it is easy to see that the notion of Weil restriction is compatible with base change; i.e., if $T \longrightarrow S$ is a morphism of base change, and if we write $T' := S' \times_S T$, then, for any S'-scheme X', there is a canonical isomorphism

$$\mathfrak{R}_{T'/T}(X' \times_S T') \simeq \mathfrak{R}_{S'/S}(X') \times_S T$$

of functors on (Sch/T).

In the following we need the terminology of relative representability of functors; cf. Grothendieck [1], Sect. 3. Let

$$F, G : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$$

be contravariant functors, and let $u : F \longrightarrow G$ be a functorial morphism. Then, for each functorial morphism $T \longrightarrow G$, where T is an arbitrary S-scheme, the fibred product $F_T = F \times_G T$ may be viewed as a functor from $(\mathrm{Sch}/T)^0$ to (Sets). One says that F is *relatively representable* over G via u if, for each $T \longrightarrow G$, the projection $F_T \longrightarrow T$ is a morphism in (Sch/S); i.e., if each F_T is representable by a T-scheme. Many notions on morphisms between schemes can easily be adapted to the context of relative representability. For example, u is called an open immersion, or a closed immersion, or a morphism of finite type, etc., if the corresponding property is true for each morphism of schemes $u_T : F_T \longrightarrow T$, obtained from $u : F \longrightarrow G$ by the "base change" $T \longrightarrow G$.

Proposition 2. *Let $u' : F' \longrightarrow G'$ be a morphism between functors from* $(\mathrm{Sch}/S')^0$ *to* (Sets).

(i) *Assume that u' is an open immersion and that $h : S' \longrightarrow S$ is proper. Then the associated morphism $h_*(u') : h_*F' \longrightarrow h_*G'$ is an open immersion.*

(ii) *Assume that u' is a closed immersion and that $h : S' \longrightarrow S$ is finite and locally free or, more generally, proper, flat, and of finite presentation. Then $h_*(u') : h_*F' \longrightarrow h_*G'$ is a closed immersion.*

Proof. Let us write $F = h_*F'$ and $G = h_*G'$, and let $T \longrightarrow G$ be a morphism, where T is an arbitrary S-scheme. Setting $T' := T \times_S S'$, we claim that $T \longrightarrow G$ factors canonically through h_*T'. Indeed, we have a canonical morphism $T \longrightarrow h_*T'$.

Furthermore, $T \longrightarrow G$ corresponds to a morphism $T' \longrightarrow G'$ and, hence, to a morphism $h_*T' \longrightarrow h_*G' = G$. That the composition with $T \longrightarrow h_*T'$ yields $T \longrightarrow G$ is easily verified with the help of 4.1/1. Consequently, we can view F_T as being obtained from $F_{h_*T'}$ by means of the base change $T \longrightarrow h_*T'$, a fact to be used below.

Furthermore, since h_* commutes with fibred products, there are isomorphisms

$$h_*F'_{T'} \simeq F \times_G h_*T' \simeq F_{h_*T'} ,$$

and we can look at the canonical commutative diagram

$$\begin{array}{ccc}
F'_{T'} & \longrightarrow & T' \\
 & & \downarrow \\
F_T & \longrightarrow & T \\
\downarrow & & \downarrow \\
F_{h_*T'} & \longrightarrow & h_*T' \; .
\end{array}$$

In order to prove assertion (i), it has to be shown that the morphism in the middle row, which is obtained from the one in the lower row by the base change $T \longrightarrow h_*T'$, is an open immersion of schemes. We know already that the upper row is an open immersion of schemes; let U' be the image of $F'_{T'}$ in T', and set $V' := T' - U'$. Then V' is closed in T' and, since $T' \longrightarrow T$ is proper, its image V in T is closed again. Set $U := T - V$. Interpreting F_T as the fibred product of $F_{h_*T'}$ and T over h_*T', we have

$$F_T = \mathrm{Hom}_{S'}(\cdot \times_S S', U') \times_{\mathrm{Hom}_{S'}(\cdot \times_S S', T')} \mathrm{Hom}_S(\cdot, T) .$$

Thus, if Z is an arbitrary S-scheme, $F_T(Z)$ consists of all S-morphisms $Z \longrightarrow T$ where $Z \times_S S' \longrightarrow T'$ factors through U'; i.e., of those S-morphisms $Z \longrightarrow T$ which factor through U. Hence F_T is represented by the open subscheme U of T and assertion (i) follows.

Next, let us verify assertion (ii) for the case where h is finite and locally free. Similarly as before, let V' be the closed subscheme of T' which is given by the closed immersion $F'_{T'} \longrightarrow T'$. Then we have to find a closed subscheme V of T such that, given any S-morphism $Z \longrightarrow T$, it factors through V if and only if $Z \times_S S' \longrightarrow T'$ factors through V'. The problem is local on S, T, and Z, so we may assume that all three schemes are affine, say with rings of global sections R, A, and C. Let $R \longrightarrow R'$ be the homomorphism between rings of global sections on S and S'. We may assume R' is a free R-module of rank n. Let $e_1, \ldots, e_n$ be a basis of R' over R; then these elements give rise to a basis of $A \otimes_R R'$ over R. Furthermore, let $\mathfrak{a}' \subset A \otimes_R R'$ be the ideal corresponding to V', and fix generators a'_i, $i \in I$, of $\mathfrak{a}'$. There are equations

$$a'_i = \sum_{j=1}^{n} c_{ij} e_j , \qquad i \in I ,$$

with coefficients $c_{ij} \in A$. These coefficients generate an ideal $\mathfrak{a} \subset A$, and we claim that the associated closed subscheme $V \subset T$ is as required. Namely, consider the homomorphism $\sigma : A \longrightarrow C$ which is associated to $Z \longrightarrow T$ as well as the homomorphism $\sigma' : A \otimes_R R' \longrightarrow C \otimes_R R'$ associated to $Z \times_S S' \longrightarrow T'$. Since

$$\ker \sigma' = (\ker \sigma) \otimes_R R' = \bigoplus_{i=1}^{n} (\ker \sigma) \cdot e_i ,$$

we see that $\mathfrak{a}' \subset \ker \sigma'$ if and only if $\mathfrak{a} \subset \ker \sigma$, i.e., that Z' is mapped into V' if and only if Z is mapped into V. So it follows that V represents the functor F_T.

If, more generally, h is proper, flat, and of finite presentation, one uses techniques from the construction of Hilbert schemes as in [FGA], n°221, Sect. 3, in order to show that there is a largest closed subscheme V of T such that an S-morphism $Z \longrightarrow T$ factors through V if and only if, after base change with $h : S' \longrightarrow S$, it factors through $V' \subset T'$. □

A functor $F : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ is called a *sheaf with respect to the Zariski topology* (see 8.1) if, for each S-scheme T and for each covering $\{T_i\}$ of T, the sequence

$$\mathrm{Hom}_S(T, F) \longrightarrow \prod_i \mathrm{Hom}_S(T_i, F) \rightrightarrows \prod_{i,j} \mathrm{Hom}_S(T_i \cap T_j, F)$$

is exact. Of course, if F is a scheme, F is a sheaf in this sense.

Proposition 3. *If $F' : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ is a sheaf with respect to the Zariski topology, then the same is true for $F := h_* F'$.*

Proof. Since, for any S-scheme T, we have

$$\mathrm{Hom}_S(T, F) = \mathrm{Hom}_{S'}(T \times_S S', F') ,$$

the assertion is obvious. □

We want to apply the above results to the case where F' consists of an S'-scheme X', and give a criterion of Grothendieck for the representability of $X := h_* X' = \mathfrak{R}_{S'/S}(X')$ by an S-scheme. Then, if X is representable, it defines the Weil restriction of X'.

Theorem 4. *Let $h : S' \longrightarrow S$ be a morphism of schemes which is finite and locally free, and let X' be an S'-scheme. Assume that, for each $s \in S$ and each finite set of points $P \subset X' \otimes_S k(s)$, there is an affine open subscheme U' of X' containing P. Then $h_* X' = \mathfrak{R}_{S'/S}(X')$ is representable by an S-scheme X and, thus, the Weil restriction of X' exists.*

Proof. We may assume that S and, hence, S' are affine, say with rings of global sections R and R' and that R' is a free R-module, say with generators $e_1, \ldots, e_n$. Let us first show that $h_* X'$ is representable if X' is affine. So assume X' is affine and view it as a closed subscheme of some scheme $\mathrm{Spec}\, R'[t]$, where t is a (finite or infinite) system of indeterminates. Applying Proposition 2, it is only necessary to consider the case where $X' = \mathrm{Spec}\, R'[t]$. Consider n copies of the system t and write $t_1, \ldots, t_n$ for these systems. Then, for any R-algebra A, there is a bijection

$$\mathrm{Hom}_{R'}(R'[t], A \otimes_R R') \longrightarrow \mathrm{Hom}_R(R[t_1, \ldots, t_n], A) ,$$

which is functorial in A. In order to define this map, consider an R'-homomorphism $\sigma' : R'[t] \longrightarrow A \otimes_R R'$. The latter is determined by the image $\sigma'(t)$ of t in $A \otimes_R R'$. Using the direct sum decomposition

$$A \otimes_R R' = \bigoplus_{i=1}^{n} (A \otimes_R R) e_i ,$$

we can write

$$\sigma'(t) = \sum_{i=1}^{n} \sigma(t_i) \otimes e_i$$

with systems $\sigma(t_1), \ldots, \sigma(t_n)$ of elements in A, and we can think of σ as of a homomorphism $\sigma : R[t_1, \ldots, t_n] \longrightarrow A$. Then it is easily seen that $\sigma' \longmapsto \sigma$ defines the desired bijection. Consequently, in this case the functor $h_* X'$ is representable by the S-scheme $\mathrm{Spec}\, R[t_1, \ldots, t_n]$, and it follows that the Weil restriction $\mathfrak{R}_{S'/S}(X')$ exists.

Next, let us consider the case where X' is not necessarily affine. Let $\{U'_i\}_{i \in I}$ be the system of all affine open subschemes of X'. Then, by what we have just seen, each $h_* U'_i$ is representable by an (affine) scheme U_i, and the open immersion $U'_i \hookrightarrow X'$ gives rise to a morphism $U_i \longrightarrow h_* X'$ which is an open immersion by Proposition 2. Viewing the U'_i as open subschemes of X', we have canonical gluing data for them, and these data give rise to gluing data for the U_i. So, gluing the U_i, we obtain an S-scheme Y. Since X' is a sheaf with respect to the Zariski topology, the same is true for $h_* X'$ (see Proposition 3) and there is a functorial morphism $Y \longrightarrow h_* X'$. The latter is an open immersion by Proposition 2.

In order to show that $Y \longrightarrow h_* X'$ is an equivalence of functors, it is enough to show that each functorial morphism $a : T \longrightarrow h_* X'$, where T is an arbitrary S-scheme, factors uniquely through Y or, what amounts to the same, that the latter is the case locally in a neighborhood of each point $z \in T$. Let (z_j) be the finite family of points in $T \times_S S'$ lying over z. Furthermore, let $a' : T \times_S S' \longrightarrow X'$ be the morphism corresponding to a, and set $x_j = a'(z_j)$. By our assumption, there is an affine open subscheme $U' \subset X'$ containing all points x_j. We know already that $h_* U'$ is representable by an S-scheme U and that the canonical morphism $U \longrightarrow h_* X'$ is an open immersion; the latter factors through Y by the definition of Y. Replacing T by a suitable open subscheme containing z, we may assume that $a' : T' \longrightarrow X'$ factors through U'. Then $a : T \longrightarrow h_* X'$ factors through U and, hence, through Y. The factorization is unique due to the fact that $Y \longrightarrow h_* X'$ is an open immersion. □

We want to mention some general properties of Weil restrictions, assuming that we are in the situation of Theorem 4.

Proposition 5. *Let $S' \longrightarrow S$ be a morphism of schemes which is finite and locally free, and let X' be an S'-scheme. Assume that the Weil restriction $X = \mathfrak{R}_{S'/S}(X')$ exists as an S-scheme, and consider the following properties for relative schemes:*

(a) *quasi-compact.*

(b) *separated,*

(c) *locally of finite type,*
(d) *locally of finite presentation,*
(e) *finite presentation,*
(f) *proper,*
(g) *flat,*
(h) *smooth.*

Then the above properties carry over from X' to X under the following additional assumptions:

property (a) *if S is locally noetherian or if $S' \longrightarrow S$ is étale,*
properties (b), (c), (d), (e), *and* (h) *without any further assumptions, and*
properties (f) *and* (g) *if $S' \longrightarrow S$ is étale.*

Proof. Let us begin with properties which carry over from X' to X without any additional assumptions, say with property (b). Since the Weil restriction of the diagonal morphism $X' \longrightarrow X' \times_{S'} X'$ yields the diagonal morphism $X \longrightarrow X \times_S X$ and since the Weil restriction respects closed immersions by Proposition 2, we see that X is separated if X' is separated.

Next, let us look at properties (c) and (d). That they carry over from X' to X follows from the construction of Weil restrictions in the affine case. Namely, if X' is a closed subscheme of the affine n-space $\mathbb{A}^n_{S'}$, and if $S' \longrightarrow S$ is a finite and free morphism of affine schemes, say of degree d, then it follows from Proposition 2 that X is a closed subscheme of $\mathfrak{R}_{S'/S}(\mathbb{A}^n_{S'}) \simeq \mathbb{A}^m_S$ where $m = nd$. So X is locally of finite type if the same is true for X'. Furthermore, the proof of Proposition 2 shows that the ideal defining X as a closed subscheme of $\mathbb{A}^m_S$ is finitely generated if the same is true for X' as a closed subscheme of $\mathbb{A}^n_{S'}$. So it follows that X is locally of finite presentation if the same is true for X'. The latter result can also be obtained by functorial arguments using the characterization [EGA IV_3], 8.14.2, of morphisms which are locally of finite presentation.

If X' satisfies property (e), we can view it as an S-scheme of finite presentation. Using a limit argument, we may assume that S is noetherian. Then X is locally of finite presentation, since property (d) carries over from X' to X, and quasi-compact over S since, as we will see below, also property (a) carries over from X' to X if the base S is noetherian. But then X is of finite presentation over S.

Finally, the characterization of smoothness in terms of the lifting property 2.2/6 shows by functorial reasons that X satisfies property (h) if X' does.

Now assume that $S' \longrightarrow S$ is étale and finite. In order to show that X satisfies properties (a), (f), or (g) if X' does, we may work locally on S, say in a neighborhood of a point $s \in S$. Furthermore, Weil restrictions commute with base change on S. So we may replace S by an étale neighborhood of s. But then, since locally up to étale base change étale morphisms are open immersions, see 2.3/8, we are reduced to the case where S' consists of a finite disjoint sum $\coprod S_i$ of copies S_i of S and where $S' \longrightarrow S$ is the canonical map. Then, in terms of fibred products over S,

$$\mathfrak{R}_{S'/S}(X') \simeq \prod_i \mathfrak{R}_{S_i/S}(X' \times_{S'} S_i) \simeq \prod_i X' \times_{S'} S_i \,,$$

and it is trivial that X satisfies properties (a), (f), or (g) if X' does.

It remains to show that, under appropriate conditions, property (a) carries over from X' to X, a fact which is already known if $S' \longrightarrow S$ is étale. We claim that it is also true for radicial morphisms. To verify this, it is enough to prove that, for S' radicial over S, the Weil restriction $\mathfrak{R}_{S'/S}$ transforms any affine open covering (U'_j) of X' into an affine open covering $(\mathfrak{R}_{S'/S}(U'_j))$ of X. Looking at fibres over S, we may assume that S is the spectrum of a field K. Then S' consists of a finite-dimensional local K-algebra K' whose residue field is purely inseparable over K. Now let (U'_j) be an affine open covering of X'. To see that the sets $\mathfrak{R}_{K'/K}(U'_j)$ really cover X, consider a geometric point $\operatorname{Spec} E \longrightarrow X$ where E is a field over K. Then the scheme $\operatorname{Spec}(E \otimes_K K')$ consists of a single point and the corresponding morphism $\operatorname{Spec}(E \otimes_K K') \longrightarrow X'$ must factor through a member of the open covering (U'_j) of X'. Consequently, $\operatorname{Spec} E \longrightarrow X$ factors through a member of the family $(\mathfrak{R}_{K'/K}(U'_j))$ which justifies our claim.

Now assume that the base S is locally noetherian. In order to show that X satisfies property (a) if X' does, we may assume that S is noetherian. We will conclude by using a noetherian argument and a stratification of S. Let η be a generic point of S. Restricting ourselves to a neighborhood of η, we can assume that S is irreducible and, since quasi-compactness can be tested after killing nilpotent elements of structure sheaves, that S is reduced. Furthermore, we can assume that S and S' are affine, say $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$. The fibre S'_η is the spectrum of the finite-dimensional K-algebra $K' = R' \otimes_R K$ where $K = k(\eta) = Q(R)$. Let L be the maximal étale K-subalgebra between K and K'. It is obtained as follows. Decompose K' into a finite direct product $\prod K'_i$ of local K-algebras K'_i and, for each i, choose a maximal separable extension field L_i between K and K'_i. Then the residue field of K'_i is purely inseparable over L_i and we have $L = \prod L_i$. Set $T := \operatorname{Spec}(R' \cap L)$ so that $S' \longrightarrow S$ factors through T. Over the generic point η, the finite morphism $T \longrightarrow S$ is étale. Thus, using the openness of the étale locus, we know that $T \longrightarrow S$ is étale over an open neighborhood of η. Restricting to this neighborhood, we may assume that $T \longrightarrow S$ is étale everywhere. Furthermore, for each $a \in K'$, there is an integer n such that a^n belongs to L. This property carries over to the fibres of $S' \longrightarrow T$ so that the latter morphism is radicial. Since $X = \mathfrak{R}_{T/S}(\mathfrak{R}_{S'/T}(X'))$, we see by what we have proved above for étale and for radicial morphisms that, working over a neighborhood of η, the scheme X is quasi-compact if X' is.

The argument just given shows that the original morphism $X \longrightarrow S$ is quasi-compact over a dense open subset of S if X' is quasi-compact over S'. Looking at the complement S_1 of this set and viewing it as a scheme with respect to the canonical reduced structure, we can perform the base change $S_1 \longrightarrow S$. It follows in the same way that $X \times_S S_1 \longrightarrow S_1$ is quasi-compact over a dense open subset of S_1. Continuing this way, the procedure will stop after finitely many steps due to the noetherian hypothesis. Thus, finally, it is seen that X is quasi-compact over S. □

We want to add, again in the situation of Theorem 4, that, for any S-scheme X, the canonical morphism $X \longrightarrow \mathfrak{R}_{S'/S}(X \times_S S')$ is a closed immersion, provided X and, thus, $\mathfrak{R}_{S'/S}(X \times_S S')$ are separated. This follows by means of descent from the fact that the composition of canonical morphisms

$$X \times_S S' \longrightarrow \mathfrak{R}_{S'/S}(X \times_S S') \times_S S' \longrightarrow X \times_S S'$$

is the identity on $X \times_S S'$.

Finally, let us state how Néron models behave with respect to Weil restrictions.

Proposition 6. *Let $S' \longrightarrow S$ be a finite and flat morphism of Dedekind schemes. Let* $\operatorname{Spec} K$ *and* $\operatorname{Spec} K'$ *denote the schemes of generic points of S and S'. Furthermore, consider a torsor X' (under a smooth S'-group scheme G') which is a Néron model of the scheme of generic fibres $X' \times_{S'} \operatorname{Spec} K'$. Then the Weil restriction $X = \mathfrak{R}_{S'/S}(X')$ exists as an S-scheme and is a Néron model of the scheme of generic fibres* $X \times_S \operatorname{Spec} K$.

Proof. Using the quasi-projectivity of torsors over Dedekind schemes (cf. 6.4/1), the existence of $X = \mathfrak{R}_{S'/S}(X')$ as an S-scheme follows from Theorem 4. Furthermore, it follows from Proposition 5 that X is separated, of finite type, and smooth. Finally, that X satisfies the Néron mapping property is a formal consequence of the definition of Weil restrictions, namely of the equation

$$\operatorname{Hom}_S(Z, X) = \operatorname{Hom}_{S'}(Z \times_S S', X') .$$

□

Chapter 8. The Picard Functor

Following Grothendieck's treatment [FGA], we introduce the relative Picard functor $\operatorname{Pic}_{X/S}$ and treat the notion of the rigidified relative Picard functor. The main purpose of this chapter is the presentation of various results on the representability of $\operatorname{Pic}_{X/S}$. We explain Grothendieck's theorem on the representability of $\operatorname{Pic}_{X/S}$ by a scheme and point out improvements due to Mumford [2] as well as those due to Altman and Kleiman [1]. In Section 8.3, we discuss the main steps of M. Artin's approach [5] to the representability of $\operatorname{Pic}_{X/S}$ by an algebraic space; for details, the reader is referred to his paper. At the end of the chapter, there is a collection of some results on smoothness as well as on finiteness properties of $\operatorname{Pic}_{X/S}$, as can be found in [SGA 6].

8.1 Basics on the Relative Picard Functor

For any scheme X, we denote by $\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ the group of isomorphism classes of invertible sheaves on X. It is called the absolute *Picard group* of X. Fixing a base scheme S and an S-scheme X, we can consider the contravariant functor

$$P_{X/S} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}) , \qquad T \longmapsto \operatorname{Pic}(X \times_S T) ,$$

from the category (Sch/S) of S-schemes to the category of sets, which factors through the category of commutative groups. Using the procedure of sheafification, we want to associate a functor with $P_{X/S}$ which, under certain conditions, is representable; namely, the so-called *relative Picard functor.*

To begin with, let us discuss a necessary condition for a functor $F : (\mathrm{Sch}/S)^0 \longrightarrow$ (Sets) to be representable. Let $\mathfrak{M}$ be a class of morphisms in (Sch/S) which is stable under composition and under fibred products and which contains all isomorphisms. Then F is called a *sheaf with respect to* $\mathfrak{M}$ or an $\mathfrak{M}$*-sheaf* if, for any family of S-schemes $(T_i)_{i \in I}$, the canonical morphism

$$F(\coprod T_i) \longrightarrow \prod F(T_i)$$

is an isomorphism and if, for all morphisms $T' \longrightarrow T$ in $\mathfrak{M}$, the sequence

$$F(T) \longrightarrow F(T') \rightrightarrows F(T'')$$

is exact (where $T'' = T' \times_T T'$ and where the double arrows on the right are induced by the two projections from T'' onto T'). For example, we can consider the class $\mathfrak{M} = \mathfrak{M}_{\mathrm{Zar}}$ of all morphisms in (Sch/S) of type $\coprod T_i \longrightarrow T$, where the maps $T_i \longrightarrow T$

are open immersions and where $\{T_i\}_{i\in I}$ is an open covering of T. If F is a sheaf with respect to $\mathfrak{M}_{\mathrm{Zar}}$, it is said that F is a *sheaf with respect to the Zariski topology.* To give an equivalent condition, one can require that, for all open coverings $\{T_i\}_{i\in I}$ of T, the canonical sequence

$$F(T) \longrightarrow \prod_i F(T_i) \rightrightarrows \prod_{i,j} F(T_i \times_T T_j)$$

is exact.

There are further topologies of more general type; cf. [SGA 3_{I}], Exp. IV, 6.3.1. We mention the fpqc-topology, the fppf-topology, and the étale topology. If top is any of the abbreviations

fpqc (= faithfully flat and quasi-compact),

fppf (= faithfully flat and of finite presentation), or

ét (= étale surjective),

we write $\mathfrak{M}_{\mathrm{top}}$ for the class of all morphisms in (Sch/S) which are of type top and say that a functor $F : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ is a sheaf with respect to the top-topology (or, simply, with respect to top), if it is a sheaf with respect to both $\mathfrak{M}_{\mathrm{Zar}}$ *and* $\mathfrak{M}_{\mathrm{top}}$.

Proposition 1. *Let F be a representable contravariant functor from* (Sch/S) *to* (Sets). *Then F is a sheaf with respect to* fpqc *and, hence, with respect to* fppf, ét, *and* Zar.

Proof. If F is represented by an S-scheme X, we have $F(T) = \mathrm{Hom}_S(T, X)$. Since morphisms to X can be defined locally, it follows for any open covering $\{T_i\}$ of T that the canonical sequence

$$\mathrm{Hom}_S(T, X) \longrightarrow \prod_i \mathrm{Hom}_S(T_i, X) \rightrightarrows \prod_{i,j} \mathrm{Hom}_S(T_i \times_T T_j, X)$$

is exact. So F is a sheaf with respect to the Zariski topology.

Furthermore, for any S-morphism $T' \longrightarrow T$ which is fpqc, the canonical sequence

$$\mathrm{Hom}_S(T, X) \longrightarrow \mathrm{Hom}_S(T', X) \rightrightarrows \mathrm{Hom}_S(T'', X)$$

is exact; namely, it is isomorphic to the sequence

$$\mathrm{Hom}_T(T, X_T) \longrightarrow \mathrm{Hom}_{T'}(T', X_{T'}) \rightrightarrows \mathrm{Hom}_{T''}(T'', X_{T''})$$

which, by descent theory, is exact, as shown in the proof of 6.1/6. Thus F is a sheaf with respect to fpqc. $\square$

Returning to the functor

$$P_{X/S} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}), \qquad T \longmapsto \mathrm{Pic}(X \times_S T),$$

it is easily seen that, in general, $P_{X/S}$ is not a sheaf, even with respect to the Zariski topology. As a consequence, we cannot expect its representability. Indeed, if $P_{X/S}$ were a sheaf with respect to the Zariski topology, a line bundle on $X \times_S T$ would be trivial as soon as it trivializes over (the pull-back of) an open covering of T.

However, this is not the case. So if we want to deal with a functor from which representability can be expected, we have to sheafify $P_{X/S}$; this can be done by using standard methods from sheaf theory.

In order to explain the procedure of sheafification, let us, again, consider a functor $F : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ and a class $\mathfrak{M}$ of morphisms in (Sch/S) which is stable under composition and under fibred products and which contains all isomorphisms. To give a sheafification of F (within the context of sheaves with respect to $\mathfrak{M}$) means to construct a morphism $F \longrightarrow F^\dagger$ into a sheaf $F^\dagger$ such that each morphism from F into an arbitrary sheaf G (always with respect to $\mathfrak{M}$) admits a unique factorization through $F^\dagger$. The construction of $F^\dagger$ is straightforward. Let $T' \longrightarrow T$ be a morphism in $\mathfrak{M}$ and denote by $\bar{H}^0(T'/T, F)$ the subset of $F(T')$ consisting of all elements ξ which are characterized by the following property: if ξ_1 and ξ_2 are the "pull-backs" of ξ with respect to the two projections from $T'' = T' \times_T T'$ onto T', there is a morphism $\tilde{T} \longrightarrow T''$ in $\mathfrak{M}$ such that the images of ξ_1 and ξ_2 with respect to $F(T'') \longrightarrow F(\tilde{T})$ coincide in $F(\tilde{T})$. If T' varies over (Sch/S), the sets $\bar{H}^0(T'/T, F)$ form an inductive system. Provided $\mathfrak{M}$ is not "too big", the direct limit of this system exists, and we can set

$$F^\dagger(T) := \varinjlim \bar{H}^0(T'/T, F) .$$

It is verified without difficulties that $F^\dagger$ is a sheaf with respect to $\mathfrak{M}$ and that the canonical morphism $F \longrightarrow F^\dagger$ defines $F^\dagger$ as a sheafification of F.

The direct limits which have been used to define the sheaf $F^\dagger$ exist if we take for $\mathfrak{M}$ any of the classes $\mathfrak{M}_{\mathrm{Zar}}$, $\mathfrak{M}_{\mathrm{\acute{e}t}}$, or $\mathfrak{M}_{\mathrm{fppf}}$, whereas in the case $\mathfrak{M} = \mathfrak{M}_{\mathrm{fpqc}}$ some precautionary measures, like working in a fixed universe, are necessary. However, since the class $\mathfrak{M}_{\mathrm{fpqc}}$ is quite big, it may happen that sheafifications with respect to $\mathfrak{M}_{\mathrm{fpqc}}$ depend on the choice of the universe. It is for this reason that, when working with sheafifications, we will generally use the class $\mathfrak{M}_{\mathrm{fppf}}$ instead of $\mathfrak{M}_{\mathrm{fpqc}}$.

Now, in order to construct a sheafification of the functor

$$P_{X/S} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}), \qquad T \longmapsto \mathrm{Pic}(X \times_S T),$$

say with respect to the fppf-topology, one first sheafifies $P_{X/S}$ with respect to $\mathfrak{M}_{\mathrm{fppf}}$. The resulting sheaf P_1 might not be a sheaf with respect to $\mathfrak{M}_{\mathrm{Zar}}$ since morphisms in $\mathfrak{M}_{\mathrm{Zar}}$ are not necessarily quasi-compact and, thus, not necessarily fppf. However, if T is affine, any morphism $\coprod T_i \longrightarrow T$ in $\mathfrak{M}_{\mathrm{Zar}}$ which corresponds to a finite open covering $\{T_i\}$ of T by basic open subschemes $T_i \subset T$ is fppf. Hence P_1 is already an fppf-sheaf on affine schemes. Therefore we can sheafify P_1 with respect to $\mathfrak{M}_{\mathrm{Zar}}$ without destroying sheaf properties with respect to $\mathfrak{M}_{\mathrm{fppf}}$ on affine schemes. It follows that the resulting functor is a sheaf with respect to the fppf-topology; it is the fppf-*sheaf associated to* $P_{X/S}$. Since $P_{X/S}$ is a group functor, the associated fppf-sheaf can be viewed as a group functor, too. In the same way, one can proceed with any other of the topologies introduced above.

Definition 2. *The* fppf-*sheaf associated to the functor*

$$P_{X/S} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}), \qquad T \longmapsto \mathrm{Pic}(X \times_S T),$$

is called the relative Picard functor of X over S; it is denoted by $\mathrm{Pic}_{X/S}$. *For any S-scheme T, we call* $\mathrm{Pic}_{X/S}(T)$ *the relative Picard group of* $X \times_S T$ *over T.*

Using the structural morphism $f: X \longrightarrow S$ as well as the notion of higher direct images of f, we can define the relative Picard functor also by the formula

$$\mathrm{Pic}_{X/S}(T) = H^0(T, R^1 f_*(\mathbb{G}_m))$$

which has to be read with respect to the fppf-topology; note that $\mathbb{G}_m$ is the sheaf which associates to each scheme Z the group of units $\Gamma(Z, \mathcal{O}_Z^*)$. We will see below that the restriction to the fppf-topology in place of the fpqc-topology is not too serious since we are mainly interested in the case where $f: X \longrightarrow S$ is proper and fppf.

Sometimes it is useful to have an explicit description of elements of relative Picard groups. So consider an element $\xi \in \mathrm{Pic}_{X/S}(S)$ and assume for simplicity that S is affine or, more generally, quasi-compact. Otherwise one has to work locally with respect to an open affine covering of S. Then, in the quasi-compact case, ξ is represented by a line bundle $\xi' \in \mathrm{Pic}(X \times_S S')$ where S' is fppf over S. Furthermore, there must be an fppf-morphism $\tilde{S} \longrightarrow S'' = S' \times_S S'$ such that the pull-back of ξ' with respect to $\tilde{S} \longrightarrow S'' \longrightarrow S'$ is the same for both projections from S'' to S'. Conversely, each $\xi' \in \mathrm{Pic}(X \times_S S')$ satisfying the latter condition gives rise to an element $\xi \in \mathrm{Pic}_{X/S}(S)$. Two such elements $\xi_i' \in \mathrm{Pic}(X \times_S S_i')$, $i = 1, 2$, with S_i' fppf over S represent the same element $\xi \in \mathrm{Pic}_{X/S}(S)$ if and only if there exists an fppf-morphism $\tilde{S} \longrightarrow S_1' \times_S S_2'$ such that, on $\tilde{S}$, the pull-back of ξ_1' coincides with the pull-back of ξ_2'. Also it should be noted that, due to the sheaf property of $\mathrm{Pic}_{X/S}$, an element $\xi \in \mathrm{Pic}_{X/S}(S)$ is trivial if it is induced by the pull-back to X of a line bundle on S. The converse is not true, in general.

Proposition 3. *Assume that* $f: X \longrightarrow S$ *is proper and of finite presentation. Consider an element* $\xi \in \mathrm{Pic}_{X/S}(S)$ *which is induced by a line bundle* $\mathcal{L}$ *on X. Then* ξ *is trivial if and only if there is an open covering* $\{S_i\}$ *of S such that* $\mathcal{L}$ *trivializes over* $X \times_S S_i$ *for each i.*

Proof. The if-part of the assertion follows from the sheaf properties of $\mathrm{Pic}_{X/S}$. So it remains to justify the only-if-part. The direct image $f_*(\mathcal{O}_X)$ is a quasi-coherent $\mathcal{O}_S$-algebra. Assuming S to be affine and interpreting $f: X \longrightarrow S$ as a limit of morphisms of finite type between noetherian schemes, we can use the Stein factorization $X \xrightarrow{g} T \xrightarrow{h} S$ of f, where g satisfies $g_*(\mathcal{O}_X) = \mathcal{O}_T$ and where h, being a limit of finite morphisms, is integral. Furthermore, since the fibres of g are the connected components of the fibres of f, it follows that the fibres of h are set-theoretically finite. Now assume that $\mathcal{L}$ gives rise to the trivial element $\xi \in \mathrm{Pic}_{X/S}(S)$. We claim that the canonical homomorphism $g^*(g_*(\mathcal{L})) \longrightarrow \mathcal{L}$ is an isomorphism. Using descent, this fact can be tested after base change with an fppf-morphism. For example, we can assume that, after such a base change, $\mathcal{L}$ becomes trivial. Since the formation of $g_*(\mathcal{L})$ commutes with flat base change, the above isomorphism has only to be established for the trivial bundle $\mathcal{L}$. But then the claim follows from the fact that $g_*(\mathcal{O}_X) = \mathcal{O}_T$. So we see that $\mathcal{L}$ is the pull-back of the line bundle $g_*(\mathcal{L})$ on T. The latter is locally trivial over T. Since $h: T \longrightarrow S$ is integeral and, thus, a closed map, and since its fibres are set-theoretically finite, it follows that $g_*(\mathcal{L})$ is locally trivial also over S. Hence $\mathcal{L}$ is locally trivial over S. □

We assume in the following that $f: X \longrightarrow S$ is quasi-compact and quasi-separated. Then the Leray spectral sequence associated to f and $\mathbb{G}_m$ (see [SGA 4_{II}], Exp. V, § 3) gives the exact sequence

$$0 \longrightarrow H^1(S, f_*(\mathbb{G}_m)) \longrightarrow H^1(X, \mathbb{G}_m) \longrightarrow \mathrm{Pic}_{X/S}(S) \longrightarrow H^2(S, f_*(\mathbb{G}_m)) \longrightarrow H^2(X, \mathbb{G}_m)$$

where the cohomology groups are meant with respect to the fppf-topology. Since the descent with respect to fpqc-morphisms turns line bundles into line bundles, it follows that the group $H^1(X, \mathbb{G}_m)$ is the same for the fpqc-, the fppf-, the étale, and even for the Zariski topology. So we may use the Zariski topology and see $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$. Thus the obstruction of representing an element of $\mathrm{Pic}_{X/S}(S)$ by an element of $\mathrm{Pic}(X)$ is given by an element in $H^2(S, f_*(\mathbb{G}_m))$ which becomes zero in $H^2(X, \mathbb{G}_m)$. Just as in the case of $H^1(X, \mathbb{G}_m)$, one shows that $H^1(S, f_*(\mathbb{G}_m))$ is independent of the topologies mentioned above if $f_*(\mathcal{O}_X) = \mathcal{O}_S$ or, by means of the Stein factorization, if f is proper. In particular, we have $H^1(S, f_*(\mathbb{G}_m)) = \mathrm{Pic}(S)$ if $f_*(\mathcal{O}_X) = \mathcal{O}_S$.

In order to determine the cohomology group $H^2(X, \mathbb{G}_m)$, one can use the étale topology instead of the fppf-topology; cf. Grothendieck [3], pp. 171–183. The same is true for the cohomology group $H^2(S, f_*(\mathbb{G}_m))$ if $f_*(\mathcal{O}_X) = \mathcal{O}_S$ or, without this assumption, if f is proper. Namely, by means of the Stein factorization, it is possible to reduce to the case where $f_*(\mathcal{O}_X) = \mathcal{O}_S$. So, for example, if f is proper, the above exact sequence shows that the relative Picard functor $\mathrm{Pic}_{X/S}$ can be constructed by using the étale topology in place of the fppf-topology. In particular, the formula

$$\mathrm{Pic}_{X/S}(T) = H^0(T, R^1 f_*(\mathbb{G}_m))$$

remains valid if, on the right-hand side the fppf-topology is replaced by the étale topology.

The cohomology group $H^2(X, \mathbb{G}_m)$ is called the (cohomological) *Brauer group* of X. In particular, if we assume $f_*(\mathcal{O}_X) = \mathcal{O}_S$, the obstructions of representing elements in $\mathrm{Pic}_{X/S}(S)$ by line bundles on X are given by elements of the Brauer group $\mathrm{Br}(S)$ which become zero in the Brauer group $\mathrm{Br}(X)$. All obstructions of this type disappear if the map $H^2(S, \mathbb{G}_m) \longrightarrow H^2(X, \mathbb{G}_m)$ is injective; for example, if $f: X \longrightarrow S$ has a section or if the Brauer group $\mathrm{Br}(S)$ vanishes itself. For an affine scheme $S = \mathrm{Spec}\, R$, the group $Br(S)$ is zero in each of the following situations:

(a) R is a separably closed field.

(b) R is the field of fractions of a henselian discrete valuation ring with algebraically closed residue field; cf. Grothendieck [3], Thm. 1.1, or Milne [1], Chap. III, 2.22.

(c) R is a strictly henselian valuation ring; cf. Grothendieck [3], Prop. 2.1, or Milne [1], Chap. IV, 1.7 and 2.12.

The equation $f_*(\mathcal{O}_X) = \mathcal{O}_S$ is compatible with flat base change. We say that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds *universally* if the equation is true after *any* base change over S. Using this terminology, we want to summarize the above considerations.

Proposition 4. *Let $f: X \longrightarrow S$ be quasi-compact and quasi-separated and assume that f satisfies $f_*(\mathcal{O}_X) = \mathcal{O}_S$ (resp. that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally). Then, for each S-scheme T which is flat over S (resp. for each S-scheme T), the canonical sequence*

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(X \times_S T) \longrightarrow \mathrm{Pic}_{X/S}(T) \longrightarrow \mathrm{Br}(T) \longrightarrow \mathrm{Br}(X \times_S T)$$

is exact. If, in addition, f admits a section, the sequence

$$0 \longrightarrow \mathrm{Pic}(T) \longrightarrow \mathrm{Pic}(X \times_S T) \longrightarrow \mathrm{Pic}_{X/S}(T) \longrightarrow 0$$

is exact.

In particular, in the latter case, we can identify the relative Picard functor $\mathrm{Pic}_{X/S}$ in the usual way with the functor

$$(\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}), \qquad T \longmapsto \mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T).$$

If the existence of a global section is replaced by the condition that $f: X \longrightarrow S$ has local sections, one can still say that the formula

$$\mathrm{Pic}_{X/S}(T) = H^0(T, R^1 f_*(\mathbb{G}_m))$$

remains valid if one considers the Zariski topology on the right-hand side.

In order to see, in the above situation, that the relative Picard functor $\mathrm{Pic}_{X/S}$ is a sheaf even with respect to the fpqc-topology and in order to prepare the discussion of rigidificators, we want to look at the situation from another point of view. We assume that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally and that f admits a section $\varepsilon: S \longrightarrow X$. For any line bundle $\mathcal{L}$ on X, let us call an isomorphism $\alpha: \mathcal{O}_S \xrightarrow{\sim} \varepsilon^*(\mathcal{L})$ a *rigidification* of $\mathcal{L}$. Furthermore, the pair $(\mathcal{L}, \alpha)$ will be referred to as a line bundle which is rigidified along the section ε. Then we can look at the functor $(P, \varepsilon): (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ which associates to each S-scheme T the set $(P, \varepsilon)(T)$ of isomorphism classes of line bundles on $X_T = X \times_S T$ which are rigidified along the section $\varepsilon_T: T \longrightarrow X_T$. The functor (P, ε) has the advantage that it is automatically a sheaf with respect to the Zariski topology. Namely, using the fact that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ is true universally, one shows easily that rigidified line bundles do not admit non-trivial automorphisms; hence the terminology of rigidifications is justified. Furthermore, it follows from descent theory that (P, ε) is a sheaf even with respect to the fpqc-topology. Namely, consider a sequence

$$(P, \varepsilon)(T) \longrightarrow (P, \varepsilon)(T') \rightrightarrows (P, \varepsilon)(T''),$$

where $T' \longrightarrow T$ is an fpqc-morphism and where $T'' = T' \times_T T'$. The map on the left-hand side is injective by 6.1/4. To show the exactness of the sequence, fix an element $(\mathcal{L}', \alpha') \in (P, \varepsilon)(T')$ whose images in $(P, \varepsilon)(T'')$ coincide. Then we have an isomorphism $p_1^* \mathcal{L}' \xrightarrow{\sim} p_2^* \mathcal{L}'$ between the two pull-backs of $\mathcal{L}'$ to T'' which is compatible with rigidifications. Hence this isomorphism is automatically a descent datum, and the descent is effective by 6.1/4. Thus the above sequence is exact, and (P, ε) is a sheaf with respect to fpqc. For each line bundle $\mathcal{L}$ on X, the bundle $\mathcal{L} \otimes f^*(\varepsilon^*(\mathcal{L}^{-1}))$ has a rigidification. Therefore we have

$$(P, \varepsilon)(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$$

for all S-schemes T. Since (P, ε) is a sheaf with respect to the fpqc-topology and, thus, with respect to the fppf-topology, it is canonically isomorphic to the relative Picard functor $\mathrm{Pic}_{X/S}$. Thereby we see once more that the second assertion of Proposition 4 is true.

Using the above argument, it can easily be shown that the relative Picard functor $\mathrm{Pic}_{X/S}$ which has been defined within the framework of the fppf-topology is even a sheaf with respect to the fpqc-topology, provided $f: X \longrightarrow S$ is fppf and satisfies $f_*(\mathcal{O}_X) = \mathcal{O}_S$ universally. Namely, we may perform a base change with X over S and thereby assume that f has a section. Then, by considering rigidifications, it follows that $\mathrm{Pic}_{X/S}$ is a sheaf with respect to fpqc.

If the assumptions that the equation $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally and that there is a section $\varepsilon: S \longrightarrow X$ are not satisfied, it is sometimes useful to introduce a generalization of the notion of rigidifications so that, similarly as above, one can deal with rigidified line bundles.

Definition 5. *Let $f: X \longrightarrow S$ be proper, flat, and of finite presentation. Then a subscheme $Y \subset X$, which is finite, flat, and of finite presentation over S, is called a rigidificator of f or, more precisely, of the relative Picard functor $\mathrm{Pic}_{X/S}$ if*

$$(\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}), \qquad T \longmapsto \Gamma(X_T, \mathcal{O}_{X_T}),$$

is a subfunctor of the functor

$$(\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}), \qquad T \longmapsto \Gamma(Y_T, \mathcal{O}_{Y_T});$$

i.e., if the map $\Gamma(X_T, \mathcal{O}_{X_T}) \longrightarrow \Gamma(Y_T, \mathcal{O}_{Y_T})$, which is derived from the inclusion $Y_T \hookrightarrow X_T$, is injective for all S-schemes T.

If $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally, it is immediately clear that, for each section $\varepsilon: S \longrightarrow X$ of f, the closed subscheme $\varepsilon(S) \subset X$ is a rigidificator of f. Furthermore, let us mention without proof two non-trivial examples where rigidifications exist; cf. Raynaud [6], Prop. 2.2.3.

Proposition 6. *Let $f: X \longrightarrow S$ be as in Definition 5.*

(a) *If the fibres of f do not have embedded components, f admits a rigidificator locally over S with respect to the étale topology.*

(b) *If S is the spectrum of a discrete valuation ring, f has a rigidificator.*

Let Y be a rigidificator of $f: X \longrightarrow S$. Then an invertible sheaf on X which is rigidified along Y is defined as a pair $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible sheaf on X, and where α is an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}_{|Y}$. Rigidified line bundles do not admit non-trivial automorphisms. Therefore the functor

$$(\mathrm{Pic}_{X/S}, Y): (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}),$$

which associates to an arbitrary S-scheme T the set of isomorphism classes of line bundles on X_T which are rigidified along Y_T, is a sheaf with respect to the Zariski topology and, by descent theory, even with respect to the fpqc-topology. Furthermore, $(\mathrm{Pic}_{X/S}, Y)$ is canonically a group functor.

In order to relate the functor $(\mathrm{Pic}_{X/S}, Y)$ to the relative Picard functor $\mathrm{Pic}_{X/S}$, it is necessary to look at rigidificators from another point of view. However, before we can do this, we have to discuss a basic result on the direct image of $\mathcal{O}_X$-modules which are locally of finite presentation; by the latter we mean (quasi-coherent) $\mathcal{O}_X$-modules which, locally, are isomorphic to the cokernel of a homomorphism of type $\mathcal{O}_X^m \longrightarrow \mathcal{O}_X^n$. Furthermore, we need the concept of cohomological flatness. Assume that $f: X \longrightarrow S$ is proper and of finite presentation, and consider an $\mathcal{O}_X$-module $\mathcal{F}$ of locally finite presentation, which is flat over S. Then $\mathcal{F}$ is said to be *cohomologically flat over S in dimension* 0 if the formation of the direct image $f_*(\mathcal{F})$ commutes with base change. If the condition is true for $\mathcal{F} = \mathcal{O}_X$, we say that f itself is cohomologically flat in dimension 0. The latter is the case if f is flat and if the geometric fibres of f are reduced; cf. [EGA III_2], 7.8.6.

Theorem 7. *Let $f: X \longrightarrow S$ be a proper morphism which is finitely presented. Furthermore, let $\mathcal{F}$ be an $\mathcal{O}_X$-module of locally finite presentation which is S-flat. Then there exists an $\mathcal{O}_S$-module $\mathcal{Q}$ of locally finite presentation, unique up to canonical isomorphism, such that there is an isomorphism of functors*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{M}),$$

which is functorial for all quasi-coherent $\mathcal{O}_S$-modules $\mathcal{M}$. In particular, there is an isomorphism of functors

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{M}).$$

The $\mathcal{O}_S$-module $\mathcal{Q}$ is locally free if and only if $\mathcal{F}$ is cohomologically flat over S in dimension 0. *In the latter case, $\mathcal{Q}$ and $f_*(\mathcal{F})$ are dual to each other and, in particular, $f_*(\mathcal{F})$ is locally free.*

We will not repeat the *proof* of the theorem from [EGA III_2], 7.7.6. But to give some idea, we want to show how the assertions follow from the theorem on cohomology and base change as contained in Mumford [3], Chap. II, § 5. We may assume that S is affine, say $S = \operatorname{Spec} A$. Then the theorem on cohomology and base change says there is a finite complex

$$K^{\cdot}: 0 \longrightarrow K^0 \xrightarrow{\varphi} K^1 \longrightarrow K^2 \longrightarrow \ldots \longrightarrow K^n \longrightarrow 0$$

of finitely generated projective A-modules (we may assume of free A-modules, after restriction of S) as well as an isomorphism of functors

$$H^p(X, \mathcal{F} \otimes_A M) \simeq H^p(K^{\cdot} \otimes_A M), \qquad p \geq 0,$$

on the category of A-modules M. (Using Mumford's version of the base change, one has remove the noetherian hypothesis by a limit argument; furthermore, the above functors have to be considered on the category of all A-modules M and not just on the category of all A-algebras B.) Dualizing the map $\varphi: K^0 \longrightarrow K^1$ gives an exact sequence

$$0 \longleftarrow \operatorname{coker} \varphi^* \longleftarrow (K^0)^* \xleftarrow{\varphi^*} (K^1)^*,$$

and we claim there is a functorial isomorphism

$$(*) \qquad H^0(K^{\cdot} \otimes_A M) = \ker(\varphi \otimes M) \xrightarrow{\sim} \operatorname{Hom}_A(\operatorname{coker} \varphi^*, M)$$

of functors in M. Namely, applying the functor $\operatorname{Hom}_A(\cdot, M)$, which is left-exact, to the preceding exact sequence yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(\operatorname{coker} \varphi^*, M) \longrightarrow \operatorname{Hom}_A((K^0)^*, M) \longrightarrow \operatorname{Hom}_A((K^1)^*, M).$$

Then we compare it with the exact sequence

$$0 \longrightarrow \ker(\varphi \otimes M) \longrightarrow K^0 \otimes_A M \xrightarrow{\varphi \otimes M} K^1 \otimes_A M.$$

The canonical homomorphisms $K^i \otimes_A M \longrightarrow \operatorname{Hom}_A((K^i)^*, M)$, $i = 1, 2$, are isomorphisms since K^0 and K^1 are free, and there is an isomorphism

$$H^0(K^{\cdot} \otimes_A M) \xrightarrow{\sim} \operatorname{Hom}_A(\operatorname{coker} \varphi^*, M),$$

which is functorial in M. Hence the existence of the functorial isomorphism $(*)$ is proved. Writing $Q = \operatorname{coker} \varphi^*$ and using the theorem on cohomology and base change, the resulting functorial isomorphism

$$H^0(X, \mathcal{F} \otimes_A M) \xrightarrow{\sim} \operatorname{Hom}_A(Q, M)$$

implies the main assertion of our theorem. Since the tensor product is right-exact and since Hom is left-exact, the isomorphism $(*)$ shows that $\mathcal{F}$ is cohomologically flat over S in dimension 0 if and only if $Q = \operatorname{coker} \varphi^*$ is a projective, i.e., locally free A-module. If the latter is the case, $\ker \varphi$ is locally free since it is the dual of $\operatorname{coker} \varphi^*$. □

If $f: X \longrightarrow S$ is proper, finitely presented, and flat, the assertion of the above theorem holds for the $\mathcal{O}_X$-module $\mathcal{F} = \mathcal{O}_X$. Restricting the resulting functorial isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{M})$$

to quasi-coherent $\mathcal{O}_S$-modules of type $\mathcal{M} = \mathcal{O}_T$ which are obtained from morphisms $T \longrightarrow S$, one ends up with functorial isomorphisms

$$\Gamma(X_T, \mathcal{O}_{X_T}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{O}_T) \xrightarrow{\sim} \operatorname{Hom}_S(T, V)$$

where V denotes the S-scheme corresponding to the symmetric $\mathcal{O}_S$-algebra $\mathcal{S}ym_{\mathcal{O}_S}(\mathcal{Q})$ of $\mathcal{Q}$. Dropping the middle term, we get a functorial isomorphism between functors on the category of all S-schemes T. The scheme V is also referred to as the *total space of the module* $\mathcal{Q}$. We say that V is locally free if this is true for $\mathcal{Q}$ as an $\mathcal{O}_S$-module. The latter is equivalent to the fact that V is smooth over S. So we can state the following result.

Corollary 8. *Let $f: X \longrightarrow S$ be proper, finitely presented, and flat, and let $\mathcal{Q}$ be the $\mathcal{O}_S$-module associated to $f_*(\mathcal{O}_X)$ in the sense of Theorem 7. Then the functor*

$$(\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}), \qquad T \longmapsto \Gamma(X_T, \mathcal{O}_{X_T})$$

is represented by the total space V of $\mathcal{Q}$. Furthermore, V is locally free if and only if f is cohomologically flat in dimension 0.

If, in addition to the above assumptions, f is finite, it is automatically cohomologically flat in dimension 0. In particular, the functor of global sections of a rigidificator is always represented by the total space of a module which is locally free. Using the assertion of the corollary, we can give a further characterization of rigidificators.

Proposition 9. *Let $f: X \longrightarrow S$ be proper, finitely presented, and flat, and consider a subscheme $Y \subset X$ which is finite, flat, and of finite presentation over S. Let V_X (resp. V_Y) be the S-scheme, which, as in Corollary 8, represents the functor of global sections on X (resp. Y). Then the following conditions are equivalent:*

(a) *Y is a rigidificator of f.*

(b) *The morphism $V_X \longrightarrow V_Y$, which is induced by the inclusion $Y \hookrightarrow X$, is a closed immersion.*

Proof. Let $\mathcal{Q}$ (resp. $\mathcal{Q}'$) denote the $\mathcal{O}_S$-module which is obtained by means of Theorem 7 from $f: X \longrightarrow S$ (resp. $Y \longrightarrow S$). Then, for all S-schemes T such that $\mathcal{O}_T$ is a quasi-coherent $\mathcal{O}_S$-module, the inclusion $Y \hookrightarrow X$ gives rise to a sequence

$$(*) \qquad 0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{O}_T) \longrightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{Q}', \mathcal{O}_T)\,.$$

The latter is exact for all T if and only if Y is a rigidificator of f. Now the sequence $(*)$ corresponds to a sequence

$$(**) \qquad \mathcal{Q}' \longrightarrow \mathcal{Q} \longrightarrow 0$$

of $\mathcal{O}_X$-modules which is exact if and only if $(*)$ is exact for all T. On the other hand, $(**)$ yields a sequence between associated symmetric $\mathcal{O}_S$-algebras

$$(***) \qquad \mathcal{S}ym_{\mathcal{O}_S}(\mathcal{Q}') \longrightarrow \mathcal{S}ym_{\mathcal{O}_S}(\mathcal{Q}) \longrightarrow 0$$

which is exact if and only if it is exact in degree 1, i.e., if and only if $(**)$ is exact. This verifies the assertion of the proposition. □

As before, let $f: X \longrightarrow S$ be proper, finitely presented, and flat, and let V be the S-scheme representing the functor $T \longmapsto \Gamma(X_T, \mathcal{O}_{X_T})$ of global sections on X. Then V may be viewed as a functor to the category of rings and thus is a ring scheme. We claim:

Lemma 10. *The subfunctor of units $T \longmapsto \Gamma(X_T, \mathcal{O}_{X_T}^*)$ is represented by an open subscheme $V^* \subset V$. In particular, V^* is a group scheme.*

Proof. The assertion is clear if f is cohomologically flat in dimension 0. Namely, then V is locally free and we can use a norm argument. In the general case, one views V and V^* as functors and shows that the injection $V^* \hookrightarrow V$ is relatively representable by open immersions. In order to do this, consider an S-scheme T and a T-valued point $g: T \longrightarrow V$ as well as the associated cartesian diagram

$$\begin{tikzcd} V^* \times_V T \arrow[r, hook] \arrow[d] & T \arrow[d, "g"] \\ V^* \arrow[r, hook] & V\,. \end{tikzcd}$$

Then g corresponds to a global section in the structure sheaf of $X \times_S T$. Let U' be the maximal open subset of $X \times_S T$ where g is invertible. Since f is proper, the complement of U' projects onto a closed subset F of T. Therefore its complement $U := T - F$ is an open subscheme of T, and it is easily verified that $V^* \times_V T \longrightarrow T$ is represented by the open immersion $U \hookrightarrow T$. □

The canonical map $\mathcal{O}_S \longrightarrow f_*(\mathcal{O}_X)$ defines a morphism $\mathbb{G}_a \longrightarrow V$ which is a closed immersion as can be seen by using arguments as in the proof of Proposition 9. Restricting to the subschemes of units yields an immersion of group schemes $\mathbb{G}_m \longrightarrow V^*$ which is a closed immersion again. It is easily seen that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ holds universally if and only if the map $\mathbb{G}_a \longrightarrow V$ or, equivalently, the map $\mathbb{G}_m \longrightarrow V^*$ is an isomorphism.

Finally, let Y be a rigidificator of $f: X \longrightarrow S$ and, as in Proposition 9, let V_X and V_Y denote the schemes representing the functors of global sections on X and on Y. Then the closed immersion $V_X \hookrightarrow V_Y$ gives rise to an immersion $V_X^* \hookrightarrow V_Y^*$, and there is a canonical map $V_Y^* \longrightarrow (\mathrm{Pic}_{X/S}, Y)$ to the Picard functor $(\mathrm{Pic}_{X/S}, Y)$ of line bundles which are rigidified along Y. Namely, fixing an S-scheme T, a global invertible section a on $Y \times_S T$ is mapped to the pair $(\mathcal{O}_{X_T}, \alpha)$ where the isomorphism $\alpha: \mathcal{O}_{X_T|Y_T} \xrightarrow{\sim} \mathcal{O}_{X_T|Y_T}$ is the multiplication by a. Adding the canonical map $(\mathrm{Pic}_{X/S}, Y) \longrightarrow \mathrm{Pic}_{X/S}$, one obtains the sequence

$$0 \longrightarrow V_X^* \hookrightarrow V_Y^* \longrightarrow (\mathrm{Pic}_{X/S}, Y) \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow 0\,.$$

Proposition 11. *The preceding sequence is exact in terms of sheaves with respect to the étale topology.*

The *proof* is straightforward; see Raynaud [6], 2.1.2 and 2.4.1. It is shown in the same article that $(\mathrm{Pic}_{X/S}, Y)$ is representable by an algebraic space; cf. our discussion of the representability of Picard functors in 8.3. Thus, even if $\mathrm{Pic}_{X/S}$ is not representable (by a scheme or by an algebraic space), but if there exists a rigidificator Y, there is a representable object which closely dominates the relative Picard functor.

8.2 Representability by a Scheme

There are two types of results concerning the representability of the relative Picard functor $\mathrm{Pic}_{X/S}$; namely, results on the representability by schemes and results on the representability by algebraic spaces. If one wants $\mathrm{Pic}_{X/S}$ to be a scheme, one has to ask strong conditions for the structural morphism $f: X \longrightarrow S$ whereas, if one allows to work more generally within the context of algebraic spaces, one can obtain the representability of $\mathrm{Pic}_{X/S}$ by an algebraic space under conditions which are not so restrictive and quite natural to ask.

In the present section, we will give an outline of Grothendieck's method for representing $\mathrm{Pic}_{X/S}$ by a scheme and, in the next section, we will roughly explain the idea of M. Artin's approach for representing $\mathrm{Pic}_{X/S}$ by an algebraic space. Let us start by stating the main results on the representability of $\mathrm{Pic}_{X/S}$ by a scheme.

Theorem 1 (Grothendieck [FGA], n°232, Thm. 3.1). *Let $f: X \longrightarrow S$ be projective and finitely presented. Assume that f is flat, and that the geometric fibres of f are reduced and irreducible. Then $\mathrm{Pic}_{X/S}$ is representable by a separated S-scheme which is locally of finite presentation over S.*

The proof of Theorem 1 consists mainly of methods from projective geometry. If one replaces the condition "projective" by "proper", these methods are not applicable for a general base S. Furthermore, the assumption on the fibres of f is an inevitable technical condition without which the proof cannot work. It is the very reason for getting representability by a scheme and for the fact that the representing S-scheme is separated.

To illustrate this point, let us look at an *example of Mumford.* He considered a projective flat family of geometrically reduced curves where $\mathrm{Pic}_{X/S}$ does not exist as a scheme. Namely let $S = \mathrm{Spec}\,\mathbb{R}[[t]]$, and let X be the S-subscheme of $\mathbb{P}^2_S$ given by the equation $X_1^2 + X_2^2 = tX_0^2$. One may view X as a conic which geometrically degenerates into two projective lines. The special fibre over the closed point of S is irreducible whereas, after the base change with $S' = \mathrm{Spec}\,\mathbb{C}[[t]]$, it decomposes into two lines which are conjugated under the Galois group $\mathbb{Z}/2\mathbb{Z}$ of S' over S. We claim that the Picard functor $\mathrm{Pic}_{X'/S'}$ is a scheme. Indeed, it is a disjoint union of subschemes representing the subfunctors $\mathrm{Pic}^d_{X'/S'}$, $d \in \mathbb{Z}$, of $\mathrm{Pic}_{X'/S'}$ which are given by line bundles of total degree d. Furthermore, each $\mathrm{Pic}^d_{X'/S'}$ is obtained by gluing copies of S' along the generic point; namely by gluing copies $S'_{a,b}$ with $a, b \in \mathbb{Z}$ and $a + b = d$ where the decompositions $d = a + b$ correspond to the possibilities of degenerations of a line bundle of degree d on the generic fibre into a line bundle with partial degrees a and b on the components of the special fibre. In particular, $\mathrm{Pic}_{X'/S'}$ is not separated and there are orbits of the Galois action on $\mathrm{Pic}_{X'/S'}$ which are not contained in an open affine subscheme. So, the descent datum given by the Galois action cannot be effective, and hence $\mathrm{Pic}_{X/S}$ is not representable by a scheme over S. A closer look at this example shows that the very reason for this is the fact that the irreducible components of the fibres of f are not geometrically irreducible. The same can be read from the following generalization of Grothendieck's result:

Thm 1: geom. irred. Fasern
Thm 2: schwächer: Komponenten der Fasern geom. irred.

Theorem 2 (Mumford, unpublished). *Let $f: X \longrightarrow S$ be flat, projective, and finitely presented with geometrically reduced fibres. Assume that the irreducible components of the fibres of f are geometrically irreducible. Then $\mathrm{Pic}_{X/S}$ is representable by a (not necessarily separated) S-scheme which is locally of finite presentation over S.*

If the base scheme S is a field, one can prove the representability of $\mathrm{Pic}_{X/S}$ under weaker assumptions than those mentioned in Theorem 1. This was first done by Grothendieck for the projective case; cf. [FGA], n°232, Sect. 6. Later on Murre and Oort treated the proper case.

Theorem 3 (Murre [1] and Oort [1]). *Let X be a proper scheme over a field k. Then $\mathrm{Pic}_{X/k}$ is representable by a scheme which is locally of finite type over k.*

The theorem of Murre can also be deduced from the results on the representability of $\mathrm{Pic}_{X/S}$ by an algebraic space; cf. Section 8.3. Namely, a group object in the category of algebraic spaces over a field is representable by a scheme.

Finally, we want to introduce the notion of universal line bundles which is quite convenient to work with when $\mathrm{Pic}_{X/S}$ is representable. We assume that the structural morphism $f: X \longrightarrow S$ has a section ε and that $f_*\mathcal{O}_X = \mathcal{O}_S$ holds universally. In this case $\mathrm{Pic}_{X/S}$ is isomorphic to the functor

$$(P, \varepsilon) : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$$

which associates to each S-scheme S' the set of isomorphism classes of line bundles on $X' = X \times_S S'$ which are rigidified along the induced section $\varepsilon' = \varepsilon \otimes \mathrm{id}_{S'}$; cf. Section 8.1. If $\mathrm{Pic}_{X/S}$ is a scheme, it also represents the functor (P, ε). So the identity on $\mathrm{Pic}_{X/S}$ gives rise to a line bundle $\mathscr{P}$ on $X \times_S \mathrm{Pic}_{X/S}$ which is canonically rigidified along the induced section. $\mathscr{P}$ is called the *universal line bundle* for $(X/S, \varepsilon)$. That this terminology is justified can be seen if we write down explicitly the condition of (P, ε) being representable:

Proposition 4. *Let $f: X \longrightarrow S$ be finitely presented and flat, and let ε be a section of f. Assume that $f_*\mathcal{O}_X = \mathcal{O}_S$ holds universally. If $\mathrm{Pic}_{X/S}$ is representable by a scheme, the universal line bundle $\mathscr{P}$ for $(X/S, \varepsilon)$ has the following property:*

For any S-scheme S', and for any line bundle $\mathscr{L}'$ on $X' = X \times_S S'$ which is rigidified along the induced section ε', there exists a unique morphism $g: S' \longrightarrow \mathrm{Pic}_{X/S}$ such that $\mathscr{L}'$, as a rigidified line bundle, is isomorphic to the pull-back of $\mathscr{P}$ under the morphism $\mathrm{id}_X \times g$.

Note that $f_*\mathcal{O}_X = \mathcal{O}_S$ holds universally under the assumptions of Theorem 1; cf. [EGA III_2], 7.8.6.

Next we turn to the proof of Theorem 1. Since the relative Picard functor is a sheaf for the Zariski topology, its representability is a local problem on S. So we may assume that X is a closed subscheme of the projective space $\mathbb{P}^n_S$. In order to state what the proof yields in this special case, we have to introduce some further notions.

Following Altmann and Kleiman [1], a morphism of schemes $f: X \longrightarrow S$ is called *strongly projective* (resp. *strongly quasi-projective*) if it is finitely presented and if there exists a locally free sheaf $\mathscr{E}$ on S of constant finite rank such that X is S-isomorphic to a closed subscheme (resp. subscheme) of $\mathbb{P}(\mathscr{E})$. Let $\mathcal{O}_X(1)$ be the canonical (relatively) very ample line bundle on X. For any polynomial $\Phi \in \mathbb{Q}[t]$, one introduces the subfunctor $\mathrm{Pic}^{\Phi}_{X/S}$ of $\mathrm{Pic}_{X/S}$ which is induced by the line bundles with Hilbert polynomial Φ (with respect to $\mathcal{O}_X(1)$) on the fibres of X over S; cf. [EGA III_1], 2.5.3 for the definition of Hilbert polynomials. Then one can state the following stronger version of Theorem 1, which clearly suggests that Grothendieck's result deals with a problem inside the category of (quasi-) projective S-schemes.

Theorem 5. *Let $f: X \longrightarrow S$ be strongly projective, and let S be quasi-compact. Assume that f is flat, and that the geometric fibres of f are reduced and irreducible. Then, for every $\Phi \in \mathbb{Q}[t]$, the functor $\mathrm{Pic}^{\Phi}_{X/S}$ is representable by a strongly quasi-projective S-scheme. Furthermore, $\mathrm{Pic}_{X/S}$ is represented by the disjoint union of all $\mathrm{Pic}^{\Phi}_{X/S}$, where Φ ranges over $\mathbb{Q}[t]$.*

In the following we want to sketch the main steps of the *proof of Theorem* 5; in particular, we want to point out where the specific assumptions of the theorem are employed. The proof itself decomposes into three parts:

I) The notion of relative Cartier divisors gives rise to a functor

$$\mathrm{Div}_{X/S} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}) ,$$

which associates to an S-scheme S' the set of all relative Cartier divisors of the S'-scheme $X' := X \times_S S'$. There is a canonical morphism

$$\mathrm{Div}_{X/S} \longrightarrow \mathrm{Pic}_{X/S}$$

which is relatively representable. We will show a slightly weaker version of the latter statement which is enough for our purposes.

II) We will show that the functor $\mathrm{Div}_{X/S}$ is representable by an S-scheme. More precisely, we introduce Hilbert polynomials with respect to the fixed very ample line bundle $\mathcal{O}_X(1)$, and we look at the subfunctor $\mathrm{Div}^{\Phi}_{X/S}$ which consists of all relative Cartier divisors with Hilbert polynomial Φ. Then we will show that $\mathrm{Div}^{\Phi}_{X/S}$ is an open subfunctor of $\mathrm{Div}_{X/S}$ and that $\mathrm{Div}^{\Phi}_{X/S}$ is a strongly quasi-projective S-scheme. Furthermore, $\mathrm{Div}_{X/S}$ is the disjoint union of all schemes $\mathrm{Div}^{\Phi}_{X/S}$, where Φ ranges over $\mathbb{Q}[t]$. This part is the hardest of the whole proof, since the representability of the Hilbert functor is involved.

III) For suitable polynomials Φ, the functor $\mathrm{Pic}^{\Phi}_{X/S}$ is a quotient (as a sheaf for the fppf-topology) of an open subscheme of $\mathrm{Div}_{X/S}$ with respect to a proper smooth equivalence relation. We will show that such a quotient is representable by a scheme. Hence, $\mathrm{Pic}^{\Phi}_{X/S}$ is representable in such a special case. For general Φ, there exists an integer n_Φ such that the translate of $\mathrm{Pic}^{\Phi}_{X/S}$ by the element associated to $\mathcal{O}_X(n_\Phi)$ is of the type as treated in the special case. So $\mathrm{Pic}^{\Phi}_{X/S}$ is representable again. More precisely, we will see that it is representable by a strongly quasi-projective S-scheme. Furthermore, $\mathrm{Pic}^{\Phi}_{X/S}$ is an open and closed subfunctor of $\mathrm{Pic}_{X/S}$, so $\mathrm{Pic}_{X/S}$ is represented by the disjoint union of all schemes $\mathrm{Pic}^{\Phi}_{X/S}$ where Φ ranges over $\mathbb{Q}[t]$.

Let us start with part I. An *effective Cartier divisor* on a scheme X is a closed subscheme D of X such that its defining sheaf of ideals $\mathcal{J}$ is an invertible $\mathcal{O}_X$-module; i.e., for each $x \in X$, the ideal $\mathcal{J}_x$ is generated by a regular element of $\mathcal{O}_X$. We denote by $\mathcal{O}_X(D)$ the associated line bundle

$$\mathcal{O}_X(D) = \mathcal{J}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{O}_X) ,$$

and by $s_D \in \Gamma(X, \mathcal{O}_X(D))$ the global section associated to the inclusion $\mathcal{J} \hookrightarrow \mathcal{O}_X$. We refer to s_D as the canonical section of $\mathcal{O}_X(D)$. It corresponds to the canonical inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$. Thus, an effective Cartier divisor gives rise to a pair $(\mathcal{L}, s)$ consisting of a line bundle $\mathcal{L}$ and a global section $s \in \Gamma(X, \mathcal{L})$ which induces a

regular element s_x on each stalk $\mathcal{L}_x$, $x \in X$; i.e., the map $i_s : \mathcal{O}_x \longrightarrow \mathcal{L}_x$ sending the unit element 1_x of $\mathcal{O}_x$ to s_x is injective. Two pairs $(\mathcal{L}, s)$ and $(\mathcal{L}', s')$ are called equivalent if there exists an isomorphism $\varphi : \mathcal{L} \longrightarrow \mathcal{L}'$ such that $\varphi(s)$ and s' differ by a factor which is a global section of $\mathcal{O}_X^*$. Associating to a pair $(\mathcal{L}, s)$ the subscheme D of X which is defined by the sheaf of ideals $\mathcal{L}^{-1}$ viewed as a subsheaf of $\mathcal{O}_X$ via the morphism $i_s \otimes \mathcal{L}^{-1}$, we obtain a bijection between the set of all effective Cartier divisors on X and the set of all equivalence classes of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is a line bundle on X, and where s is a global section of $\mathcal{L}$ inducing a regular element on each stalk of $\mathcal{L}$. We denote by $\Gamma(X, \mathcal{L})^*$ the subset of $\Gamma(X, \mathcal{L})$ consisting of all global sections of $\mathcal{L}$ which induce regular elements on each stalk $\mathcal{L}_x$, $x \in X$. Thus the set of effective Cartier divisors D on X inducing the same line bundle $\mathcal{L}$ corresponds bijectively to the set $\Gamma(X, \mathcal{L})^*/\Gamma(X, \mathcal{O}_X^*)$.

Now let $f: X \longrightarrow S$ be locally of finite presentation. An *effective relative Cartier divisor* on X over S is an effective Cartier divisor D on X which is flat over S. Further characterizations of effective relative Cartier divisors are given by the following lemma.

Lemma 6. *Let $\mathcal{J}$ be a quasi-coherent sheaf of ideals of $\mathcal{O}_X$ which is locally of finite presentation, and let D be the closed subscheme of X defined by $\mathcal{J}$. Let x be a point of D, and set $s = f(x)$. Then the following conditions are equivalent:*

(i) *$\mathcal{J}$ is invertible at x (i.e., $\mathcal{J}_x$ is generated by a regular element), and D is flat over S at x.*

(ii) *X and D are flat over S at x, and the restriction D_s of D onto the fibre X_s over s is an effective Cartier divisor on X_s at x.*

(iii) *X is flat over S at x, and $\mathcal{J}_x$ is generated by an element f_x which induces a regular element on X_s at x.*

Proof. To show the assertion (i) $\Longrightarrow$ (ii), let h be a local section of $\mathcal{J}$ which generates $\mathcal{J}_x$. Then h is a regular element of $\mathcal{O}_{X,x}$, and the multiplication by h gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{D,x} \longrightarrow 0 .$$

After tensoring with the residue field $k(s)$ of s over $\mathcal{O}_{S,s}$, we obtain the sequence

$$0 \longrightarrow \mathcal{O}_{X_s,x} \longrightarrow \mathcal{O}_{X_s,x} \longrightarrow \mathcal{O}_{D_s,x} \longrightarrow 0 .$$

Due to the flatness of D over S, this sequence is exact. Thus, $\tilde{h}$ gives rise to a regular element of $\mathcal{O}_{X_s,x}$ and, hence, D_s is an effective Cartier divisor on X_s. In order to show that X is flat over S at x, we may use a limit argument ([EGA IV_3], 8.5.5 and 11.5.5.2) and thereby assume that S is locally noetherian. Looking at the long exact Tor-sequence, the flatness of D yields

$$h \cdot \mathrm{Tor}_n^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, k(s)) = \mathrm{Tor}_n^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, k(s))$$

for $n \geqq 1$. Since S ist locally noetherian, and since X is locally of finite type over S, the modules $\mathrm{Tor}_n^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, k(s))$ are finitely generated over $O_{X,x}$. But then Nakayama's lemma implies $\mathrm{Tor}_n^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, k(s)) = 0$ for n $\geqq 1$, because $x \in D$. Hence X is flat over S at x by Bourbaki [2], Chap. III, § 5, n°2, Thm. 1.

The assertion (ii) $\Longrightarrow$ (iii) follows from Nakayama's lemma, and the remaining implication (iii) $\Longrightarrow$ (i) is a consequence of [EGA IV$_3$], 11.3.7. □

It is clear from condition (ii) that the notion of effective relative Cartier divisors is stable under any base change $S' \longrightarrow S$. Thus, there is a functor

$$\mathrm{Div}_{X/S} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})\,, \qquad S' \longmapsto \mathrm{Div}(X'/S')$$

where $\mathrm{Div}(X'/S')$ denotes the set of all effective relative Cartier divisors of $X' = X \times_S S'$ over S'. Associating to an effective relative Cartier divisor D the line bundle $\mathcal{O}_X(D)$, we obtain the canonical morphism

$$\mathrm{Div}_{X/S} \longrightarrow \mathrm{Pic}_{X/S}\,, \qquad D \longmapsto \mathcal{O}_X(D)\,.$$

As a first step towards the representability of $\mathrm{Pic}_{X/S}$, one proves that this morphism is relatively representable. Recall, this means that for each morphism $T \longrightarrow \mathrm{Pic}_{X/S}$ from an S-scheme T to $\mathrm{Pic}_{X/S}$, the morphism

$$\mathrm{Div}_{X/S} \times_{\mathrm{Pic}_{X/S}} T \longrightarrow T$$

obtained from $\mathrm{Div}_{X/S} \longrightarrow \mathrm{Pic}_{X/S}$ by the base change $T \longrightarrow \mathrm{Pic}_{X/S}$ is a morphism of schemes. However, we will show the latter only under the assumption that the map $T \longrightarrow \mathrm{Pic}_{X/S}$, as an element of $\mathrm{Pic}_{X/S}(T)$, is given by a line bundle on $X \times_S T$. This is enough for our application, because in part III we will apply it to the case where $T = \mathrm{Div}_{X/S}$ and where the map $T \longrightarrow \mathrm{Pic}_{X/S}$ is the canonical one. On the other hand, each map $T \longrightarrow \mathrm{Pic}_{X/S}$ corresponds to a line bundle on $X \times_S T$ if f has a section; cf. 8.1/4. So in this case we will really get the relative representability of $\mathrm{Div}_{X/S} \longrightarrow \mathrm{Pic}_{X/S}$.

Proposition 7. *Let $f : X \longrightarrow S$ be as in Theorem 5, and let T be an S-scheme. Let $\mathcal{L}$ be a line bundle on $X_T = X \times_S T$, and denote by $T \longrightarrow \mathrm{Pic}_{X/S}$ the morphism corresponding to $\mathcal{L}$. Then there exists an $\mathcal{O}_T$-module $\mathcal{F}$, which is locally of finite presentation, such that $\mathrm{Div}_{X/S} \times_{\mathrm{Pic}_{X/S}} T$ is represented by the projective T-scheme $\mathbb{P}(\mathcal{F})$.*

Furthermore, there is a canonical way to choose $\mathcal{F}$. If $\mathcal{L}$ is cohomologically flat in dimension zero, then $f_*(\mathcal{L})$ and $\mathcal{F}$ are locally free, and $\mathcal{F}$ is isomorphic to the dual of $f_*(\mathcal{L})$.

Proof. We may assume $T = S$. The fibred product $\mathrm{Div}_{X/S} \times_{\mathrm{Pic}_{X/S}} S$ is isomorphic to the functor $D_{\mathcal{L}} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ which associates to an S-scheme S' the set of all relative Cartier divisors D' on X'/S' such that $\mathcal{O}_{X'}(D')$ and $\mathcal{L}'$ give rise to the same element in $\mathrm{Pic}_{X/S}(S')$, where $\mathcal{L}'$ denotes the pull-back of $\mathcal{L}$ to X'. By Proposition 8.1/3 the latter condition is equivalent to the fact that $\mathcal{O}_{X'}(D')$ and $\mathcal{L}'$ are isomorphic locally over S'. Hence, as we have shown during our general discussion of Cartier divisors, there is a bijection

$$\Gamma(S', (f'_*\mathcal{L}')^*/f'_*(\mathcal{O}^*_{X'})) \longrightarrow D_{\mathcal{L}}(S')$$

where f' is obtained from f by the base change $S' \longrightarrow S$, and where $(f'_*\mathcal{L}')^*$ denotes the subsheaf of $(f'_*\mathcal{L}')$ consisting of all sections which induce regular elements on every fibre X_s of f. Thus, we have a bijection

$$\Gamma(S, (f_*\mathcal{L})^*/\mathcal{O}^*_S) \longrightarrow D_{\mathcal{L}}(S)$$

which is compatible with base change. Since f is proper and flat, there exists an $\mathcal{O}_S$-module $\mathcal{F}$ of locally finite presentation such that there is an isomorphism

$$(*) \qquad f_*\mathcal{L} \longrightarrow \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$$

which is compatible with any base change $S' \longrightarrow S$; see Theorem 8.1/7. Furthermore, $\mathcal{F}$ is canonically determined by $\mathcal{L}$. Since the geometric fibres of f are reduced and irreducible, the local sections of $(f_*\mathcal{L})^*$ coincide with the local sections of $f_*\mathcal{L}$ which do not induce the zero section on any fibre X_s. Interpreting them as local homomorphisms $\mathcal{F} \longrightarrow \mathcal{O}_S$ via $(*)$ and applying Nakayama's lemma, they correspond to those local homomorphisms $\mathcal{F} \longrightarrow \mathcal{O}_S$ which are surjective. Thus, the sections of $(f_*\mathcal{L})^*/\mathcal{O}^*_S$ correspond bijectively to the set of quasi-coherent quotients of $\mathcal{F}$ which are invertible, and hence to the sections of the projective bundle $\mathbb{P}(\mathcal{F})$; cf. [EGA II], 4.2.3. Since all maps under consideration are compatible with base change, $\mathcal{F}$ is as required. The last statement of the proposition has already been mentioned in 8.1/7. □

Thereby we have finished part I. Next, we discuss part II. The representability of $\mathrm{Div}_{X/S}$ will be derived from the representability of the Hilbert functor. The latter is defined as follows. For any S-scheme X denote by $\mathrm{Hilb}(X/S)$ the set of all closed subschemes D of X which are proper, finitely presented, and flat over S. Then

$$\mathrm{Hilb}_{X/S} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})\,, \qquad S' \longrightarrow \mathrm{Hilb}(X \times_S S'/S')$$

is a functor, the so-called *Hilbert functor of X over S*. We see from Lemma 6 that $\mathrm{Div}_{X/S}$ is an open subfunctor of $\mathrm{Hilb}_{X/S}$ if X is proper, finitely presented, and flat over S. Thus the representability of $\mathrm{Div}_{X/S}$ follows from the representability of $\mathrm{Hilb}_{X/S}$. We want to mention that, for the representability of $\mathrm{Hilb}_{X/S}$ by a scheme, it is essential that X is quasi-projective over S. Namely, there is an example by Hironaka of a proper and smooth manifold of dimension 3 over a field on which the group $\mathbb{Z}/2\mathbb{Z}$ acts freely. But the quotient with respect to this action does not exist in the category of schemes; cf. Hironaka [1]. One shows that, in this situation, the Hilbert functor cannot be represented by a scheme; namely, the equivalence relation given by the group action constitutes a closed subscheme R of $X \times_S X$ which is proper and flat with respect to the second projection. Thus R gives rise to an element $g \in \mathrm{Hilb}_{X/S}(X)$ and, if $\mathrm{Hilb}_{X/S}$ were representable as a scheme, the image of the morphism $X \longrightarrow \mathrm{Hilb}_{X/S}$ given by g would serve as a quotient of X under the group action.

For showing the representability of $\mathrm{Hilb}_{X/S}$, it is convenient to look at a more general situation. Given an $\mathcal{O}_X$-module $\mathcal{F}$ which is locally of finite presentation, one introduces the functor

$$\mathrm{Quot}_{(\mathcal{F}/X/S)} : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$$

which associates to an S-scheme S' the set of quotients $\mathcal{G}'$ of the pull-back $\mathcal{F}'$ of $\mathcal{F}$

to $X' = X \times_S S'$ where $\mathscr{G}'$ is required to be locally of finite presentation over $\mathcal{O}_{X'}$, to be flat over S', and to have proper support over S'. The key result on the representability of the functor $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ is the following theorem of Grothendieck (cf. [FGA], n°221, Thm. 3.1); the strengthening from the projective to the strongly projective case is due to Altman and Kleiman [1], Thm. 2.6.

Theorem 8. *Let $f: X \longrightarrow S$ be strongly quasi-projective, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module which is locally of finite presentation. Fix a (relatively) very ample line bundle $\mathcal{O}_X(1)$ associated to an embedding of X into a projective bundle over S. Assume that $\mathscr{F}$ is isomorphic to a quotient of an $\mathcal{O}_X$-module of the form $f^*\mathscr{B} \otimes \mathcal{O}_X(v)$ for some $v \in \mathbb{Z}$, where $\mathscr{B}$ is a locally free $\mathcal{O}_S$-module with a constant finite rank. Then $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ is represented by a separated S-scheme which is a disjoint union of strongly quasi-projective S-schemes.*

If, in addition, f is proper, then $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ is a disjoint union of strongly projective S-schemes.

Note that, for $\mathscr{F} = \mathcal{O}_X$, the functors $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ and $\mathrm{Hilb}_{X/S}$ coincide. Furthermore, $\mathrm{Div}_{X/S}$ is a quasi-compact open subfunctor of $\mathrm{Hilb}_{X/S}$ if X is proper, finitely presented, and flat over S. Thus, if $\mathrm{Hilb}_{X/S}$ is represented by a disjoint union of strongly quasi-projective S-schemes, so is $\mathrm{Div}_{X/S}$.

When a very ample line bundle $\mathcal{O}_X(1)$ is fixed, $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ can be covered in a canonical way by open subfunctors which will correspond to quasi-compact open subschemes of $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ (resp. of $\mathrm{Hilb}_{X/S}$). Namely, for any $\mathcal{O}_X$-module $\mathscr{G}$ which is locally of finite presentation and has proper support, and for any point $s \in S$, one has the Hilbert polynomial $\chi(\mathscr{G}_s)(t)$; its value at any $n \in \mathbb{Z}$ is given by the Euler-Poincaré characteristic

$$\chi(\mathscr{G}_s(n)) = \sum_{i=0}^{\infty} (-1)^i \dim_{k(s)} H^i(X_s, \mathscr{G}_s(n))$$

of $\mathscr{G}(n)$ over the fibre X_s, where we have written $\mathscr{G}_s(n)$ for the restriction of $\mathscr{G} \otimes \mathcal{O}_X(n)$ to X_s. The Hilbert polynomial has rational coefficients; cf. [EGA III_1], 2.5.3. Furthermore, when $\mathscr{G}$ is flat over S, it is locally constant as a function of $s \in S$; cf. [EGA III_2], 7.9.11. So, for a polynomial $\Phi \in \mathbb{Q}[t]$, let $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ be the subfunctor of $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ consisting of all quotients with a fixed Hilbert polynomial Φ. In the same way, one introduces the subfunctor $\mathrm{Hilb}^{\Phi}_{X/S}$ of $\mathrm{Hilb}_{X/S}$. It is clear that $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ constitutes an open and closed subfunctor of $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ and that the subfunctors $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ cover $\mathrm{Quot}_{(\mathscr{F}/X/S)}$ if Φ ranges over $\mathbb{Q}[t]$. Thus, it suffices to prove the following theorem.

Theorem 8'. *Let X be S-isomorphic to a finitely presented subscheme of $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is a locally free $\mathcal{O}_S$-module of constant finite rank. Denote by $f: X \longrightarrow S$ the structural morphism and by $\mathcal{O}_X(1)$ the canonical (relatively) very ample line bundle on X. Let $\mathscr{F}$ be isomorphic to a quotient of $(f^*\mathscr{B}) \otimes \mathcal{O}_X(v)$ where $v \in \mathbb{Z}$ and where $\mathscr{B}$ is a locally free sheaf on S of constant finite rank, and assume that $\mathscr{F}$ is locally of finite presentation. Furthermore, fix a polynomial $\Phi \in \mathbb{Q}[t]$. Then, there exists an integer m_0 satisfying the following property:*

For each $m \geq m_0$, the map

$$\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)} \longrightarrow \mathrm{Grass}_{\Phi(m)}(\mathscr{B} \otimes \mathscr{S}ym_{v+m}(\mathscr{E})),$$

which associates to an element $\mathscr{G}' \in \mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}(S')$ the direct image $f_*(\mathscr{G}'(m))$, constitutes a functor which is relatively representable by a quasi-compact immersion. In particular, $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ is representable by a strongly quasi-projective S-scheme.

If, in addition, X is a closed subscheme of $\mathbb{P}(\mathscr{E})$, the immersion of above is closed and $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ is strongly projective over S.

For a locally free $\mathcal{O}_S$-module $\mathscr{L}$ and a non-negative integer r, we denote by $\mathrm{Grass}_r(\mathscr{L})$ the contravariant functor from (Sch/S) to (Sets) which associates to an S-scheme S' the set of locally free quotients of $\mathscr{L} \otimes \mathcal{O}_{S'}$ of rank r. Then $\mathrm{Grass}_r(\mathscr{L})$ is representable by a closed subscheme of $\mathbb{P}(\mathscr{D})$, where $\mathscr{D}$ is the r-th exterior power of $\mathscr{L}$; cf. Grothendieck [2], § 2. Since we have not restricted ourselves to polynomials $\Phi \in \mathbb{Q}[t]$ which take values $\Phi(m)$ in the non-negative integers for large integers m, we define $\mathrm{Grass}_r(\mathscr{L})$ by the empty functor if $r \in \mathbb{Q} - \mathbb{N}$. Note that $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ is the empty functor if the polynomial Φ does not take values $\Phi(m)$ in the non-negative integers for large integers m.

For $\mathscr{F} = \mathcal{O}_X$, one has $\mathrm{Quot}^{\Phi}_{(\mathcal{O}_X/X/S)} = \mathrm{Hilb}^{\Phi}_{X/S}$. If X is proper and flat over S, we know that $\mathrm{Div}_{X/S}$ is an open subfunctor of $\mathrm{Hilb}_{X/S}$. So we denote by $\mathrm{Div}^{\Phi}_{X/S}$ the induced subfunctor of $\mathrm{Hilb}^{\Phi}_{X/S}$. Thus, Theorem 8' implies the following corollary.

Corollary 9. *Let $f: X \longrightarrow S$ be strongly projective (resp. strongly quasi-projective), and let $\Phi \in \mathbb{Q}[t]$. Then $\mathrm{Hilb}^{\Phi}_{X/S}$ is representable by a strongly projective (resp. strongly quasi-projective) S-scheme.*

If, in addition, X is proper and flat over S, then $\mathrm{Div}^{\Phi}_{X/S}$ is representable by a strongly quasi-projective S-scheme.

Now let us give an *outline of the proof of Theorem* 8'. First one reduces to the case where X is the projective space $\mathbb{P}(\mathscr{E})$ associated to a locally free sheaf $\mathscr{E}$ of constant rank $e + 1$ on S, and where $\mathscr{F}$ is isomorphic to $f^*\mathscr{B}(v) := (f^*\mathscr{B}) \otimes \mathcal{O}_X(v)$ for some locally free sheef $\mathscr{B}$ on S which has constant rank b over S. Namely, $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ is a locally closed (resp. closed) subfunctor of $\mathrm{Quot}^{\Phi}_{(f^*\mathscr{B}(v)/\mathbb{P}(\mathscr{E})/S)}$ of finite presentation. In the latter case, there is a canonical isomorphism

$$\mathscr{B} \otimes \mathscr{S}ym_{v+m}(\mathscr{E}) \xrightarrow{\sim} f_*(\mathscr{F}(m))$$

for $m \in \mathbb{Z}$; cf. [EGA III_1], 2.1.15. We assume this situation from now on. Then a key point is the following observation of Mumford which simplifies the original proof of Grothendieck; cf. Mumford [2], Lecture 14.

Proposition 10. *There exists a constant m_0 depending on the integers e, b, v and on the coefficients of Φ, such that the following property is satisfied:*

Let S' be an S-scheme, and let $\mathscr{G}' \in \mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}(S')$. Denote by $\mathscr{K}'$ the kernel of the canonical map $\mathscr{F}' \longrightarrow \mathscr{G}'$. Then, for all $m \geq m_0$, the $\mathcal{O}_{X'}$-module $\mathscr{K}'(m)$ is generated by the local sections of $f'_*(\mathscr{K}'(m))$, and $R^i f'_*(\mathscr{K}'(m)$ vanishes for $i \geq 1$. The same is true for $\mathscr{F}'(m)$ and $\mathscr{G}'(m)$.

A detailed proof of this proposition can be found in [SGA 6], Exp. XIII, § 1, for the case where S' defines a geometric point of S. The general case follows then by the theory of cohomology and base change; cf. Mumford [3], § 5.

Going back to the proof of Theorem 8', keep the notation of Proposition 10. Then, for $m \geqq m_0$ and for each S-scheme S', the canonical map

$$f'_*(\mathscr{F}'(m)) \longrightarrow f'_*(\mathscr{G}'(m))$$

is surjective. Since $R^i f'_* \mathscr{G}'(m)$ vanishes for $m \geqq m_0$ and $i \geqq 1$, the direct image $f'_*(\mathscr{G}'(m)$ is a locally free $\mathscr{O}_{S'}$-module of rank $\Phi(m)$, due to [EGA III$_2$], 7.9.9. Thus, we get the canonical morphism

$$\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)} \longrightarrow \mathrm{Grass}_{\Phi(m)}(f_*(\mathscr{F}(m))$$

associating to a flat quotient $\mathscr{G}'$ of $\mathscr{F}'$ on X' the direct image $f_*(\mathscr{G}'(m))$. Moreover, one can reconstruct the subsheaf $\mathscr{H}'$ of $\mathscr{F}'$ from the canonical surjective map

$$f'_*(\mathscr{F}'(m)) \longrightarrow f'_*(\mathscr{G}'(m)) .$$

Thus, one can view $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ as a subfunctor of the Graßmannian functor $\mathrm{Grass}_{\Phi(m)}(\mathscr{B} \otimes \mathscr{S}\mathit{ym}_{\nu+m}(\mathscr{E}))$ which associates to an S-scheme S' the set of all locally free quotients of $f'_*(\mathscr{F}'(m))$ of rank $\Phi(m)$. It remains to see that the monomorphism

$$\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)} \longrightarrow \mathrm{Grass}_{\Phi(m)}(\mathscr{B} \otimes \mathscr{S}\mathit{ym}_{\nu+m}(\mathscr{E}))$$

is representable by a quasi-compact immersion. So denote by G the S-scheme $\mathrm{Grass}_{\Phi(m)}(\mathscr{B} \otimes \mathscr{S}\mathit{ym}_{\nu+m}(\mathscr{E}))$ and by $\mathscr{Q}$ the universal quotient of $\mathscr{B} \otimes \mathscr{S}\mathit{ym}_{\nu+m}(\mathscr{E})$. The latter is a quotient (as an $\mathscr{O}_G$-module) of the pull-back $(\mathscr{B} \otimes \mathscr{S}\mathit{ym}_{\nu+m}(\mathscr{E}))_G$ of $\mathscr{B} \otimes \mathscr{S}\mathit{ym}_{\nu+m}(\mathscr{E})$ to G, which is locally free of rank $\Phi(m)$. Let $\mathscr{F}_G$ be the pull-back of $\mathscr{F}$ on $X_G = X \times_S G$, and let $f_G : X_G \longrightarrow G$ be the map obtained from f by the base change $G \longrightarrow S$. By using the canonical isomorphism

$$(\mathscr{B} \otimes \mathscr{S}\mathit{ym}_{\nu+m}(\mathscr{E}))_G \longrightarrow (f_G)_*(\mathscr{F}_G(m)) ,$$

we obtain a canonical map

$$(f_G)_*(\mathscr{F}_G(m)) \longrightarrow \mathscr{Q} .$$

The kernel of this map generates a subsheaf $\mathscr{H}_G(m)$ of $\mathscr{F}_G(m)$. Denote by $\mathscr{H}_G$ the $\mathscr{O}_{X_G}$-module $\mathscr{H}_G(m) \otimes \mathscr{O}_{X_G}(-m)$ and by $\mathscr{G}_G$ the quotient $\mathscr{F}_G/\mathscr{H}_G$. By reducing to a noetherian base scheme S, one shows that there exists a (unique) subscheme Z of G such that a morphism $T \longrightarrow G$ factors through Z if and only if the pull-back $\mathscr{G}_T$ of $\mathscr{G}_G$ on $X \times_S T$ is flat over T and has Hilbert polynomial Φ on the fibres over T; cf. [FGA], n°221, Sect. 3. Furthermore, the inclusion $Z \hookrightarrow G$ is finitely presented. Hence, $\mathrm{Quot}^{\Phi}_{(\mathscr{F}/X/S)}$ is represented by Z which is strongly quasi-projective over S. Finally, Z is strongly projective because the valuative criterion is satisfied by [EGA IV$_2$], 2.8.1. □

Thereby we have finished part II. Finally we come to part III. We begin by recalling some definitions on equivalence relations in categories. Let C be a category such that direct products $X_1 \times X_2$ and fibred products $X_1 \times_Y X_2$ exist in C. A *C-equivalence relation* on an object X of C is a representable subfunctor R of $X \times X$ such that, for each object T of C, the subset

$$R(T) \subset X(T) \times X(T)$$

is the graph of an equivalence relation on $X(T)$. Denote by $p_i : R \longrightarrow X$ the projection onto the i-th factor, $i = 1, 2$. A *categorical quotient of X with respect to the equivalence relation R* is a pair (Z, u) consisting of an object Z of C and a morphism $u : X \longrightarrow Z$ satisfying $up_1 = up_2$ such that, for any morphism $v : X \longrightarrow Y$ satisfying $vp_1 = vp_2$, there exists a unique morphism $\overline{v} : Z \longrightarrow Y$ such that $v = \overline{v}u$. If it exists, it is uniquely determined, and we will usually denote it by X/R. Furthermore, due to the definition of a fibred product, there is a canonical morphism

$$i : R \longrightarrow X \times_{X/R} X .$$

R is called an *effective equivalence relation on X* if the categorical quotient X/R exists and if the canonical morphism i is an isomorphism. In this case, X/R is called an *effective quotient*. Quite often, the canonical morphism i is not an isomorphism; this means that the equivalence relation given by the fibred product $X \times_{X/R} X$ over the categorical quotient X/R, is usually larger than the given relation R.

In the following, we consider the category of S-schemes. Then one can look at quotients also from the sheaf-theoretical point of view. Due to Proposition 8.1/1, any S-scheme X is a sheaf with respect to the fppf-topology (or the fpqc-topology). So, one can ask for the quotient of X with respect to R in the category of sheaves for the fppf-topology. Using the procedure of sheafification, one easily shows that such a quotient exists and that it is effective. Let us denote it by (X/R). Furthermore let us assume that the categorical quotient (in the category of S-schemes) X/R exists. So, viewing X and X/R as sheaves for the fppf-topology, one obtains canonical morphisms

$$X \longrightarrow (X/R) \longrightarrow X/R .$$

If (X/R) is represented by a scheme, (X/R) is the effective quotient of X with respect to R (for the category of S-schemes), and the canonical morphism $(X/R) \longrightarrow X/R$ is an isomorphism.

Example 11. *Let $f : X \longrightarrow Y$ be an* fppf-*morphism of S-schemes. Denote by $R(f)$ the subscheme $X \times_Y X$ of $X \times_S X$. Then $R(f)$ is an effective equivalence relation on X and (Y, f) is the effective quotient of X with respect to $R(f)$ in the category of S-schemes as well as in the category of sheaves for the* fppf-*topology.*

Proof. Since f is an fppf-morphism, Y is the quotient (in the category of sheaves for the fppf-topology) of X with respect to $R(f)$. Hence the assertion follows from what has been said before. □

For any property P applicable to morphisms, an equivalence relation R on an S-scheme X is said to satisfy the property P if P holds for the projections $p_i : R \longrightarrow X$.

We need the following general theorem on the existence of effective quotients with respect to proper flat equivalence relations.

Theorem 12. *Let $f : X \longrightarrow S$ be strongly quasi-projective, and let R be a proper flat equivalence relation on X which is finitely presented. Assume that the fibres of the projection $p_2 : R \longrightarrow X$ have only a finite number of Hilbert polynomials with respect to an embedding of X into $\mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a locally free $\mathcal{O}_S$-module of constant finite rank. Then R is effective, the quotient map $q : X \longrightarrow X/R$ is strongly projective and faithfully flat, and X/R is strongly quasi-projective over S.*

In particular, X/R is the effective quotient of X with respect to R in the category of sheaves for the fppf-*topology.*

The *proof* is easily done by using the existence of the Hilbert scheme; cf. Altman and Kleiman [1], § 2. Namely, set $H = \coprod \mathrm{Hilb}^{\Phi}_{X/S}$ where Φ ranges over the finitely many Hilbert polynomials of p_2; then H exists as a scheme and is strongly quasi-projective over S; cf. Corollary 9. Let D be the universal subscheme of $X \times_S H$. The projection $p : D \longrightarrow H$ is proper, flat, and finitely presented, and the equivalence relation R is a subscheme of $X \times_S X$ which is proper, flat, and finitely presented with respect to the second projection p_2. So, using the universal property of the Hilbert scheme, there is a unique morphism $g : X \longrightarrow H$ such that

$$R = (\mathrm{id}_X \times g)^* D \; .$$

Now the idea is to realize the quotient as the image of g.

For an S-scheme T and for points $x_1, x_2 \in X(T)$, write $x_1 \sim x_2$ whenever $(x_1, x_2) \in R(T)$. Then one shows

$$(*) \qquad x_1 \sim x_2 \Leftrightarrow g x_1 = g x_2 \Leftrightarrow (x_1, g x_2) \in D(T) \; .$$

Namely, set $R_i = (\mathrm{id}_X, x_i)^* R$ for $i = 1, 2$. Due to the definition of $\mathrm{Hilb}_{X/S}$, we have $g x_1 = g x_2$ if and only if for all T-schemes T', the set $R_1(T')$ coincides with $R_2(T')$ viewing both as subsets of $(X \times_S T)(T')$. Since R is an equivalence relation, the latter is equivalent to $(x_1, \mathrm{id}_T) \in R_2(T)$ and hence to $x_1 \sim x_2$. Thus, the first equivalence is clear. Due to the definition of g, the condition $(x_1, g x_2) \in D(T)$ is equivalent to $(x_1, x_2) \in R(T)$. Then the second equivalence is also clear.

Now, denote by Γ_g the graph of $g : X \longrightarrow H$. Since H is separated over S, the graph Γ_g is closed in $X \times_S H$. Furthermore, because Γ_g is isomorphic to X, it is of finite presentation over S. Since Γ_g is contained in D due to $(*)$, it is a closed subscheme of D. Moreover, Γ_g is of finite presentation over D, since D is of finite presentation over S. We want to show that Γ_g descends to a closed subscheme Z of H which is of finite presentation over H. So look at the projection $p : D \longrightarrow H$. Due to the definition of $\mathrm{Hilb}_{X/S}$, the map p is faithfully flat, proper, and finitely presented. Consider the canonical descent datum on D. In order to show Γ_g descends to a closed subscheme Z of H which is of finite presentation over H, it suffices to show that the closed subschemes $\Gamma_g \times_H D$ and $D \times_H \Gamma_g$ of $D \times_H D$ coincide. The latter is easily checked by looking at T-valued points and by using the equivalence $(*)$. The map $g : X \longrightarrow H$ factors through Z and, identifying X with Γ_g, the map $g : X \longrightarrow H$ is obtained from $p : D \longrightarrow H$ by the base change $Z \longrightarrow H$. Hence, $g : X \longrightarrow Z$ is faithfully flat, and strongly projective over Z, since D, being proper and strongly quasi-projective over H, is strongly projective over H. Because of $(*)$, we have a canonical isomorphism

$$R \longrightarrow X \times_Z X \; .$$

Then (Z, g) is an effective quotient of X with respect to R as explained in Example 11. Finally, $Z \longrightarrow S$ is strongly quasi-projective because Z is a closed subscheme of the strongly quasi-projective S-scheme H. □

Now we want to explain how the *proof of Theorem* 5 can be derived from the results we have discussed up to now. Let Φ be a polynomial with rational coefficients. Since the Hilbert polynomial of any $\mathcal{O}_X$-module, which is locally of finite presentation over X and flat over S, is locally constant, $\mathrm{Pic}^{\Phi}_{X/S}$ is an open and closed subfunctor of $\mathrm{Pic}_{X/S}$. Thus, it remains to show that $\mathrm{Pic}^{\Phi}_{X/S}$ is representable by a strongly quasi-projective S-scheme.

In order to do this, we need the notion of bounded families of coherent sheaves on the fibres of X over S. So, let S be a quasi-compact scheme and let X be an S-scheme of finite presentation. Let Λ be a family of isomorphism classes of coherent sheaves on the fibres of X over S; i.e., for each $s \in S$ and for each extension field K of $k(s)$, we are given a family of coherent sheaves $\mathscr{F}_K$ on X_K. Two sheaves $\mathscr{F}_K$ and $\mathscr{F}'_{K'}$ belong to the same class if there exist $k(s)$-embeddings of K and K' into a field L such that $\mathscr{F}_K \otimes_K L$ and $\mathscr{F}'_{K'} \otimes_{K'} L$ are isomorphic on X_L. The family Λ is called bounded if there exists an S-scheme T of finite presentation and a sheaf $\mathscr{F}$ on $X_T = X \times_S T$ which is locally of finite presentation such that Λ is contained in the family $(\mathscr{F}_{k(t)}; t \in T)$. There is the following proposition, cf. [SGA 6], Exp. XIII, Thm. 1.13.

Proposition 13. *Let S be quasi-compact, and let $X \longrightarrow S$ be strongly projective. Let Λ be a family of coherent sheaves on the fibres of X over S. Then the followig conditions are equivalent:*

(i) *Λ is bounded.*

(ii) *The set of Hilbert polynomials $\chi(\mathscr{F}_K)(t)$ is finite where $\mathscr{F}_K$ ranges over the elements of the family Λ, and there exist integers $n \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that Λ is contained in the family of all classes of quotients of $\mathcal{O}_X(n)^N$.*

Furthermore we need the following result; cf. [SGA 6], Exp. XIII, Lemma 2.11.

Proposition 14. *Under the assumption of Theorem 5, a family Λ of line bundles $\mathscr{L}_K$ on the fibres of X over S is bounded if and only if the set of Hilbert polynomials $\chi(\mathscr{L}_K)(t)$ is finite.*

Now consider the morphism

$$\mathrm{Div}_{X/S} \longrightarrow \mathrm{Pic}_{X/S} \; .$$

Fix the polynomial Φ, and denote by $D(\Phi)$ the inverse image of $\mathrm{Pic}^{\Phi}_{X/S}$ in $\mathrm{Div}_{X/S}$. It is clear that $D(\Phi)$ is a disjoint union of connected components of $\mathrm{Div}_{X/S}$. Then it follows from Proposition 14 that there are only finitely many connected components of $\mathrm{Div}_{X/S}$ which are involved. Thus, due to Corollary 9, we see that $D(\Phi)$ is strongly quasi-projective over S.

Let us assume for a moment that the following condition on $\mathrm{Pic}^{\Phi}_{X/S}$ is satisfied: for any S-scheme S' and for any line bundle $\mathcal{L}'$ on $X' = X \times_S S'$ which induces an element of $\mathrm{Pic}^{\Phi}_{X/S}$, we have

$$\begin{aligned} R^i f'_*(\mathcal{L}'(n)) &= 0 \quad \text{for} \quad i > 0 \quad \text{and} \quad n \geqq 0\,, \quad \text{and} \\ f'_*(\mathcal{L}'(n)) &\neq 0 \quad \text{for } n \geqq 0\,. \end{aligned}$$

Note that such line bundles are cohomologically flat in dimension zero. Furthermore, in this case, the map $D(\Phi) \longrightarrow \mathrm{Pic}^{\Phi}_{X/S}$ is an epimorphism (in terms of sheaves for the fppf-topology). Let $\mathcal{L}$ be the line bundle on $X \times_S D(\Phi)$ which corresponds to the universal (relative) Cartier divisor on $X \times_S D(\Phi)$. Then the map $D(\Phi) \longrightarrow \mathrm{Pic}^{\Phi}_{X/S}$ is induced by $\mathcal{L}$. If $f(\Phi)$ is the base change of f by $D(\Phi) \longrightarrow S$, the direct image of $\mathcal{L}$ under $f(\Phi)$ is locally free of rank $\Phi(0)$. Due to Proposition 7, the morphism

$$D(\Phi) \times_{\mathrm{Pic}^{\Phi}_{X/S}} D(\Phi) \longrightarrow D(\Phi)$$

is representable by the flat (even smooth) strongly projective morphism

$$\mathbb{P}(\mathcal{F}) \longrightarrow D(\Phi)\,,$$

where $\mathcal{F}$ is the dual of the direct image of $\mathcal{L}$ under $f(\Phi)$, since $\mathcal{L}$ is cohomologically flat in dimension zero. Now in order to show the representability of $\mathrm{Pic}^{\Phi}_{X/S}$, consider the following diagram

$$\begin{array}{ccc} D(\Phi) \times_{\mathrm{Pic}^{\Phi}_{X/S}} D(\Phi) & \longrightarrow & D(\Phi) \\ \downarrow & & \downarrow \\ D(\Phi) & \longrightarrow & \mathrm{Pic}^{\Phi}_{X/S}\,. \end{array}$$

It says that $\mathrm{Pic}^{\Phi}_{X/S}$ is isomorphic to the quotient (as sheaf for the fppf-topology) of $D(\Phi)$ by a proper and flat equivalence relation. Thus $\mathrm{Pic}^{\Phi}_{X/S}$ is representable by a strongly quasi-projective S-scheme; cf. Theorem 12.

Now it remains to remove the special assumption on $\mathrm{Pic}^{\Phi}_{X/S}$ which has been made above. If n is an integer, we denote by $\mathrm{Pic}^{\Phi}_{X/S} + n\xi$ the functor which associates to an S-scheme S' the subset

$$\{\mathcal{L}'(n)\,;\, \mathcal{L}' \in \mathrm{Pic}^{\Phi}_{X/S}(S')\}$$

of $\mathrm{Pic}_{X/S}(S')$. Note that this functor is of the form $\mathrm{Pic}^{\Psi}_{X/S}$ for a suitable polynomial $\Psi \in \mathbb{Q}[t]$. It suffices to show that there exists an integer n such that $\mathrm{Pic}^{\Phi}_{X/S} + n\xi$ fulfills the above assumptions. However, since $\mathrm{Pic}^{\Phi}_{X/S}$ is bounded due to Proposition 14, the latter follows from Propositions 13 and 10 by base change theory.

Thus we have finished part III, and thereby we conclude our discussion of Theorem 5.

8.3 Representability by an Algebraic Space

The most restrictive assumption in Grothendieck's theorem 8.2/1 on the representability of $\mathrm{Pic}_{X/S}$ is that the geometric fibres of $f: X \longrightarrow S$ have to be reduced and irreducible. As we have seen in the preceding section by looking at Mumford's example, even if X is projective and flat over S, there is an obstruction to $\mathrm{Pic}_{X/S}$ being a scheme, which is located in the fibres of f. However, in Mumford's example, there exists a surjective étale extension $S' \longrightarrow S$ such that the functor $\mathrm{Pic}_{X/S} \times_S S'$ is representable by a scheme over S'. Working within the category of algebraic spaces (the definition is given below), we can say that $\mathrm{Pic}_{X/S}$ is representable, since this category is stable under quotients by étale equivalence relations. This example suggests that, in comparison with Grothendieck's theorem, the assumptions on the S-scheme X can be weakened if one wants to represent $\mathrm{Pic}_{X/S}$ by an algebraic space.

Theorem 1 (M. Artin [5], Thm. 7.3). *Let $f: X \longrightarrow S$ be a morphism of algebraic spaces which is proper, flat, and finitely presented. Then, if f is cohomologically flat in dimension zero, the relative Picard functor $\mathrm{Pic}_{X/S}$ is represented by an algebraic space over S.*

A proper and flat morphism f is cohomologically flat in dimension zero if, for example, the geometric fibres of f are reduced; cf. [EGA III_2], 7.8.6. Furthermore, let us mention that there is a converse of Theorem 1 when the base S is reduced.

Remark 2. *Let $f: X \longrightarrow S$ be a morphism of schemes which is proper, flat, and finitely presented. Assume that S is reduced. Then $\mathrm{Pic}_{X/S}$ is an algebraic space if and only if f is cohomologically flat in dimension zero.*

Namely, in order to show the cohomological flatness of f when $\mathrm{Pic}_{X/S}$ is an algebraic space, one has only to verify that the dimension of $H^0(X_s, \mathcal{O}_{X_s})$ is locally constant on S; cf. [EGA III_2], 7.8.4. Then one can assume that S is a discrete valuation ring. Hence, the assertion follows from Raynaud [6], Prop. 5.2.

As we will see below, the method for the proof of Theorem 1 is completely different from the one used in the last section. It does not involve projective methods nor does it make use of the representability of the Hilbert functor or of the functor of relative Cartier divisors. Also we want to mention that the theorem does not contain 8.2/1. Only for the case where the base scheme S is a field, 8.2/1 and 8.2/3 are corollaries of Theorem 1, since a group object in the category of algebraic spaces over a field is represented by a scheme.

If, in the situation of Theorem 1, f is not cohomologically flat in dimension zero, the only option which is left is to work with rigidificators (cf. 8.1/5), and one can look for the representability of rigidified relative Picard functors; cf. Section 8.1.

Theorem 3 (Raynaud [1], Thm. 2.3.1). *Let $f: X \longrightarrow S$ be a proper, flat, and finitely presented morphism of algebraic spaces, and let Y be a rigidificator for $\mathrm{Pic}_{X/S}$. Then* *the rigidified Picard functor* $(\mathrm{Pic}_{X/S}, Y)$ *is representable by an algebraic space over* S, *and there exists a universal rigidified line bundle on* $(\mathrm{Pic}_{X/S}, Y)$.

The proofs of these theorems make use of a general principle for the construction of algebraic spaces which is due to M. Artin; cf. [5], Thm. 3.4. Namely, there is a criterion describing a necessary and sufficient condition for the representability of contravariant functors from (Sch/S) to (Sets) by algebraic spaces. It is for this criterion that the category of algebraic spaces yields a natural environment for questions on the representability of contravariant functors from (Sch/S) to (Sets). Within the category of algebraic spaces one can carry out many of the fundamental constructions, as contained in [FGA], under more general conditions, and one achieves results on the representability of certain functors under quite general assumptions.

Before we explain the criterion, let us briefly mention the basic definitions concerning algebraic spaces. As an introduction to the theory of algebraic spaces, we refer to M. Artin [3]. A detailed treatment can be found in Knutson [1].

In the following, let S be a scheme. Sometimes, for technical reasons, when we want to apply the approximation theorem 3.6/16, we have to assume that the base scheme S is locally of finite type over a field or over an excellent Dedekind ring.

Definition 4. *A (locally separated) algebraic space* X *over* S *is a functor*

$$X : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$$

with the following properties:

(i) *X is a sheaf with respect to the étale topology.*

(ii) *There exists a morphism* $\tau : U \longrightarrow X$ *of an S-scheme U, which is locally of finite presentation, to X such that τ is relatively representable by étale surjective morphisms of schemes.*

(iii) *The product* $U \times_X U$ *is represented by a subscheme of* $U \times_S U$ *such that the immersion* $U \times_X U \longrightarrow U \times_S U$ *is quasi-compact.*

Condition (ii) means that, for every S-scheme V and every morphism $V \longrightarrow X$, the product $U \times_X V$ is represented by a scheme and that the projection $U \times_X V \longrightarrow V$ is étale and surjective. Furthermore, it follows from (iii) that $U \times_X V \longrightarrow U \times_S V$ is a quasi-compact immersion. The algebraic space X is called *separated* over S if $U \times_X U$ is representable by a closed subscheme of $U \times_S U$.

Keeping the notations of Definition 4, the algebraic space X is the quotient of U by the equivalence relation $R = U \times_X U$ (in terms of sheaves with respect to the étale topology). Conversely, given an S-scheme U of locally finite presentation and a finitely presented subscheme R of $U \times_S U$ which defines an étale equivalence relation, one can show that the quotient of U by R (in terms of sheaves with respect to the étale topology) is an algebraic space. Thus we also could have defined algebraic spaces over S as quotients of S-schemes by étale equivalence relations.

A *morphism of algebraic spaces* over S is a morphism of functors. Viewing an algebraic space as a quotient of a scheme with respect to an étale equivalence relation, one can describe morphisms between algebraic spaces in terms of morphisms between schemes.

Proposition 5. *Let* $f : X_1 \longrightarrow X_2$ *be a morphism of algebraic spaces over S. Then, for each i, there exists a representation of X_i as a quotient of an S-scheme U_i by an étale equivalence relation (as above), and there is an S-morphism* $g : U_1 \longrightarrow U_2$ *such that one has the following commutative diagram*

$$\begin{array}{ccccc} (U_1 \times_{X_1} U_1) & \rightrightarrows & U_1 & \longrightarrow & X_1 \\ \downarrow{\scriptstyle g \times g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ (U_2 \times_{X_2} U_2) & \rightrightarrows & U_2 & \longrightarrow & X_2 . \end{array}$$

Furthermore, any morphism $g : U_1 \longrightarrow U_2$ *inducing a commutative square as on the left-hand side gives rise to a morphism* $f : X_1 \longrightarrow X_2$.

Associating to an S-scheme its functor of points, one gets a canonical map from the category of S-schemes to the category of algebraic spaces over S. This map gives rise to a fully faithful left exact embedding of categories. In the following, we will usually identify an S-scheme with its associated algebraic space over S.

Clearly, any property of S-schemes which is local for the étale topology, carries over to the context of algebraic spaces. One just requires that the property under consideration holds for the scheme U in Definition 4. This applies to the properties of being reduced, normal, regular, locally noetherian, etc.. Similarly, any property of morphisms of schemes which is local for the étale topology (on the source and on the target) carries over to the category of algebraic spaces. Thus, the properties of being flat, étale, locally of finite type, locally of finite presentation, etc. are defined. In particular, an algebraic space is provided with an étale topology in a natural way; a basis for this topology is given by the family of S-schemes U which are étale over X. The structure sheaves $\mathcal{O}_U$, where U is a scheme mapping étale to X, induce a sheaf (with respect to the étale topology) $\mathcal{O}_X$ on the algebraic space X. This sheaf is called the structure sheaf of X.

A morphism $Y \longrightarrow X$ of algebraic spaces over S is called an immersion (resp. open immersion, resp. closed immersion) if $Y \longrightarrow X$ is relatively representable by an immersion (resp. open immersion, resp. closed immersion). Thus, the notions of open and of closed subspaces of X are defined in the obvious way as equivalence classes of immersions. In particular, X carries a Zariski topology.

An algebraic space X over S is called quasi-compact if there exists a surjective étale morphism $U \longrightarrow X$ where U is a quasi-compact scheme. A morphism $X \longrightarrow Y$ of algebraic spaces is called quasi-compact if for any quasi-compact scheme V over Y, the fibre product $X \times_Y V$ is quasi-compact. Then we define a morphism $X \longrightarrow Y$ of algebraic spaces to be of finite type if it is quasi-compact and locally of finite type; and to be of finite presentation if it is quasi-compact, quasi-separated, and locally of finite presentation.

A morphism $X \longrightarrow Y$ of algebraic spaces is called proper if it is separated, of finite type, and universally closed. The latter has to be tested on the scheme level.

We mention that there is a valuative criterion for properness; cf. Deligne and Mumford [1], Thm. 4.19.

Now let us introduce the notion of points of an algebraic space.

Definition 6. *A point x of an algebraic space X over S is a morphism $x\colon \operatorname{Spec} K \longrightarrow X$ of algebraic spaces over S, where K is a field and where x is a categorical monomorphism. The field K is called the residue field of x, usually denoted by $k(x)$.*

Two points $x_i : \operatorname{Spec} K_i \longrightarrow X$, $i = 1, 2$, are called equivalent if there is an isomorphism $\sigma : \operatorname{Spec} K_1 \longrightarrow \operatorname{Spec} K_2$ such that $x_1 = x_2\sigma$. We identify equivalent points. Since, in Definition 6, we have required x to be a monomorphism, it is easily seen that this notion of points is equivalent to the usual one when X is a scheme. Furthermore, if $U \longrightarrow X$ is a morphism where U is a scheme, then each point of U induces a point of X. So every non-empty algebraic space X over S has a point whose residue field is of finite type over S. One can even show that, for each point x of X, there exists an étale map $U \longrightarrow X$ from a scheme U and a point u of U mapping to x such that the induced extension of the residue fields $k(x) \longrightarrow k(u)$ is trivial. Such a pair (U, u) is called an étale neighborhood of (X, x) without residue field extension. By using Lemma 2.3/7, one easily sees that the family of all such étale neighborhoods is a directed inductive system. So we get the notion of a local ring at a point of an algebraic space.

Definition 7. *The local ring for the étale topology of an algebraic space X at a point x of X is defined by the inductive limit*

$$\mathcal{O}_{X,x} = \varinjlim \mathcal{O}_{U,u}$$

where the limit is taken over the family of all étale neighborhoods (U, u) of (X, x) without residue field extension.

As explained in Section 2.3, this ring is henselian. If x is a point of a scheme X, the henselization of the local ring of X at x (in terms of schemes with respect to the Zariski topology) serves as the local ring of X at x if X is viewed as an algebraic space.

Let us mention some conditions under which an algebraic space is already a scheme. So let us start with an S-scheme U and an étale equivalence relation R on U. If R is finite, then the quotient U/R (in terms of sheaves with respect to the étale topology) is represented by a scheme if and only if, for each point u of U, the set of points which, under R, are equivalent to u is contained in an affine open subscheme; cf. [FGA], n°212, Thm. 5.3. For example, if U is affine, then U/R is represented by the affine scheme defined by the kernel of the maps

$$\mathcal{O}_U(U) \rightrightarrows \mathcal{O}_R(R)\,.$$

In general, such a quotient is just an algebraic space and not necessarily a scheme, even if R is finite. But it can be shown that, for any algebraic space X over S, there exists a dense open subspace which is a scheme. If the base scheme S is a field, separated algebraic spaces over S of dimension 1 are schemes. Furthermore, group

objects in the category of algebraic spaces over a field are schemes, as one easily shows by using the results of Section 6.6.

Next we want to describe M. Artin's criterion for a functor to be an algebraic space. We begin by reviewing some notions which are needed to state the general theorem. In the following, let S be a base scheme which is *locally of finite type over a field or over an excellent Dedekind ring*, and let

$$F : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$$

be a contravariant functor. If $T = \operatorname{Spec} B$ is an affine scheme over S, we will also write $F(B)$ instead of $F(T)$.

The functor F is said to be *locally of finite presentation* over S if, for every filtered inverse system of affine S-schemes $\{\operatorname{Spec} B_i\}$, the canonical morphism

$$\varinjlim F(B_i) \longrightarrow F(\varinjlim B_i)$$

is an isomorphism. Note that, if F is an S-scheme, then F is locally of finite presentation as a functor if and only if it is locally of finite presentation as a scheme over S; cf. [EGA IV_3], 8.14.2.

Furthermore, we need some definitions concerning deformations. Let s be a point in S whose residue field is of finite type over S, let k' be a finite extension of $k(s)$, and let ζ_0 be an element of $F(k')$. An *infinitesimal deformation* of ζ_0 is a pair (A, ξ) where A is an artinian local S-scheme with residue field k', and where ξ is an element of $F(A)$ inducing $\zeta_0 \in F(k')$ by functoriality. A *formal deformation of* ζ_0 is a pair $(\bar{A}, \{\xi_n\}_{n\in\mathbb{N}})$, where $\bar{A}$ is a complete noetherian local $\mathcal{O}_S$-algebra with residue field k', where the elements $\xi_n \in F(\bar{A}/\mathfrak{m}^{n+1})$ are compatible in the sense that ξ_n induces ξ_{n-1} by functoriality, and where ξ_0 coincides with ζ_0. Here $\mathfrak{m}$ is the maximal ideal of $\bar{A}$. If the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is induced by an element $\bar{\xi} \in F(\bar{A})$ via functoriality, then $(\bar{A}, \{\xi_n\}_{n\in\mathbb{N}})$ or $(\bar{A}, \bar{\xi})$ is called an *effective formal deformation* of ζ_0. A formal deformation $(\bar{A}, \{\xi_n\}_{n\in\mathbb{N}})$ of ζ_0 is said to be *versal* (resp. *universal*) if it has the following property:

Let (B', η') be an infinitesimal deformation of ζ_0 and, for an integer n, let the $(n+1)$-st power of the maximal ideal of B' be zero. Let B be a quotient of B', and denote by $\eta \in F(B)$ the element induced by η'. Then every map

$$(\bar{A}/\mathfrak{m}^{n+1}, \xi_n) \longrightarrow (B, \eta)$$

sending ξ_n to η can be factored (resp. uniquely factored) through (B', η') in the sense of morphisms of $\mathcal{O}_S$-algebras.

We mention that, in general, the canonical map

$$(*) \qquad F(\bar{A}) \longrightarrow \varprojlim F(\bar{A}/\mathfrak{m}^{n+1})$$

is not injective. Hence, if $(\bar{A}, \bar{\xi})$ is an effective formal deformation of ζ_0, the element $\bar{\xi} \in F(\bar{A})$ does not need to be uniquely determined by the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ even if $(\bar{A}, \bar{\xi})$ is universal. Nevertheless, the ring $\bar{A}$ is uniquely determined (up to canonical isomorphism) if $(\bar{A}, \bar{\xi})$ is a universal deformation of ζ_0. But, for most of the functors we are interested in, the map $(*)$ is bijective for any noetherian complete local $\mathcal{O}_S$-algebra $\bar{A}$. For example, this is the case for the Hilbert functor $\mathrm{Hilb}_{X/S}$ or for the relative Picard functor $\mathrm{Pic}_{X/S}$ if X is proper over S, as one can show by using

Grothendieck's existence theorem on formal sheaves; cf. [EGA III_1], § 5. In particular, in these cases any formal deformation is effective.

Now let X be an algebraic space over S, and let x be a point of X which is of finite type over S. Denote by $k(x)$ the residue field of x and by ζ_0^x the inclusion of x into X. Let $\bar{A}^x$ be the completion of the local ring of X at x with respect to the maximal ideal, and let

$$\bar{\xi}^x : \operatorname{Spec} \bar{A}^x \longrightarrow X$$

be the canonical morphism. The pair $(\bar{A}^x, \bar{\xi}^x)$ will serve as an effective formal deformation of ζ_0^x which is universal. Thus, in order to show that a contravariant functor F from (Sch/S) to (Sets) is an algebraic space, one should first look for the existence of universal deformations at all points of F which are of finite type over S. Therefore, one introduces the following notion.

A contravariant functor $F : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ is said to be *pro-representable* if the following data are given:

(a) an index set I,

(b) for each $x \in I$, an $\mathcal{O}_S$-field of finite type k^x and an element $\zeta_0^x \in F(k^x)$,

(c) for each $x \in I$, a formal deformation $(\bar{A}^x, \{\xi_n^x\}_{n \in \mathbb{N}})$ of $\zeta_0^x \in F(k^x)$,

satisfying the condition that, for each artinian local S-scheme T of finite type and for each $\eta \in F(T)$, there is a unique $x \in I$ and a unique map $T \longrightarrow \operatorname{Spec} \bar{A}^x$ sending $\{\xi_n^x\}$ to η.

Note that $(\bar{A}^x, \{\xi_n^x\}_{n \in \mathbb{N}})$ is a universal formal deformation of ζ_0^x. Furthermore, F is called *effectively pro-representable* if each sequence $\{\xi_n^x\}$ is induced by an element $\bar{\xi}^x \in F(\bar{A}^x)$. If F is effectively pro-representable, then the elements $x \in I$ are called the points of finite type of F. In the case where F is an algebraic space, the notion of points of finite type coincides with the one given in Definition 6; one associates to $x \in I$ the point of F given by the map $\zeta_0^x : \operatorname{Spec} k^x \longrightarrow F$. The universal deformations $(\bar{A}^x, \bar{\xi}^x)$ of ζ_0^x, $x \in I$, are called the *formal moduli* of F.

A morphism $\xi : X \longrightarrow F$ from an S-scheme X to the functor F is said to be *formally smooth* (resp. *formally étale*) *at a point* $x \in X$ if ξ fulfills the following lifting property: For every commutative diagram of morphisms

$$\begin{tikzcd} X \arrow[d] & Z_0 \arrow[l] \arrow[d] \\ F & Z \arrow[l] \end{tikzcd}$$

where Z is an artinian S-scheme, where Z_0 is a closed subscheme of Z defined by a nilpotent ideal, and where $Z_0 \longrightarrow X$ is a map sending Z_0 to x, there exists a factorization (resp. a unique factorization) $Z \longrightarrow X$ making the diagram commutative. One easily shows that, if $\xi : X \longrightarrow F$ is relatively representable by morphisms which are locally of finite presentation, ξ is formally étale at a point x of X if and only if, after any base change $Y \longrightarrow F$ by an S-scheme Y, the projection $X \times_F Y \longrightarrow Y$ is étale at every point of $X \times_F Y$ above x; use [EGA IV_4], 17.14.2.

Theorem 8 (M. Artin [5], Thm. 3.4). *Let S be a scheme which is locally of finite type over a field or over an excellent Dedekind ring. Let F be a functor from* $(\mathrm{Sch}/S)^0$ *to* (Sets). *Then F is an algebraic space (resp. a separated algebraic space) over S if and only if the following conditions hold:*

[0] (sheaf axiom) *F is a sheaf for the étale topology.*

[1] (finiteness) *F is locally of finite presentation.*

[2] (pro-representability) *F is effectively pro-representable.*

[3] (relative representability) *Let T be an S-scheme of finite type, and let ξ, $\eta \in F(T)$. Then the condition $\xi = \eta$ is representable by a subscheme (resp. a closed subscheme) of $T \times_S T$.*

[4] (openness of versality) *Let X be an S-scheme of finite type, and let $\xi : X \longrightarrow F$ be a morphism. If ξ is formally étale at a point $x \in X$, then it is formally étale in a neighborhood of x.*

The necessity is not difficult to show and has already been discussed when introducing the above notions. For the sufficiency which is the more interesting part, one needs an approximation argument for algebraic structures over complete local rings; cf. M. Artin [5], Thm. 1.6. The rough idea for the proof of the sufficiency is the following.

One has to find a morphism $U \longrightarrow F$ from an S-scheme which is locally of finite presentation to F such that $U \longrightarrow F$ is relatively representable by étale surjective morphisms. We will first construct an étale neighborhood for each point of F which is of finite type over S. Consider such a point x of F, and let $(\bar{A}^x, \bar{\xi}^x)$ be the formal deformation pro-representing F at x. Then one constructs an algebraization of $(\bar{A}^x, \bar{\xi}^x)$; i.e., an S-scheme X of finite type, a closed point $x \in X$ with residue field $k(x) = k^x$, and an element $\xi \in F(X)$, such that the triple (X, x, ξ) gives rise to a versal formal deformation of ζ_0^x. More precisely, there is an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \bar{A}^x$ such that ξ induces ξ_n^x in $F(\bar{A}^x/\mathfrak{m}^{n+1})$ for each $n \in \mathbb{N}$. The existence of such an algebraization follows easily from the approximation theorem 3.6/16 if the ring $\bar{A}^x$ of the formal modulus is isomorphic to a formal power series ring $\hat{\mathcal{O}}_{S,s}[[t_1, \ldots, t_n]]$, where $\hat{\mathcal{O}}_{S,s}$ is the completion of a local ring of S.—For example, this holds for the Picard functor of a relative curve.—In this case, $\bar{A}^x$ is the completion of an S-scheme X of finite type at a point x of finite type. Namely, write $\bar{A}^x$ as a union of $\mathcal{O}_S$-subalgebras B of finite type. Since F is assumed to be locally of finite presentation, the element $\bar{\xi}^x$ is represented by an element $\xi \in F(B)$ for some $\mathcal{O}_S$-subalgebra B of finite type. The inclusion $B \hookrightarrow \bar{A}^x$ yields a map $F(B) \longrightarrow F(\bar{A}^x)$ sending ξ to $\bar{\xi}^x$. Due to the approximation theorem, there is an étale neighborhood (X', x') of (X, x) without residue field extension such that there is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \bar{A}^x \arrow[d] & \operatorname{Spec} \bar{A}^x/\mathfrak{m}^2 \bar{A}^x \arrow[l] \arrow[d] \\ \operatorname{Spec} B & X' \arrow[l] \end{tikzcd}$$

sending the closed point of $\operatorname{Spec} \bar{A}^x/\mathfrak{m}^2 \bar{A}^x$ to x'. The completion $\hat{\mathcal{O}}_{X',x'}$ is still isomorphic to the ring $\bar{A}^x$. Denote by $\xi' \in F(X')$ the image of ξ under the functorial map $F(B) \longrightarrow F(X')$. Due to the versality of $(\bar{A}, \bar{\xi}^x)$, there is an automorphism $\varphi : \bar{A}^x \longrightarrow \bar{A}^x$, which is the identity modulo $\mathfrak{m}^2 \bar{A}^x$, and which sends ξ_n^x to ξ_n' for each

$n \in \mathbb{N}$ where ξ'_n is induced by ξ' via functoriality. Thus (X', x', ξ') is the required algebraization.

Now, let I be the set of points of F which are of finite type over S and, for $x \in I$, denote by (U^x, u^x, ξ^x) an algebraization of the formal modulus $(\bar{A}^x, \bar{\xi}^x)$. One easily shows that $\xi^x : U^x \longrightarrow F$ is formally étale at u^x. Due to condition [4], after shrinking U^x we may assume that ξ^x is étale at every point. Hence, since $U^x \longrightarrow F$ is relatively representable by condition [3], it is representable by étale maps. If we denote by U the disjoint union of the U^x, $x \in I$, the map

$$U = \coprod_{x \in I} U^x \longrightarrow F$$

is representable by étale surjective maps. Furthermore, due to condition [3], the equivalence relation $U \times_F U \longrightarrow U \times_S U$ is relatively representable by a subscheme (resp. by a closed subscheme) of $U \times_S U$. Thereby we see that F is an algebraic space as asserted in Theorem 8. □

Conditions [0] and [1] are natural, and they are satisfied quite often. For conditions [2] and [3], it is convenient to suppose that there is a deformation theory for the functor F so that one can rewrite the conditions in terms of deformation theory. Then it is often possible to decide whether a functor is pro-representable or relatively representable. Condition [4] is the one which is most difficult to verify, but it can also be interpreted by infinitesimal methods. We mention that there is a general theorem by M. Artin which relates the representability of a functor admitting a deformation theory to a list of conditions which can be checked in specific situations; for instance for the Hilbert functor or the relative Picard functor; cf. M. Artin [5], Thm. 5.4. Since many technical details are involved, we omit precise statements here.

To end our discussion, we want to indicate the procedure of proof for Theorem 1. Details can be found in M. Artin [5], Section 7; see also the appendix of M. Artin [7]. Since X is assumed to be of finite presentation over S, one can reduce to the case where the base scheme S is of finite type over the integers $\mathbb{Z}$. Then one applies the general criterion for a functor to be an algebraic space. The deformation theory for $\mathrm{Pic}_{X/S}$ is given by the exponential map. If $f : X \longrightarrow S$ is cohomologically flat in dimension zero, the deformation theory for $\mathrm{Pic}_{X/S}$ fulfills all conditions which are required in the list of the general statement. Thus $\mathrm{Pic}_{X/S}$ is pro-representable. Due to Grothendieck's existence theorem on formal sheaves, [EGA III_1], § 5, one obtains formal moduli for $\mathrm{Pic}_{X/S}$, i.e., $\mathrm{Pic}_{X/S}$ is effectively pro-representable. Then, due to M. Artin's approximation theorem, the formal moduli are algebraizable, and hence one gets local models for the space which will represent $\mathrm{Pic}_{X/S}$. Since these local models are unique up to étale morphism, they can be glued together to form an algebraic space over S.

Finally let us mention that the definition of algebraic spaces is not generalized by allowing flat equivalence relations of finite type in place of étale ones. This is due to the following fact; cf. M. Artin [7], Cor. 6.3.

If U is an S-scheme of finite type over a noetherian base scheme S, and if R is a flat equivalence relation of finite type on U, then the quotient U/R in terms of sheaves for the fppf-topology is represented by an algebraic space.

As a corollary, one obtains the following useful assertion.

Proposition 9. *Let H and G be group objects in the category of algebraic spaces over S and let $H \longrightarrow G$ be an immersion. Assume that H is flat over S. Then the quotient G/H in terms of sheaves for the fppf-topology is represented by an algebraic space.*

8.4 Properties

In this section we want to collect some results concerning the smoothness and certain finiteness properties of $\mathrm{Pic}_{X/S}$. Let us start with a theorem which is contained in [FGA], n°236, Thm. 2.10, for the case where $\mathrm{Pic}_{X/S}$ is a scheme; but it is immediately clear that it remains true if $\mathrm{Pic}_{X/S}$ is an algebraic space.

Theorem 1. *Let $f : X \longrightarrow S$ be a proper and flat morphism which is locally of finite presentation. Assume that f is cohomologically flat in dimension zero so that $\mathrm{Pic}_{X/S}$ is an algebraic space. Then the following assertions hold.*

(a) *There is a canonical isomorphism*

$$\mathrm{Lie}(\mathrm{Pic}_{X/S}) \xrightarrow{\sim} R^1 f_* \mathcal{O}_X$$

where $\mathrm{Lie}(\mathrm{Pic}_{X/S})$ *is the Lie algebra of* $\mathrm{Pic}_{X/S}$.

(b) *If S is the spectrum of a field K, then*

$$\dim_K \mathrm{Pic}_{X/K} \le \dim_K H^1(X, \mathcal{O}_X)\,,$$

and equality holds if and only if $\mathrm{Pic}_{X/K}$ *is smooth over K. In particular, the latter is the case if the characteristic of K is zero.*

Proof. (a) Write $\mathcal{O}_S[\varepsilon]$ for the $\mathcal{O}_S$-algebra of the dual numbers over $\mathcal{O}_S$, and set $S[\varepsilon] = \mathrm{Spec}(\mathcal{O}_S[\varepsilon])$. Then one can interpret $\mathrm{Lie}(\mathrm{Pic}_{X/S})$ as the subfunctor of $\mathrm{Hom}_S(S[\varepsilon]), \mathrm{Pic}_{X/S})$ consisting of all morphisms which, modulo ε, reduce to the unit section of $\mathrm{Pic}_{X/S}$. Setting $X[\varepsilon] = X \times_S S[\varepsilon]$, one obtains the exact sequence

$$\begin{array}{c} 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}^*_{X[\varepsilon]} \longrightarrow \mathcal{O}^*_X \longrightarrow 0 \\ h \longmapsto 1 + h \cdot \varepsilon \end{array}$$

Since f is cohomologically flat in dimension zero, the canonical map $f_* \mathcal{O}_{X[\varepsilon]} \longrightarrow f_* \mathcal{O}_X$ is surjective. Therefore the sequence of sheaves with respect to the étale-topology

$$0 \longrightarrow R^1 f_* \mathcal{O}_X \longrightarrow R^1 f_* \mathcal{O}^*_{X[\varepsilon]} \longrightarrow R^1 f_* \mathcal{O}^*_X \longrightarrow R^2 f_* \mathcal{O}_X$$

is exact. Since $\mathrm{Lie}(\mathrm{Pic}_{X/S})$ corresponds to the kernel of the map $R^1f_*\mathcal{O}^*_{X[\varepsilon]} \longrightarrow R^1f_*\mathcal{O}^*_X$, it can be identified with $R^1f_*\mathcal{O}_X$.

(b) follows from (a) and 2.2/15. □

Proposition 2. *Let $f: X \longrightarrow S$ be a proper and flat morphism which is locally of finite presentation. Let s be a point of S such that $H^2(X_s, \mathcal{O}_{X_s}) = 0$. Then there exists an open neighborhood U of s such that $\mathrm{Pic}_{X/S}|_U$ is formally smooth over U.*

In particular, in the case of a relative curve X over S, both $\mathrm{Pic}_{X/S}$ and $(\mathrm{Pic}_{X/S}, Y)$, where Y is a rigidificator for $\mathrm{Pic}_{X/S}$, are formally smooth over S.

Proof. Due to the semicontinuity theorem [EGA III_2], 7.7.5, there exists an open neighborhood U of s such that $H^2(X_s, \mathcal{O}_{X_s}) = 0$ for all $s \in U$. We may assume $U = S$. In order to prove that $\mathrm{Pic}_{X/S}$ is formally smooth over S, we have to establish the lifting property for $\mathrm{Pic}_{X/S}$. So consider an affine S-scheme Z and a subscheme Z_0 of Z which is defined by an ideal $\mathcal{N}$ of $\mathcal{O}_Z$ satisfying $\mathcal{N}^2 = 0$. Then we have to show that the map

$$R^1(f \times_S Z)_*\mathcal{O}^*_{X\times_S Z} \longrightarrow R^1(f \times_S Z_0)_*\mathcal{O}^*_{X\times_S Z_0}$$

is surjective. The cokernel of this map is a subsheaf of the $\mathcal{O}_Z$-module $R^2(f \times_S Z)_*(\mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{O}_X)$. The latter vanishes, since $H^2(X_s, \mathcal{O}_{X_s}) = 0$ for all $s \in S$; use [EGA III_2], 7.7.10 and 7.7.5 (II). Thus we see that $\mathrm{Pic}_{X/S}$ satisfies the lifting property and, hence, is formally smooth over S.

In the case of a relative curve X over S, the assumption $H^2(X_s, \mathcal{O}_{X_s}) = 0$ is satisfied at all $s \in S$, so $\mathrm{Pic}_{X/S}$ is formally smooth over S. Furthermore, since there is no obstruction to lifting a rigidification, we see that $(\mathrm{Pic}_{X/S}, Y)$ is formally smooth over S, too. □

Now we will concentrate on finiteness assertions for $\mathrm{Pic}_{X/S}$. When proving Grothendieck's theorem 8.2/1, we had seen in 8.2/5 that $\mathrm{Pic}^{\Phi}_{X/S}$ is quasi-projective over S. But if we impose stronger conditions on the fibres of X, we can expect better results.

Theorem 3 ([FGA], n°236, Thm. 2.1). *Let $f: X \longrightarrow S$ be a proper (resp. projective) morphism which is locally of finite presentation. Assume that the geometric fibres of X are reduced and irreducible. Then $\mathrm{Pic}_{X/S}$ is a separated algebraic space (resp. separated scheme) over S.*

If, in addition, $f: X \longrightarrow S$ is smooth, then each closed subspace Z of $\mathrm{Pic}_{X/S}$ which is of finite type over S is proper (resp. projective) over S. In particular, if S consists of a field K, the identity component $\mathrm{Pic}^0_{X/K}$ of $\mathrm{Pic}_{X/K}$ is a proper scheme over K.

Proof. $\mathrm{Pic}_{X/S}$ is an algebraic space over S, due to 8.3/1. If X is projective over S, we know from 8.2/1 that $\mathrm{Pic}_{X/S}$ is a scheme over S and from 8.2/5 that each closed subspace Z which is of finite type over S is quasi-projective over S. The remaining assertions follow by using the valuative criteria for separatedness and properness.

Indeed, we may assume that S is the spectrum of a discrete valuation ring R, and that X admits a section over S. For showing the separatedness, we have to verify that a line bundle $\mathscr{L}$ on X which is trivial on the generic fibre is trivial. There exists a global section $f \in \Gamma(X, \mathscr{L})$ which generates $\mathscr{L}$ on the generic fibre. Since the local ring $\mathcal{O}_{X,\eta}$ of X at the generic point η of the special fibre is a discrete valuation ring such that the extension $R \longrightarrow \mathcal{O}_{X,\eta}$ is of ramification index 1, we may assume that f generates $\mathscr{L}$ at η. Then it is clear that f generates $\mathscr{L}$ on X and that $\mathscr{L}$ is trivial. Next assume that X is smooth over S. For the properness, we have to show that each line bundle on the generic fibre of X extends to a line bundle on X. Since the local rings of X are regular, the notions of Cartier divisor and Weil divisor coincide. Obviously, Weil divisors on the generic fibre of X extend to Weil divisors on X. So, each line bundle on the generic fibre extends to a line bundle on X.

If S consists of a field K, then $\mathrm{Pic}_{X/K}$ is a scheme by 8.2/3. Since any connected K-group scheme is of finite type as soon as it is locally of finite type, we see that $\mathrm{Pic}^0_{X/K}$ is of finite type and, thus, proper over K. □

Next we want to discuss finiteness assertions for $\mathrm{Pic}_{X/S}$ under more general assumptions. Since, in general, $\mathrm{Pic}_{X/S}$ will have infinitely many connected components, it cannot be of finite type over S. So the best one can expect is that there exists an open and closed subgroup $\mathrm{Pic}^{\tau}_{X/S}$ of $\mathrm{Pic}_{X/S}$ which is of finite type over S and which has the property that the quotient of $\mathrm{Pic}_{X/S}$ by $\mathrm{Pic}^{\tau}_{X/S}$ has geometric fibres which are finitely generated as abstract groups. We want to introduce the subgroup $\mathrm{Pic}^{\tau}_{X/S}$.

If S consists of a field, we know that the relative Picard functor $\mathrm{Pic}_{X/S}$ is a group scheme. Let $\mathrm{Pic}^0_{X/S}$ be its identity component. Then we set

$$\mathrm{Pic}^{\tau}_{X/S} = \bigcup_{n>0} n^{-1}(\mathrm{Pic}^0_{X/S})$$

where $n: \mathrm{Pic}_{X/S} \longrightarrow \mathrm{Pic}_{X/S}$ is the multiplication by n. Due to continuity, $\mathrm{Pic}^{\tau}_{X/S}$ is an open subscheme of $\mathrm{Pic}_{X/S}$.

For a general base S, we introduce $\mathrm{Pic}^0_{X/S}$ (resp. $\mathrm{Pic}^{\tau}_{X/S}$) as the subfunctor of $\mathrm{Pic}_{X/S}$ which consists of all elements whose restrictions to all fibres X_s, $s \in S$, belong to $\mathrm{Pic}^0_{X_s/k(s)}$ (resp. $\mathrm{Pic}^{\tau}_{X_s/k(s)}$). If $\mathrm{Pic}_{X/S}$ is an algebraic space (resp. a scheme), and if it is smooth over S along the unit section, then $\mathrm{Pic}^0_{X/S}$ is represented by an open subspace (resp. an open subscheme) of $\mathrm{Pic}_{X/S}$, cf. [EGA IV_3], 15.6.5.

Theorem 4 ([SGA 6], Exp. XIII, Thm. 4.7). *Let $f: X \longrightarrow S$ be a proper morphism which is locally of finite presentation, and let S be quasi-compact. Then*

(a) *The canonical inclusion $\mathrm{Pic}^{\tau}_{X/S} \hookrightarrow \mathrm{Pic}_{X/S}$ is relatively representable by an open and quasi-compact immersion.*

(b) *If $X \longrightarrow S$ is projective and if its geometric fibres are reduced and irreducible, the immersion $\mathrm{Pic}^{\tau}_{X/S} \hookrightarrow \mathrm{Pic}_{X/S}$ is open and closed.*

(c) *$\mathrm{Pic}^{\tau}_{X/S}$ is of finite type over S in the sense that the family of isomorphism classes of line bundles on the fibres of X which belong to $\mathrm{Pic}^{\tau}_{X/S}$ is bounded.*

The hardest part of the theorem is assertion (c). One can reduce it to the case where X is a closed subscheme of a projective space $\mathbb{P}^n_S$. In this case, one shows that all elements of $\mathrm{Pic}^{\tau}_{X/S}$ have the same Hilbert polynomial (with respect to the S-ample line bundle belonging to the embedding of X into $\mathbb{P}^n_S$), and then the assertion can be deduced from 8.2/5.

Next, we want to look at the special case where X is an abelian S-scheme, i.e., a smooth and proper S-group scheme with connected fibres.

Theorem 5. *Let A be a projective abelian S-scheme.*

(a) *Then $\mathrm{Pic}^{\tau}_{A/S}$ is a projective abelian S-scheme. It is denoted by A^* and is called the dual abelian scheme of A. In particular, A^* coincides with the identity component of $\mathrm{Pic}_{A/S}$.*

(b) *The Poincaré bundle on $A \times_S A^*$ gives rise to a canonical isomorphism $\iota : A \longrightarrow A^{**}$ where A^{**} is the dual abelian scheme of A^*.*

A proof of (a) can be found in Mumford [1], Corollary 6.8. For (b), since A and A^{**} are flat over S, it suffices to treat the case where S consists of an algebraically closed field. In this case, the assertion follows from Mumford [3], Section 13, p. 132.

In 1.2/8 we have seen that an abelian scheme over a Dedekind scheme is the Néron model of its generic fibre. Now, using the above theorem, one can show a much stronger mapping property for abelian schemes than the one required for Néron models.

Corollary 6. *Let A be an abelian S-scheme. Then any rational S-morphism $\varphi : T \dashrightarrow A$ from an S-scheme T to A is defined everywhere if T is regular.*

Proof. We may assume $T = S$. Then A is projective over S; cf. Murre [2], p. 16. Due to Theorem 5, we can identify A and A^{**}. So the map φ corresponds to a line bundle on $A^* \times_S S'$ where S' is a dense open subscheme of S. Since $S = T$ is regular and since $A^* \longrightarrow S$ is smooth, the scheme A^* is regular. So the line bundle extends to a line bundle on A^* and, thus, gives rise to an extension $S \longrightarrow A^{**}$ of $\varphi_{|S'}$. □

Now let us return to the general situation of a proper morphism $X \longrightarrow S$ of schemes. We want to discuss the group of connected components of $\mathrm{Pic}_{X/S}$ over a geometric point of S. Let s be a point of S and let $\bar{s}$ be a geometric point of S such that $k(\bar{s})$ is an algebraic closure of the residue field $k(s)$ at s. The group of connected components of $\mathrm{Pic}_{X_{\bar{s}}/k(\bar{s})}$ is called the *Néron-Severi group* of the geometric fibre $X_{\bar{s}} = X \times_S k(\bar{s})$ of X over s. It is denoted by $\mathrm{NS}_{X/S}(\bar{s})$ so that

$$\mathrm{NS}_{X/S}(\bar{s}) = \mathrm{Pic}_{X_{\bar{s}}/k(\bar{s})}(k(\bar{s})) \,/\, \mathrm{Pic}^0_{X_{\bar{s}}/k(\bar{s})}(k(\bar{s})) \, .$$

Theorem 7. ([SGA 6], Exp. XIII, Thm. 5.1). *Let $f : X \longrightarrow S$ be a proper morphism which is locally of finite presentation, and assume that S is quasi-compact. Then the Néron-Severi groups $\mathrm{NS}_{X/S}(\bar{s})$ of the geometric fibres of X are finitely generated. Their ranks as well as the orders of their torsion subgroups are bounded simultaneously.*

Remark 8. The Néron-Severi group is of arithmetical nature; i.e., the set of points where the Néron-Severi group is of a fixed type is not necessarily constructible.

For example, let $E \longrightarrow S$ be an elliptic curve with a non-constant j-invariant over an irreducible base S which is of finite type over a field. Then there are infinitely many geometric points $\bar{s}$ of S such that the geometric fibre $E_{\bar{s}}$ has complex multiplication, and there are infinitely many geometric points such that the geometric fibre $E_{\bar{s}}$ has no complex multiplication. Now consider the product $X = E \times_S E$. If $E_{\bar{s}}$ has no complex multiplication, the rank of the Néron-Severi group of $X_{\bar{s}}$ is 3. If $E_{\bar{s}}$ has complex multiplication, the rank of the Néron-Severi group of $X_{\bar{s}}$ is at least 4.

Chapter 9. Jacobians of Relative Curves

The chapter consists of two parts. In the first four sections we study the representability and structure of $\mathrm{Pic}_{X/S}$ for a relative curve X over a base S. Then, in the last three sections, we work over a base S consisting of a discrete valuation ring R with field of fractions K and, applying these results, we investigate the relationship between $\mathrm{Pic}_{X/S}$ and the Néron model of the Jacobian J_K of the generic fibre X_K.

The chapter begins with a discussion of the degree of divisors on relative curves. Then we give a detailed analysis of the Jacobian J_K of a proper curve X_K over a field, showing that the structure of J_K is closely related to geometric properties of X_K. The next two sections deal with the representability of Jacobians over a more general base. First, imposing strong conditions on the fibres of the curve and working over a strictly henselian base, we prove the representability by a scheme, using a method which was originally employed by Weil [2] and Rosenlicht [1]; see also Serre [1]. Then we explain results due to Deligne [1] and Raynaud [6], which are valid under far weaker conditions.

In the second half of the chapter, we follow Raynaud [6] and consider a proper and flat curve X over a discrete valuation ring R, assuming in most cases that X is regular at each of its points and that the generic fibre X_K is geometrically irreducible. Let P be the open subfunctor of $\mathrm{Pic}_{X/R}$ consisting of all line bundles of total degree 0 and let Q be the biggest separated quotient of P. We show that Q is a smooth R-group scheme whose generic fibre coincides with the Jacobian J_K of the generic fibre X_K. Thus if J is a Néron model of J_K, there is a canonical R-morphism $Q \longrightarrow J$. Without assuming the existence of J, we can prove under quite general conditions that, for example, if the residue field of R is perfect, then Q is already a Néron model of J_K. Thereby it is seen that the relative Picard functor leads to a second possibility of constructing Néron models. Also there are important situations where the identity component of $\mathrm{Pic}^0_{X/R}$ is already a separated scheme and where the canonical morphism $\mathrm{Pic}^0_{X/R} \longrightarrow J^0$ is an isomorphism. More precisely, we will see that the coincidence of $\mathrm{Pic}^0_{X/R}$ and J^0 is related to the fact that X has rational singularities.

In the above cases where Q is already a Néron model of J_K, it is possible to compute explicitly the group of components (of the special fibre) of this model, using the intersection form on X. In Section 9.6, we explain the general approach and carry out some computations in particular cases.

9.1 The Degree of Divisors

Let X be a proper curve over a field K. If x is a closed point of X and if f is a regular element of $\mathcal{O}_{X,x}$, define the vanishing order of f at x by

$$\mathrm{ord}_x(f) := l_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f))$$

where $l_{\mathcal{O}_{X,x}}$ denotes the length of $\mathcal{O}_{X,x}$-modules. If, for example, x is a regular point of X, the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring and $\mathrm{ord}_x(f)$ corresponds to the order of f in $\mathcal{O}_{X,x}$ (with respect to the canonical valuation on $\mathcal{O}_{X,x}$). Since we have

$$\mathrm{ord}_x(f \cdot g) = \mathrm{ord}_x(f) + \mathrm{ord}_x(g)$$

for a product of regular elements $f, g \in \mathcal{O}_{X,x}$, we can define

$$\mathrm{ord}_x(f/g) = \mathrm{ord}_x(f) - \mathrm{ord}_x(g)$$

for any element f/g of the total ring of fractions of $\mathcal{O}_{X,x}$.

Now let D be a Cartier divisor on X. For a closed point $x \in X$, we set

$$\mathrm{ord}_x(D) = \mathrm{ord}_x(f_x/g_x)$$

where f_x/g_x is a local representation of D in a neighborhood of x. We can associate to D the Weil divisor

$$\sum_{x \in X} \mathrm{ord}_x(D) \cdot x \,.$$

The *degree of a Cartier divisor* D is defined by

$$\deg(D) = \sum_{x \in X} \mathrm{ord}_x(D) \cdot [k(x) : K] \,.$$

The degree function is additive, i.e.,

$$\deg(D_1 + D_2) = \deg(D_1) + \deg(D_2) \,.$$

If D is effective, we can write

$$\deg(D) = \dim_K H^0(X, \mathcal{O}_D)$$

where $\mathcal{O}_D$ denotes the structure sheaf of the subscheme associated to D. Thus we see that the degree of a Cartier divisor on X is not altered by a base change with a field extension K'/K.

Assuming for a moment that X is reduced, we can consider the normalization $\tilde{X} \longrightarrow X$ of X. Then one can pull back Cartier divisors D on X to Cartier divisors $\tilde{D}$ on $\tilde{X}$. We claim that

$$\deg(D) = \deg(\tilde{D}) \,.$$

Indeed, it suffices to justify the following assertion. Let $U = \mathrm{Spec}(A)$ be an affine open subscheme of X and let $\tilde{A}$ be the normalization of $\tilde{A}$. Then, for each regular element f of A, one has

$$\dim_K(A/(f)) = \dim_K(\tilde{A}/(f)) \,.$$

In order to prove this, look at the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f_A"] & \tilde{A} \arrow[r] \arrow[d, "f_{\tilde{A}}"] & C \arrow[r] \arrow[d, "f_C"] & 0 \\
0 \arrow[r] & A \arrow[r] & \tilde{A} \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where the vertical maps are given by the multiplication with f. Since f is a regular element of both A and $\tilde{A}$, there is a long exact sequence

$$0 \longrightarrow \ker(f_C) \longrightarrow A/(f) \longrightarrow \tilde{A}/(f) \longrightarrow C/f \cdot C \longrightarrow 0\,.$$

Using $\dim_K(C) < \infty$, it follows that $\dim_K(\ker(f_C)) = \dim_K(C/f \cdot C)$. Hence, the assertion is evident.

A Cartier divisor D on an arbitrary proper curve X is called *principal* if there exists a meromorphic function f on X such that $D = \operatorname{div}(f)$. For a principal divisor D, we have $\deg(D) = 0$. Two Cartier divisors D_1 and D_2 are said to be *linearly equivalent* if the difference $D_1 - D_2$ is principal. So we see that the degree of a Cartier divisor D is not altered if we replace D by a divisor which is linearly equivalent to D. Since each line bundle $\mathscr{L}$ on X corresponds to a Cartier divisor D which is unique up to linear equivalence, one can define the *degree of a line bundle* $\mathscr{L}$ by setting $\deg(\mathscr{L}) := \deg(D)$. The degree plays an important role in the Riemann-Roch formula.

Theorem 1. *Let X be a proper curve over a field K, and let $\mathscr{L}$ be a line bundle on X. Then the Euler-Poincaré characteristic*

$$\chi(\mathscr{L}) = \dim_K H^0(X, \mathscr{L}) - \dim_K H^1(X, \mathscr{L})$$

of $\mathscr{L}$ is related to the Euler-Poincaré characteristic of $\mathcal{O}_X$ by the formula

$$\chi(\mathscr{L}) = \deg(\mathscr{L}) + \chi(\mathcal{O}_X)\,.$$

Proof. One proceeds as in the case of a smooth curve by looking at an exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D \longrightarrow 0$$

where D is an effective Cartier divisor on X such that $\mathscr{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X(E)$ with an effective Cartier divisor E on X. Furthermore, one has the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_E \longrightarrow 0\,.$$

Calculating the Euler-Poincaré characteristic of both sequences, the assertion follows immediately from $\mathscr{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \cong \mathcal{O}_X(E)$ and $\deg \mathscr{L} = \deg E - \deg D$. □

Man erhält: $\chi(\mathscr{L}) = \chi(\mathcal{O}_X) + (\chi(\mathcal{O}_E) - \chi(\mathcal{O}_D))$ und $\chi(\mathcal{O}_E) = \dim H^0(X, \mathcal{O}_E) - \dim H^1(X, \mathcal{O}_E) = \deg(D) - 0$, da $H^1(X, \mathcal{O}_E) = H^1(E, i^*\mathcal{O}_E) = 0$ (siehe Hartshorne III, Lemma 2.10 & Theor 2.7)

If $H^0(X, \mathcal{O}_X) = K$, for example, if X is geometrically reduced and connected, the Euler-Poincaré characteristic of $\mathcal{O}_X$ is given by $\chi(\mathcal{O}_X) = 1 - p_a$, where $p_a = \dim_K H^1(X, \mathcal{O}_X)$ is the arithmetic genus of the curve X.

If $X \longrightarrow S$ is a relative curve and if $\mathscr{L}$ is a line bundle on X, one can restrict $\mathscr{L}$ to the fibres of X over S. So, for each $s \in S$, we get a line bundle $\mathscr{L}_s$ on the fibre X_s, and the degree $\deg(\mathscr{L}_s)$ of $\mathscr{L}_s$ on the fibre X_s gives rise to a $\mathbb{Z}$-valued function on S.

Proposition 2. *Let $X \longrightarrow S$ be a flat proper S-curve of finite presentation and let $\mathscr{L}$ be a line bundle on X. For $s \in S$, denote by $\mathscr{L}_s$ the restriction of $\mathscr{L}$ to the curve X_s. Then the degree function*

$$\deg : S \longrightarrow \mathbb{Z}\,, \qquad s \longmapsto \deg(\mathscr{L}_s)$$

is locally constant on S.

Proof. The Euler-Poincaré characteristic of a flat family of coherent sheaves is locally constant on the base; cf. [EGA III$_2$], 7.9.4. Thus, using the Riemann-Roch formula, one sees that the degree function must be locally constant on S. □

Now let us return to the situation we started with. Let X be a proper curve over a field with (reduced) irreducible components $X_1, \ldots, X_r$. If $\mathscr{L}$ is a line bundle on X, we can restrict $\mathscr{L}$ to each component X_i, $i = 1, \ldots, r$, and we define the *partial degree* of $\mathscr{L}$ on X_i by

$$\deg_{X_i}(\mathscr{L}) = \deg(\mathscr{L}|_{X_i})\,.$$

In order to explain the relationship between the total degree and the partial degrees, we need the notion of multiplicities of irreducible components.

Definition 3. *Let X be a scheme of finite type over a field K, let $\bar{K}$ be an algebraic closure of K, and set $\bar{X} = X \otimes_K \bar{K}$. Denote by $X_1, \ldots, X_r$ the (reduced) irreducible components of X and, for $i = 1, \ldots, r$, let $\eta_i \in X$ be the generic point corresponding to X_i. The multiplicity of X_i in X is the length of the artinian local ring $\mathcal{O}_{X,\eta_i}$. We denote it by d_i; so*

$$d_i = l(\mathcal{O}_{X,\eta_i})\,.$$

The geometric multiplicity of X_i in X is the length of the artinian local ring $\mathcal{O}_{\bar{X},\bar{\eta}_i}$ where $\bar{\eta}_i$ is a point of $\bar{X}$ lying above η_i. We denote it by δ_i; so

$$\delta_i = l(\mathcal{O}_{\bar{X},\bar{\eta}_i})$$

If X is irreducible, we talk about the multiplicity (resp. the geometric multipicity) of X, thereby meaning the multiplicity (resp. the geometric multiplicity) of X in X. Furthermore, we denote by

$$e_i = l(\mathcal{O}_{\bar{X}_i,\bar{\eta}_i})$$

the geometric multiplicity of X_i.

Note that the definition is independent of the choice of $\bar{\eta}_i$, since all points of $\bar{X}$ above η_i are conjugated under the action of the Galois group of $\bar{K}$ over K. There are some elementary relations between the different notions of multiplicities which are easy to verify.

Lemma 4. *Keeping the notations of Definition 3, one has*

(a) $\delta_i = e_i \cdot d_i$ *for $i = 1, \ldots, r$.*

(b) $\delta_i = e_i$ *if and only if X is reduced at the point η_i.*

(c) $e_i = 1$ *if the characteristic of K is zero; otherwise it is a power of the characteristic of K.*

(a) Sei (A, I) lokal, [illegible]
A/I ist der einzige [illegible]
$A \supset I \supset I^2 \supset \ldots \supset I^n = 0$
$n = d_i$, $e_i = l((A/I) \otimes$ [illegible]
$A \otimes \bar{K} \supset I \otimes \bar{K} \supset \ldots \supset$ [illegible]
auffüllen mit e_i [illegible]
$\Rightarrow \delta_i = l(A \otimes \bar{K}) =$ [illegible]
(b) $d_i = 1 \Leftrightarrow A$ integ[illegible]
(c) Ist $K' \supset K$ separabel [illegible]
$(A/I) \otimes K'$ enthält [illegible]
O das einzige Prim[illegible]
$(A/I) \otimes K'$ artinisch [illegible]
(und Bosch 7.3/2)
Ist $K' \supset K$ rein inseparabel [illegible]
endlich, so ist [illegible]
[illegible]

Using the notion of multiplicity of components, one can state a relationship between the (total) degree and the partial degrees of a line bundle.

Proposition 5. *Let X be a proper curve over a field K with (reduced) irreducible components $X_1, \ldots, X_r$. Denote by d_i the multiplicity of X_i in X, $i = 1, \ldots, r$. Then*

$$\deg(\mathscr{L}) = \sum_{i=1}^{r} d_i \cdot \deg(\mathscr{L}|_{X_i})$$

for each line bundle $\mathscr{L}$ on X.

Proof. It suffices to establish the formula for Cartier divisors D whose support does not contain any intersection point of the different components. Since both sides of the formula are additive for divisors, we have only to consider effective Cartier divisors. Then the assertion follows from the lemma below. □

Lemma 6. *Let A be a one-dimensional noetherian local ring and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal prime ideals of A. Let M be a finitely generated A-module, and let a be an element of A which is not contained in any $\mathfrak{p}_i$. Denote by a_M the multiplication by a on M and define*

$$e_A(a, M) = l_A(\operatorname{coker}(a_M)) - l_A(\ker(a_M)) .$$

Then

$$e_A(a, M) = \sum_{i=1}^{r} l_{A_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}) \cdot e_A(a, A/\mathfrak{p}_i) .$$

Proof. Note that both sides are additive for exact sequences of A-modules. So we may assume $M = A/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of A; cf. Bourbaki [2], Chap. IV, § 1, n°4, Thm. 1. If $\mathfrak{p}$ is maximal, both sides are zero. If $\mathfrak{p}$ is minimal, then $l_{A_\mathfrak{p}}(M_\mathfrak{p}) = 1$ and the localizations of M at the other minimal primes are zero. Thus, the formula is also clear in this case. □

The results about the degree of line bundles which are presented in the following will be used in Section 9.4 to establish the representability of $\operatorname{Pic}_{X/S}$ if X is a relative curve over a discrete valuation ring. Furthermore, they will be of interest in Section 9.5 where we will discuss the relationship between the Picard functor and Néron models of Jacobians.

Lemma 7. *Let K be a separably closed field. Let X be an irreducible K-scheme of finite type of dimension r and let δ be the geometric multiplicity of X. Then, for each closed point $x \in X$ and for each system of parameters $f = (f_1, \ldots, f_r)$ of the local ring $\mathcal{O}_{X,x}$, the following assertions hold:*

(a) $\dim_K \mathcal{O}_{X,x}/(f) \geq \delta$.
(b) *If f is a regular sequence,* $\dim_K \mathcal{O}_{X,x}/(f)$ *is a multiple of* δ.
(c) *If* $\dim_K \mathcal{O}_{X,x}/(f) = \delta$, *then f is a regular sequence.*

Furthermore, there exist x and f such that $\dim_K \mathcal{O}_{X,x}/(f) = \delta$.

Proof. After shrinking X, we may assume that f gives rise to a quasi-finite morphism

$$\varphi : X \longrightarrow Y := \mathbb{A}_K^r .$$

Denote by $\bar{K}$ the algebraic closure of K and by $\bar{\varphi}$ the morphism $\varphi \otimes_K \bar{K}$. Since K is assumed to be separably closed, there exists a unique point $\bar{x}$ of $\bar{X} = X \otimes_K \bar{K}$ above x. Consider now the henselization Y' of $\bar{Y} := \mathbb{A}_{\bar{K}}^r$ at the origin. Let X' be the local component of $\bar{X} \times_{\bar{Y}} Y'$ above $\bar{x}$. Then the map

$$\varphi' : X' \longrightarrow Y'$$

obtained from $\bar{\varphi}$ via base change is finite. Furthermore, φ' is flat if and only if f is a regular sequence; cf. [EGA 0_{IV}], 15.1.14 and 15.1.21. The local rings of X' at generic points are artinian of length δ and the generic points of X' lie above the generic point of Y'. Hence, the degree of X' over Y' is a non-zero multiple of δ. So, by Nakayama's lemma, the degree of the closed fibre of φ' is greater or equal to δ. Since the degree of the closed fibre is equal to $\dim_K \mathcal{O}_{X,x}/(f)$, we see that assertion (a) is true.

If f is a regular sequence, X' is flat over Y'. Then the degree of the special fibre of φ' is equal to the degree of X' over Y'. Thus, assertion (b) is clear.

If the degree of the special fibre is δ, it is equal to the degree of X' over Y'; then $\mathcal{O}_{X'}(X')$ is free over $\mathcal{O}_{Y'}(Y')$ and, hence, flat. This shows that f is a regular sequence; so assertion (c) is true.

Next we want to show that the value δ can be attained. After replacing X by a dense open subset, we may assume that $\bar{X}_{\mathrm{red}}$ is smooth over $\bar{K}$. So the module $\Omega^1_{\bar{X}_{\mathrm{red}}/\bar{K}}$ is locally free. Furthermore, since $\Omega^1_{\bar{X}_{\mathrm{red}}/\bar{K}}$ is a quotient of $\Omega^1_{\bar{X}/\bar{K}}$, we may assume that there exist elements $a_1, \ldots, a_r \in \Gamma(X, \mathcal{O}_X)$ such that the images of the differentials $da_1, \ldots, da_r$ in $\Omega^1_{\bar{X}_{\mathrm{red}}/\bar{K}}$ give rise to a basis of this module. Consider now the morphism

$$a := (a_1, \ldots, a_r) : X \longrightarrow Y := \mathbb{A}_K^r$$

given by the functions $a_1, \ldots, a_r$. The restriction of the induced map $\bar{a} : \bar{X} \longrightarrow \bar{Y}$ to $\bar{X}_{\mathrm{red}}$ is étale. After replacing X and Y by dense open subsets, we may assume that a is finite and flat. Let x be a point of X such that $a(x)$ is a rational point of Y. We may assume that $a(x)$ is the origin. Then $f := (a_1, \ldots, a_r)$ is as required. Namely, using notations as above, we have to show that the degree of the finite and flat morphism $\varphi' : X' \longrightarrow Y'$ is δ. Since the induced morphism

$$\varphi'_{\mathrm{red}} : X'_{\mathrm{red}} \xrightarrow{\sim} Y'$$

is an isomorphism, the degree of φ' coincides with the length of the local ring $\mathcal{O}_{X',\eta'}$ at the generic point η' of X', which is equal to δ. □

As a corollary of Lemma 7, we get a relation between the geometric multiplicity of a component X_i of X and the partial degree $\deg_{X_i}(\mathscr{L})$ of a line bundle $\mathscr{L}$ on X.

Corollary 8. *Let X be a proper curve over a field K and let $X_1, \ldots, X_r$ be its (reduced) irreducible components. Let $\mathscr{L}$ be a line bundle on X. Denote by e_i the geometric multiplicity of X_i, $i = 1, \ldots, r$. Then the partial degree $\deg_{X_i}(\mathscr{L})$ of $\mathscr{L}$ on X_i is a multiple of e_i for $i = 1, \ldots, r$.*

Proof. We may assume that $X = X_i$ is reduced and irreducible, and we may assume that $\mathscr{L} = \mathcal{O}_X(D)$ is associated to an effective Cartier divisor D on X which is concentrated at a single point x. Let f be a regular element of $\mathcal{O}_{X,x}$ which represents D at x, so we have

$$\dim_K \mathcal{O}_{X,x}/(f) = \deg(\mathcal{L}) = \deg_{X_i}(\mathcal{L}) .$$

Due to Lemma 7, if K is separably closed, the geometric multiplicity $\delta_i = e_i$ of $X = X_i$ divides $\dim_K \mathcal{O}_{X,x}/(f) = \deg(\mathcal{L})$. In the general case, consider a separable closure K' of K. The irreducible component $X = X_i$ decomposes into the irreducible components X'_{ij} of $X' = X \otimes_K K'$, but the geometric multiplicities e_{ij} of X'_{ij} coincide with e_i. Thus we see that e_i divides $\deg_{X'_{ij}}(\mathcal{L} \otimes_K K')$, for all j. Now it follows from Proposition 5 that e_i divides $\deg(\mathcal{L}) = \deg_{X_i}(\mathcal{L})$, since the degree function is compatible with extensions of the base field. □

If X is a scheme of finite presentation over a strictly henselian base S, Lemma 7 can be used to show the existence of subschemes of X which are finite and flat over S and which have small degrees over S.

Corollary 9. *Let S be a strictly henselian local scheme, let s be its closed point, and let X be a flat S-scheme which is locally of finite presentation. Let X_0 be an irreducible component of the special fibre X_s of X and let δ be the geometric multiplicity of X_0 in X_s. Then there exists an S-immersion $a: Z \longrightarrow X$, where Z is finite and flat over S of rank δ and where $a_s(Z_s)$ is a point of X_0 not lying on any other irreducible component of X_s.*

Proof. Let U be an open subscheme of X such that $U_s = U \times_S k(s)$ is non-empty and contained in X_0. Due to Lemma 7, there exist a closed point x of U_s and a regular system of parameters $\bar{f}$ of $\mathcal{O}_{U_s,x} = \mathcal{O}_{U,x} \otimes_{\mathcal{O}_{S,s}} k(s)$ such that

$$\dim_{k(s)} \mathcal{O}_{U_s,x}/(\bar{f}) = \delta .$$

After restricting U, one can lift $\bar{f}$ to a sequence f of elements of $\Gamma(U, \mathcal{O}_U)$. Then f is a regular sequence of $\mathcal{O}_{U,x}$; cf. [EGA 0_{IV}], 15.1.16. After restricting U, a local component Z of $V(f)$ which contains x is finite and flat over S, so Z fulfills the assertion; cf. [EGA 0_{IV}], 15.1.16. □

Corollary 10. *Let S be a strictly henselian local scheme with closed point s, and let X be a flat curve over S which is locally of finite presentation. Let X_0 be an irreducible component of the special fibre X_s with geometric multiplicity δ in X_s. Then there exists an effective Cartier divisor Z of degree δ on X such that Z meets X_0, but no other irreducible component of X_s. Furthermore, $\deg_{X_0}(Z) = e$ where e is the geometric multiplicity of X_0.*

Corollary 9 implies the following criterion for the representability of elements of $\mathrm{Pic}_{X/S}$ by line bundles.

Proposition 11. *Let $f: X \longrightarrow S$ be a quasi-separated morphism of finite presentation such that $f_*\mathcal{O}_X = \mathcal{O}_S$. Consider S-morphisms $Z_i \longrightarrow X$, $i = 1, \dots, r$, where Z_i is finite and flat over S of degree n_i. Set $n = \gcd(n_1, \dots, n_r)$. Then, for each flat S-scheme T and for each element $\xi \in \mathrm{Pic}_{X/S}(T)$, the multiple $n \cdot \xi$ is induced by a line bundle on $X_T = X \times_S T$.*

Proof. Since n is a linear combination of $n_1, \dots, n_r$ with integer coefficients, it suffices to prove that each $n_i \cdot \xi$ is induced by a line bundle. Due to [EGA III_1], 1.4.15, and [EGA IV_1], 1.7.21, the assumption $f_*\mathcal{O}_X = \mathcal{O}_S$ remains true after flat base change. So we may assume $S = T$. The morphism $Z_i \longrightarrow X$ gives rise to a Z_i-section of $X \times_S Z_i$. So the pull-back of ξ in $\mathrm{Pic}_{X/S}(Z_i)$ is induced by a line bundle $\mathcal{L}$ on $X \times_S Z_i$; cf. 8.1/4. Then the norm of $\mathcal{L}$ with respect to the finite flat morphism $X \times_S Z_i \longrightarrow X$ gives rise to the element $n_i \cdot \xi$ in $\mathrm{Pic}_{X/S}(S)$; cf. [EGA IV_4], 21.5.6. □

As an application of Corollary 9 and Proposition 11, one obtains the following result.

Corollary 12. *Let S be a strictly henselian local scheme, let s be its closed point, and let $f: X \longrightarrow S$ be a flat morphism of finite presentation such that $f_*\mathcal{O}_X = \mathcal{O}_S$. Denote by δ the greatest common divisor of the geometric multiplicities in X_s of the irreducible components $X_1, \dots, X_r$ of X_s. Then, for each flat S-scheme T, and each element ξ of $\mathrm{Pic}_{X/S}(T)$, the multiple $\delta \cdot \xi$ is induced by a line bundle on $X \times_S T$.*

9.2 The Structure of Jacobians

In the following let X be a proper curve over a field K. Then, due to 8.2/3 and 8.4/2, $\mathrm{Pic}^0_{X/K}$ is a smooth scheme; we will also refer to it as the *Jacobian* of X. In the present section, we want to discuss the structure of $\mathrm{Pic}^0_{X/K}$ as an algebraic group depending on data furnished by the given curve X. To start with, let us mention some general results on the structure of commutative algebraic groups.

Theorem 1 (Chevalley [1] or Rosenlicht [2]). *Let K be a field and let G be a smooth and connected algebraic K-group. Then there exists a smallest (not necessarily smooth) connected linear subgroup L of G such that the quotient G/L is an abelian variety.*

If K is perfect, L is smooth and its formation is compatible with extension of the base field.

Chevalley has treated the case where K is algebraically closed and has shown that there exists a smooth connected linear subgroup L of G such that the quotient G/L is an abelian variety. If the base field is perfect, the existence of such a subgroup follows by Galois descent from the case of algebraically closed fields. It is clear that such a group is the smallest connected linear subgroup of G with abelian cokernel, and that its formation is compatible with extension of the base field.

If the base field is not perfect, there exist a finite radicial extension K' of K and a connected smooth linear K'-subgroup H' of $G' = G \otimes_K K'$ such that the quotient G'/H' is an abelian variety. Let us first show that there exists a connected linear subgroup H of G such that $H \otimes_K K'$ contains H'. Let n be the exponent of the radicial extension K'/K. Then consider the n-fold Frobenius

$$F_n : G' \longrightarrow G'^{(p^n)} = G' \times_{K'} K'^{(1/p^n)}$$

(cf. [SGA 3_{I}], Exp. $\mathrm{VII}_{\mathrm{A}}$, 4.1); the second projection is induced by the inclusion $K' \longrightarrow K'^{(1/p^n)}$. Now let H_n' be the pull-back of the subgroup $H'^{(p^n)}$ of $G'^{(p^n)}$. If $\mathscr{I}'$ is the sheaf of ideals of $\mathcal{O}_{G'}$ associated to H', the sheaf of ideals associated to H_n' is generated by the p^n-th powers of the local sections of $\mathscr{I}'$. Since K'/K is of exponent n, we see that $\mathscr{I}'$ is generated by local sections of $\mathcal{O}_G$ and, hence, that H_n' is defined over K. Now it remains to show that there exists a smallest connected linear subgroup L of G having abelian cokernel. This follows immediately from the fact that an intersection of two linear subgroups of G is linear again and has abelian cokernel if each of them has abelian cokernel. □

For an arbitrary base field K, the connected linear subgroup L does not need to be compatible with field extensions. If the base field K is perfect and the group G is commutative, one has further information on the structure of the group L.

Theorem 2 ([SGA 3_{II}], Exp. XVII, Thm. 7.2.1). *Let K be a field and let G be a smooth and connected algebraic K-group of finite type. Assume that G is commutative and linear. Then G is canonically an extension of a unipotent algebraic group by a torus.*

If, in addition, the base field K is perfect, this extension splits canonically; i.e., G is isomorphic to a product of a unipotent group and a torus.

Now we come to the discussion of the structure of $\mathrm{Pic}^0_{X/K}$. We start with a result which is a direct consequence of 8.4/2 and 8.4/3.

Proposition 3. *Let X be a proper and smooth curve over a field K. Then the Jacobian $\mathrm{Pic}^0_{X/K}$ is an abelian variety.*

If the base field K is perfect, the curve X is smooth over K if and only if it is normal. The two notions are not equivalent over arbitrary fields, so it may happen that $\mathrm{Pic}^0_{X/K}$ is not proper although X is normal.

Proposition 4. *Let X be a proper curve over a field K. Assume that X is normal, geometrically reduced, and geometrically irreducible. Then $\mathrm{Pic}^0_{X/K}$ contains neither a subgroup of type $\mathbb{G}_a$ nor a subgroup of type $\mathbb{G}_m$.*

Proof. Since, for any separable field extension K'/K, the K'-curve $X \otimes_K K'$ is normal, we may assume that K is separably closed. Then there exists a rational point on X because X is geometrically reduced. So, for any K-scheme T, elements of $\mathrm{Pic}_{X/K}(T)$ can be represented by line bundles on $X \times_K T$; cf. 8.1/4. Now, let us assume that there is a subgroup G of $\mathrm{Pic}_{X/K}$ which is of type $\mathbb{G}_a$ or $\mathbb{G}_m$. The inclusion $G \hookrightarrow \mathrm{Pic}_{X/K}$ corresponds to a line bundle $\mathscr{L}$ on $X \times_K G$. Since X is normal, the line bundle $\mathscr{L}$ is isomorphic to the pull-back of a line bundle on X; cf. Bourbaki [2], Chap. VII, § 1, n°10, Prop. 17 and 18. Hence, the map $G \longrightarrow \mathrm{Pic}_{X/K}$ which is induced by $\mathscr{L}$ must be constant. So we get a contradiction and the assertion is proved. □

Now we turn to more general cases. Let us denote by X_{red} the largest reduced subscheme of X. By functoriality, we get a canonical map

$$\mathrm{Pic}^0_{X/K} \longrightarrow \mathrm{Pic}^0_{X_{\mathrm{red}}/K}\,.$$

So we can look at the kernel and at the image of this map. The algebraic group corresponding to the kernel can easily be described by the nilradical of $\mathcal{O}_X$.

Proposition 5. *Let X be a proper curve over a field K. Then the canonical map*

$$\mathrm{Pic}_{X/K} \longrightarrow \mathrm{Pic}_{X_{\mathrm{red}}/K}$$

is an epimorphism of sheaves for the étale topology. Its kernel is a smooth and connected unipotent group which is a successive extension of additive groups of type $\mathbb{G}_a$.

Proof. Let $X' \longrightarrow X$ be a closed subscheme which is defined by a sheaf of ideals $\mathscr{N}$ of $\mathcal{O}_X$ satisfying $\mathscr{N}^2 = 0$. It suffices to show that the canonical map

$$\mathrm{Pic}_{X/K} \longrightarrow \mathrm{Pic}_{X'/K}$$

is an epimorphism of sheaves for the étale topology and that its kernel is of the type described above. Let $f: X \longrightarrow \mathrm{Spec}\, K$ be the structural morphism. The exact sequence given by the exponential map

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{O}_{X'}^* \longrightarrow 0$$

$$n \longmapsto 1 + n$$

gives rise to the exact sequence

$$R^1f_*\mathscr{N} \longrightarrow R^1f_*\mathcal{O}_X^* \longrightarrow R^1f_*\mathcal{O}_{X'}^* \longrightarrow R^2f_*\mathscr{N}$$

which has to be read as a sequence of sheaves for the étale topology. Because X is a curve, we have $R^2f_*\mathscr{N} = 0$. Hence the canonical map

$$\mathrm{Pic}_{X/K} = R^1f_*\mathcal{O}_X^* \longrightarrow \mathrm{Pic}_{X'/K} = R^1f_*\mathcal{O}_{X'}^*$$

is an epimorphism. Since, for any K-scheme T, there is a canonical isomorphism

$$H^1(X, \mathscr{N}) \otimes_K \mathcal{O}_T(T) = R^1f_*\mathscr{N}(T)\,,$$

the group functor $R^1f_*\mathscr{N}$ is represented by the vector group $H^1(X, \mathscr{N})$. Then it follows from the exact sequence above that the kernel of the map we are interested in is a quotient of the vector group $H^1(X, \mathscr{N})$. The latter is a successive extension of groups of type $\mathbb{G}_a$. So, as can easily be deduced from [SGA 3_{II}], Exp. XVII, Lemme 2.3, the kernel is as required. □

It remains to study $\mathrm{Pic}^0_{X/K}$ for reduced curves. Therefore, let us assume now that the curves under consideration are reduced. Before starting the discussion of the general case, we want to have a closer look at an example.

Definition 6. *Let S be any scheme, and let g be an integer. A semi-stable curve of genus g over S is a proper and flat morphism $f: X \longrightarrow S$ whose fibres $X_{\bar{s}}$ over geometric* *points $\bar{s}$ of S are reduced, connected, one-dimensional, and satisfy the following conditions*:

(i) *$X_{\bar{s}}$ has only ordinary double points as singularities,*

(ii) $\dim_{k(\bar{s})} H^1(X_{\bar{s}}, \mathcal{O}_{X_{\bar{s}}}) = g$.

A point x of a curve X over an algebraically closed field $\bar{K}$ is an ordinary double point if the completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ of X at x is isomorphic to the quotient $\bar{K}[[\zeta, \xi]]/(\zeta\xi)$ of the formal power series ring $\bar{K}[[\zeta, \xi]]$ in two variables. For a curve X over a field K, one can formulate the condition of X being semi-stable, without performing the base extension by an algebraic closure $\bar{K}$ of K. Namely, a geometrically connected curve X over a field K is semi-stable if and only if for each non-smooth point of X there exists an étale neighborhood which is étale over the union of the coordinate axes in $\mathbb{A}^2_{\bar{K}}$.

The interest in semi-stable curves comes from the semi-stable reduction theorem, see Deligne and Mumford [1] or Artin and Winters [1], which we want to mention without proof.

Theorem 7 (Semi-Stable Reduction Theorem). *Let R be a discrete valuation ring with fraction field K. Let X_K be a proper, smooth, and geometrically connected curve over K. Then there exist a finite separable field extension K' of K and a semi-stable curve X' over the integral closure R' of R in K' with generic fibre $X'_{K'} \cong X_K \otimes_K K'$. Furthermore, X' can be chosen to be regular.*

If X is a semi-stable curve over an algebraically closed field K, one can associate a graph $\Gamma = \Gamma(X)$ to it: the vertices of Γ are the irreducible components of X, say $X_1, \ldots, X_r$, and the edges are given by the singular points of X; namely, each singular point lying on X_i and on X_j defines an edge joining the vertices X_i and X_j. Note that $X_i = X_j$ is allowed.

Example 8. *Let X be a semi-stable curve over a field K. Then $\mathrm{Pic}^0_{X/K}$ is canonically an extension of an abelian variety by a torus T.*

More precisely, let $X_1, \ldots, X_r$ be the irreducible components of X, and let $\tilde{X}_i$ be the normalization of X_i, $i = 1, \ldots, r$. Then the canonical extension associated to $\mathrm{Pic}^0_{X/K}$ is given by the exact sequence

$$1 \longrightarrow T \hookrightarrow \mathrm{Pic}^0_{X/K} \xrightarrow{\pi^*} \prod_{i=1}^{r} \mathrm{Pic}^0_{\tilde{X}_i/K} \longrightarrow 1$$

where π^* is induced via functoriality by the morphisms $\pi_i : \tilde{X}_i \longrightarrow X$, $i = 1, \ldots, r$. The rank of the torus part T is equal to the rank of the cohomology group $H^1(\Gamma(X \otimes_K \bar{K}), \mathbb{Z})$.

Proof. Let $\pi : \tilde{X} \longrightarrow X$ be the normalization of X. The connected components of $\tilde{X}$ are the normalizations $\tilde{X}_i$ of the irreducible components X_i. They are proper and smooth over K, hence $\mathrm{Pic}^0_{\tilde{X}_i/K}$ is an abelian variety over K. Furthermore, the map π^* is compatible with field extensions. So we may assume that K is algebraically closed. Now look at the exact sequence

$$(*) \qquad 1 \longrightarrow \mathcal{O}^*_X \longrightarrow \pi_* \mathcal{O}^*_{\tilde{X}} \longrightarrow \pi_* \mathcal{O}^*_{\tilde{X}}/\mathcal{O}^*_X \longrightarrow 1 .$$

The quotient $\mathcal{Q} = \pi_* \mathcal{O}^*_{\tilde{X}}/\mathcal{O}^*_X$ is concentrated at the singular points $x_1, \ldots, x_N$ of X. The associated long exact sequence

$$\begin{aligned} 1 \longrightarrow H^0(X, \mathcal{O}^*_X) &\longrightarrow H^0(X, \pi_* \mathcal{O}^*_{\tilde{X}}) \longrightarrow H^0(X, \mathcal{Q}) \\ &\longrightarrow H^1(X, \mathcal{O}^*_X) \longrightarrow H^1(X, \pi_* \mathcal{O}^*_{\tilde{X}}) \longrightarrow 1 \end{aligned}$$

can be written in the following way

$$(**) \qquad 1 \longrightarrow K^* \longrightarrow \prod_{i=1}^{r} K^*_i \longrightarrow \prod_{j=1}^{N} K^*_j \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\tilde{X}) \longrightarrow 1$$

where

$$K^*_i = H^0(\tilde{X}_i, \mathcal{O}^*_{\tilde{X}_i}) \cong K^* \text{ and } K^*_j = (K(\tilde{x}_{j1}) \times K(\tilde{x}_{j2})^*)/K^* \cong K^*$$

if $\tilde{x}_{j1}$ and $\tilde{x}_{j2}$ are the points of $\tilde{X}$ lying above the double point x_j. Using the long exact sequence of sheaves with respect to the étale topology which is associated to $(*)$, one sees that π^* is an epimorphism, since $R^1 f_* \mathcal{Q} = 0$ where $f : X \longrightarrow \mathrm{Spec}\, K$ is the structural morphism. Furthermore, the kernel of π^* is given by the quotient of the map $R^0 f_*(\pi_* \mathcal{O}^*_{\tilde{X}}) \longrightarrow R^0 f_*(\mathcal{Q})$. The latter is a quotient of a torus and, hence a torus. The assertion concerning the rank of the torus follows from the exact sequence $(**)$. □

Now let us return to the general situation of a reduced curve over a field K. As in the theorem of Chevalley, one can expect to describe the torus part and the unipotent part of $\mathrm{Pic}^0_{X/K}$ in geometric terms, at least if the base field is perfect. So, in the following, let K be a perfect field and let X be a proper curve over K which is reduced and geometrically connected. Denote by $\tilde{X} \longrightarrow X$ the normalization of X. We want to introduce an intermediate curve X' lying between X and $\tilde{X}$.

Since there is a dense open part of X which is smooth, there exist only finitely many non-smooth points of X. We will define X' by identifying all the points of $\tilde{X}$ lying above such a non-smooth point of X. In order to explain this procedure, we can work locally. So consider a non-smooth point x of X, and let $U = \mathrm{Spec}\, A$ be an affine open neighborhood of x such that x is the only non-smooth point of U. Let $\tilde{x}_1, \ldots, \tilde{x}_n$ be the points of $\tilde{X}$ lying above x, and let $\tilde{U} = \mathrm{Spec}\, \tilde{A}$ be the inverse image of U in $\tilde{X}$. Then we define the open affine subscheme $U' = \mathrm{Spec}\, A'$ of X' lying over U by taking for A' the amalgamated sum of the maps

$$\tilde{A} \longrightarrow \prod_{i=1}^{n} k(\tilde{x}_i) \qquad \text{and} \qquad k(x) \longrightarrow \prod_{i=1}^{n} k(\tilde{x}_i) .$$

So A' consists of all elements $f \in \tilde{A}$ which take the same value $r \in k(x)$ at all points $\tilde{x}_1, \ldots, \tilde{x}_n$. These local constructions fit together to build a proper curve X', and we get canonical morphisms

$$\tilde{X} \xrightarrow{f} X' \xrightarrow{g} X .$$

The map f maps the points $\tilde{x}_1, \ldots, \tilde{x}_n$ to a single point x' of X' with residue field $k(x)$. So g does not change the residue field. Let $\tilde{\mathfrak{m}}_i \subset \tilde{A}$ be the ideal of the point $\tilde{x}_i$, $i = 1, \ldots, n$. Then we obtain the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \prod_{i=1}^{n} \mathfrak{m}_i \arrow[r] & \prod_{i=1}^{n} \mathcal{O}_{\tilde{X},\tilde{x}_i} \arrow[r] & \prod_{i=1}^{n} k(\tilde{x}_i) \arrow[r] & 0 \\
0 \arrow[r] & \mathfrak{m}' \arrow[r] \arrow[u] & \mathcal{O}_{X',x'} \arrow[r] \arrow[u] & k(x') \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where $\mathfrak{m}'$ is the maximal ideal of $\mathcal{O}_{X',x'}$. The first vertical map is bijective, and the last one corresponds to the embedding of $k(x') = k(x)$ into the product of the residue fields $k(x_i)$, $i = 1, \ldots, n$. Due to the construction, it is clear that the map $X' \longrightarrow X$ is a universal homeomorphism. Moreover, X' is the largest curve between $\tilde{X}$ and X which is universally homeomorphic to X. One shows easily that the construction of X' is compatible with field extensions, since K is perfect. The singularities of X' are as mild as possible. Namely, after base extension by an algebraic closure $\bar{K}$ of K, the singularities of $X' \otimes_K \bar{K}$ are transversal crossings of a set of smooth branches (i.e., analytically isomorphic to the crossing of the coordinate axes in $\mathbb{A}^n$ for some n).

Proposition 9. *Let X be a proper reduced curve over a perfect field K. Let $g : X' \longrightarrow X$ be the largest curve between the normalization $\tilde{X}$ of X and X which is universally homeomorphic to X. Then the canonical map*

$$\psi : \mathrm{Pic}_{X/K} \longrightarrow \mathrm{Pic}_{X'/K}$$

is an epimorphism of sheaves for the étale topology. The kernel of ψ is a connected unipotent algebraic group which is trivial if and only if the canonical map $X' \longrightarrow X$ is an isomorphism.

Proof. Let $\mathcal{P} \subset \mathcal{O}_X$ (resp. $\mathcal{Q} \subset \mathcal{O}_{X'}$) be the sheaf of (reduced) ideals defining the non-smooth locus of X (resp. of X'). There exists an integer $e \in \mathbb{N}$ such that $g_*\mathcal{Q}^e \subset \mathcal{P}$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow g_*\mathcal{O}_{X'}^* \longrightarrow (1 + g_*\mathcal{Q})/(1 + \mathcal{P}) \longrightarrow 0\,,$$

and set $\mathcal{C} := (1 + g_*\mathcal{Q})/(1 + \mathcal{P})$. It is a sheaf which is concentrated on the finitely many points of X which are not smooth; more precisely, its support consists of the points of X which are not ordinary multiple points. Let $f : X \longrightarrow \operatorname{Spec} K$ be the structural morphism. Since $R^1f_*\mathcal{C} = 0$ and $f_*\mathcal{O}_X^* = f_*g_*\mathcal{O}_{X'}^*$, the exact sequence of above gives rise to an exact sequence

$$1 \longrightarrow R^0f_*\mathcal{C} \longrightarrow R^1f_*\mathcal{O}_X^* \longrightarrow R^1f_*(g_*\mathcal{O}_{X'}^*) \longrightarrow 1$$

of sheaves for the étale topology. Thus, we see that

$$\mathrm{Pic}_{X/K} = R^1f_*\mathcal{O}_X^* \xrightarrow{\psi} \mathrm{Pic}_{X'/K} = R^1(f \circ g)_*\mathcal{O}_{X'}^* = R^1f_*(g_*\mathcal{O}_{X'}^*)$$

is an epimorphism. Due to Serre [1], Chap. V, n°15, Lemma 20, the group $R^0f_*\mathcal{C}$ and, hence, the kernel of ψ is represented by a unipotent group. For a further description of this group see Serre [1], Chap. V, n°16 and n°17. Moreover, the kernel of ψ is trivial if and only if the group $H^0(X, \mathcal{C})$ vanishes; i.e., if and only if $g_*\mathcal{Q} = \mathcal{P}$ or, equivalently, if and only if $X' \longrightarrow X$ is an isomorphism. □

Proposition 10. *Let X be a proper reduced curve over a perfect field K, and let $\bar{K}$ be an algebraic closure of K. Let $X' \longrightarrow X$ be the largest curve between the normalization $\tilde{X}$ of X and X which is universally homeomorphic to X. Then the canonical map*

$$\varphi : \mathrm{Pic}_{X'/K} \longrightarrow \mathrm{Pic}_{\tilde{X}/K}$$

is an epimorphism of sheaves for the étale topology. The kernel of φ is a torus. The latter is trivial if and only if each irreducible component of $X \otimes_K \bar{K}$ is homeomorphic to its normalization and the configuration of the irreducible components of $X \otimes_K \bar{K}$ is tree-like; i.e., $H^1(X \otimes_K \bar{K}, \mathbb{Z}) = 0$.

Proof. The proof can be done similarly as in Example 8. We may assume $X = X'$. Let $\pi : \tilde{X} \longrightarrow X$ be the normalization of X. The connected components of $\tilde{X}$ are the normalizations $\tilde{X}_i$ of the irreducible components X_i. Let x_i, $i = 1, \ldots, N$, be the singular points of X, and let $\tilde{x}_{ij}$, $j = 1, \ldots, n_i$, be the points of $\tilde{X}$ lying above $\tilde{x}_i$. Consider the exact sequence

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \pi_*\mathcal{O}_{\tilde{X}}^* \longrightarrow \pi_*\mathcal{O}_{\tilde{X}}^*/\mathcal{O}_X^* \longrightarrow 1\,.$$

The quotient $\mathcal{Q} = \pi_*\mathcal{O}_{\tilde{X}}^*/\mathcal{O}_X^*$ is concentrated at the points x_i, $i = 1, \ldots, N$. The associated long exact sequence

$$\begin{aligned}1 &\longrightarrow H^0(X, \mathcal{O}_X^*) \longrightarrow H^0(X, \pi_*\mathcal{O}_{\tilde{X}}^*) \longrightarrow H^0(X, \mathcal{Q}) \\ &\longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \pi_*\mathcal{O}_{\tilde{X}}^*) \longrightarrow 1\end{aligned}$$

can be written in the following way

$$1 \longrightarrow \Gamma^* \longrightarrow \prod_{i=1}^{r} \Gamma_i^* \longrightarrow \prod_{i=1}^{N}\left(\prod_{j=1}^{n_i} K_{ij}^*\right)\Big/K_i^* \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\tilde{X}) \longrightarrow 1$$

where $\Gamma^* = H^0(X, \mathcal{O}_X^*)$, $\Gamma_i^* = H^0(\tilde{X}_i, \mathcal{O}_{\tilde{X}}^*)$, $K_i^* = k(x_i)$, and $K_{ij}^* = k(\tilde{x}_{ij})$. As in Example 8, one shows that φ is an epimorphism for the étale topology and, moreover, that the kernel of φ is the quotient of the map $R^0f_*(\pi_*\mathcal{O}_{\tilde{X}}^*) \longrightarrow R^0f_*(\mathcal{Q})$ where $f : X \longrightarrow \operatorname{Spec} K$ is the structural morphism. The latter is a quotient of a torus and, hence, a torus.

It remains to show the last assertion. We may assume that K is algebraically closed. The kernel of φ is trivial if and only if the canonical map

$$\prod_{i=1}^{r} \Gamma_i^* \longrightarrow \prod_{i=1}^{N}\left(\prod_{j=1}^{n_i} K_{ij}^*\right)\Big/K_i^*$$

is surjective. If the map is surjective, it is clear that, for any singular point x_i of X, the points $\tilde{x}_{ij}$, $j = 1, \ldots, n_i$, lie on pairwise different components of $\tilde{X}$. Hence, each irreducible component of X is homeomorphic to its normalization. Furthermore, the surjectivity implies $H^1(X, K^*) = 0$ which is equivalent to $H^1(X, \mathbb{Z}) = 0$. The converse implication follows by similar arguments. □

Now we can deduce from Propositions 9 and 10 the structure of the linear part of $\mathrm{Pic}^0_{X/K}$.

Corollary 11. *Let X be a proper curve over a perfect field K and denote by $\tilde{X}$ the normalization of the largest reduced subscheme X_{red} of X. Then the canonical map*

$$\mathrm{Pic}_{X/K} \longrightarrow \mathrm{Pic}_{\tilde{X}/K}$$

is an epimorphism of sheaves for the étale topology. Its kernel consists of a smooth connected linear algebraic group L. The quotient of $\mathrm{Pic}^0_{X/K}$ by L is isomorphic to $\mathrm{Pic}^0_{\tilde{X}/K}$ which is an abelian variety.

Next we want to look at a reduced curve X over a perfect field K. As before, let X' denote the largest curve between X and its normalization $\tilde{X}$. Via functoriality, we get the following sequence of algebraic groups

$$\mathrm{Pic}_{X/K} \longrightarrow \mathrm{Pic}_{X'/K} \longrightarrow \mathrm{Pic}_{\tilde{X}/K} ,$$

where each map is an epimorphism of sheaves for the étale toplogy. Due to continuity, we obtain epimorphisms between the identity components

$$\mathrm{Pic}^0_{X/K} \longrightarrow \mathrm{Pic}^0_{X'/K} \longrightarrow \mathrm{Pic}^0_{\tilde{X}/K} .$$

Furthermore, if $\mathrm{Pic}^0_{X/K}$ does not contain a torus, $\mathrm{Pic}^0_{X'/K}$ does not either; for example, this can be deduced from Theorem 2. So, we obtain the following corollary.

Corollary 12. *Let X be a reduced proper curve over a perfect field K and let $\bar{K}$ be an algebraic closure of K.*

(a) *If $\mathrm{Pic}^0_{X/K}$ contains no unipotent connected subgroup, the singularities of $X \otimes_K \bar{K}$ are analytically isomorphic to the crossing of the coordinate axes in $\mathbb{A}^n$.*

(b) *If $\mathrm{Pic}^0_{X/K}$ contains no torus, each irreducible component of $X \otimes_K \bar{K}$ is homeomorphic to its normalization and the configuration of the irreducible components of $X \otimes_K \bar{K}$ is tree-like.*

(c) *If $\mathrm{Pic}^0_{X/K}$ is an abelian variety, the irreducible components of X are smooth and the configuration of the irreducible components of $X \otimes_K \bar{K}$ is tree-like.*

Finally we want to discuss the degree of line bundles belonging to $\mathrm{Pic}^0_{X/K}$. For example, if X is a connected proper and smooth curve over an algebraically closed field K, the elements of $\mathrm{Pic}^0_{X/K}(K)$ correspond to the line bundles of degree zero. Indeed, consider the universal line bundle $\mathscr{L}$ on $X \times_K \mathrm{Pic}_{X/K}$. Due to 9.1/2, the degree of the restriction $\mathscr{L}_\xi$ of $\mathscr{L}$ to the fibre over a point $\xi \in \mathrm{Pic}^0_{X/K}$ is zero. Conversely, a line bundle of degree zero is isomorphic to a line bundle $\mathcal{O}_X(D)$ where D is a Cartier divisor which can be written as

$$D = (x_1 - x_0) + \ldots + (x_n - x_0) ,$$

where $x_0, \ldots, x_n$ are closed points of X. Since X is connected, the image of the map

$$X \longrightarrow \mathrm{Pic}_{X/K} , \qquad x \longmapsto [\mathcal{O}_X(x - x_0)] ,$$

is contained in $\mathrm{Pic}^0_{X/K}$. Thus we see that each line bundle of degree zero gives ri to an element of $\mathrm{Pic}^0_{X/K}$. For arbitrary curves over fields, one has to look at th partial degrees on the irreducible components.

Corollary 13. *Let X be a proper curve over a field K and let $\bar{K}$ be an algebraic closu of K. Then $\mathrm{Pic}^0_{X/K}$ consists of all elements of $\mathrm{Pic}_{X/K}$ whose partial degree on eac irreducible component of $X \otimes_K \bar{K}$ is zero.*

Proof. We may assume that K is algebraically closed. Let $X_1, \ldots, X_r$ be the (reduced irreducible components of X. For $i = 1, \ldots, r$, let $\tilde{X}_i$ be the normalization of X_i Then consider the canonical morphism

$$\mathrm{Pic}_{X/K} \longrightarrow \mathrm{Pic}_{\tilde{X}_i/K}$$

which is defined by functoriality. Due to continuity, the identity components are mapped into each other, so we have morphisms

$$\mathrm{Pic}^0_{X/K} \longrightarrow \mathrm{Pic}^0_{\tilde{X}_i/K} .$$

Since the degree of a Cartier divisor on X_i and the degree of its pull-back on $\tilde{X}_i$ coincide, we see that the partial degrees of elements of $\mathrm{Pic}^0_{X/K}(K)$ are zero. Due to Corollary 11, the canonical morphism

$$\mathrm{Pic}_{X/K} \longrightarrow \prod_{i=1}^{r} \mathrm{Pic}_{\tilde{X}_i/K}$$

is an epimorphism and its kernel is a connected subgroup of $\mathrm{Pic}_{X/K}$. So the kernel is contained in $\mathrm{Pic}^0_{X/K}$. Since the canonical map induces an epimorphism on the identity components, we see that line bundles on X whose partial degrees are zero belong to $\mathrm{Pic}^0_{X/K}$. □

Corollary 14. *Let X be a proper curve over an algebraically closed field K with r irreducible components $X_1, \ldots, X_r$. Then the Néron-Severi group of X is a free group of rank r.*

More precisely, the map given by the partial degrees

$$\mathrm{Pic}_{X/K}/\mathrm{Pic}^0_{X/K} \longrightarrow \mathbb{Z}^r , \qquad \mathscr{L} \longmapsto (\deg_{X_1}(\mathscr{L}), \ldots, \deg_{X_r}(\mathscr{L}))$$

is injective and has finite index.

9.3 Construction via Birational Group Laws

We want to explain how the proof of Grothendieck's theorem 8.2/1 can be modified in the case of relative curves in order to recover the Jacobian variety as constructed by Serre [1] and Weil [2]. We begin by repeating what Grothendieck's approach to the representability of $\mathrm{Pic}_{X/S}$ yields in the case of a relative curve X over a scheme S.

Theorem 1. *Let $X \longrightarrow S$ be a projective and flat curve which is locally of finite presentation. If the geometric fibres of X over S are reduced and irreducible, $\mathrm{Pic}_{X/S}$ is a smooth and separated S-scheme.*

More precisely, there is a decomposition

$$\mathrm{Pic}_{X/S} = \coprod_{n \in \mathbb{Z}} (\mathrm{Pic}_{X/S})^n$$

where $(\mathrm{Pic}_{X/S})^n$ denotes the open and closed subscheme of $\mathrm{Pic}_{X/S}$ consisting of all line bundles of degree n; the scheme $(\mathrm{Pic}_{X/S})^0$ coincides with the identity component $\mathrm{Pic}^0_{X/S}$ of $\mathrm{Pic}_{X/S}$. Moreover, $(\mathrm{Pic}_{X/S})^n$ is quasi-projective over S and is a torsor under $\mathrm{Pic}^0_{X/S}$ for all $n \in \mathbb{Z}$.

Proof. The representability of $\mathrm{Pic}_{X/S}$ is due to 8.2/1; see also 8.2/5. The smoothness follows from 8.4/2. Due to 9.1/2, the degree of line bundles belonging to a fixed connected component of $\mathrm{Pic}_{X/S}$ is constant, thus $\mathrm{Pic}_{X/S}$ breaks up into the disjoint union of the $(\mathrm{Pic}_{X/S})^n$, $n \in \mathbb{Z}$. In order to show that $(\mathrm{Pic}_{X/S})^n$ is a torsor under $\mathrm{Pic}^0_{X/S}$, it remains to show that $(\mathrm{Pic}_{X/S})^n$ and $\mathrm{Pic}^0_{X/S}$ become isomorphic after faithfully flat base extension. So we may assume that X has a section over S. Then it suffices to see that $(\mathrm{Pic}_{X/S})^0$ is isomorphic to $\mathrm{Pic}^0_{X/S}$. Since the geometric fibres of X over S are irreducible and reduced, the latter follows immediately from 9.2/13. □

Let us mention some conditions under which X is projective over S.

Remark 2. Let X be a proper flat curve over S which is locally of finite presentation and whose geometric fibres are reduced and irreducible curves of genus g. Assume that X is a relative complete intersection over S. Then the relative dualizing sheaf is a line bundle. If $g \geq 2$, it is S-ample and, hence, $X \longrightarrow S$ is projective. Likewise, if $g = 0$, the dual of the relative dualizing sheaf is S-ample and, hence, $X \longrightarrow S$ is projective; moreover it is smooth. If $g = 1$, it follows that $X \longrightarrow S$ is projective locally for the étale topology on S, since $X \longrightarrow S$ admits a section through the smooth locus after étale surjective base change, and since the line bundle of all meromorphic functions having only simple poles along the given section is relatively ample.

Now we turn to a more general situation where we can construct $\mathrm{Pic}^0_{X/S}$ via birational laws. In the following let $f: X \longrightarrow S$ be a quasi-projective morphism of schemes which is of finite presentation. We want to explain some basic facts on the relationship between the n-fold symmetric product $(X/S)^{(n)}$ and the Hilbert functor $\mathrm{Hilb}^n_{X/S}$, where $\mathrm{Hilb}^n_{X/S}$ is the Hilbert functor associated to the constant polynomial n. We can say that, for any S-scheme T, the set $\mathrm{Hilb}^n_{X/S}(T)$ consists of all subschemes D of $X \times_S T$ which are finite and locally free of rank n over T. The n-fold symmetric product $(X/S)^{(n)}$ is defined as the quotient of the n-fold product of X over S by the canonical action of the symmetric group. Let us start by discussing the representability of $(X/S)^{(n)}$.

For any commutative ring A and for any A-module M, define the symmetric n-fold tensor product of M by

$$\mathrm{TS}^n_A(M) := (M^{\otimes n})^{\mathfrak{S}_n} \subset M^{\otimes n}$$

where $M^{\otimes n}$ is the n-fold tensor product of M over A and where $\mathfrak{S}_n$ is the symmetric group acting on $M^{\otimes n}$ by permuting factors. If M is a free A-module, $\mathrm{TS}^n_A(M)$ is also free and there is a canonical way to choose a basis of $\mathrm{TS}^n_A(M)$ after fixing a basis of M. Thus, we see that $\mathrm{TS}^n_A(M)$ is compatible with any base change if M is a free A-module. Since any flat A-module is a limit of finitely generated free A-modules, $\mathrm{TS}^n_A(M)$ is a flat A-module and compatible with any base change if M is flat over A. If B is an A-algebra, $\mathrm{TS}^n_A(B)$ is a subalgebra of $B^{\otimes n}$. If X and S are affine, say $S = \mathrm{Spec}\, A$ and $X = \mathrm{Spec}\, B$, the symmetric product $(X/S)^{(n)}$ is represented by $\mathrm{Spec}(\mathrm{TS}^n_A(B))$. If X is quasi-projective over S, one can establish the representability of the symmetric product $(X/S)^{(n)}$ as an S-scheme by gluing such local pieces, since any finite set of points lying on a single fibre of X/S is contained in an open affine subscheme of X. Furthermore, as we have seen above, the symmetric product $(X/S)^{(n)}$ of a flat S-scheme X is flat over S and compatible with any base change.

A polynomial law f from an A-module M to an A-module N consists of the following data: for any commutative A-algebra A', there is a map

$$f_{A'}: M \otimes_A A' \longrightarrow N \otimes_A A'$$

such that, for any morphism $u: A' \longrightarrow A''$ of commutative A-algebras, the diagram

$$\begin{array}{ccc} M \otimes_A A' & \xrightarrow{f_{A'}} & N \otimes_A A' \\ \downarrow{\scriptstyle M \otimes u} & & \downarrow{\scriptstyle N \otimes u} \\ M \otimes_A A'' & \xrightarrow{f_{A''}} & N \otimes_A A'' \end{array}$$

is commutative. A polynomial law from M to N is called homogeneous of degree n if, in addition, for any $a' \in A'$ and for any $m' \in M \otimes_A A'$, the equation

$$f_{A'}(a' \cdot m') = (a')^n \cdot f_{A'}(m') .$$

holds. For example, the map

$$\gamma^n : M \longrightarrow \mathrm{TS}^n_A(M) , \qquad m \longmapsto m \otimes \cdots \otimes m \quad (n \text{ times})$$

gives rise to a homogeneous polynomial law of degree n. Furthermore, if M is a free A-module of finite rank, the map γ^n is universal; i.e., any homogeneous polynomial law f from M to N of degree n is induced by a unique A-linear map $\varphi: \mathrm{TS}^n_A(M) \longrightarrow N$. The latter means

$$f_{A'} = (\varphi \otimes A') \circ (\gamma^n \otimes A');$$

cf. [SGA 4_{III}], Exp. XVII, 5.5.2. Since a flat A-module is a limit of free A-modules, the map γ^n is universal if M is a flat A-module.

Let us fix $S = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$ and $f: X \longrightarrow S$. For any B-module L which is free of rank n over A, there is a canonical morphism

$$\det_L : \mathrm{TS}^n_A(B) \longrightarrow A$$

which is compatible with any base change $A \longrightarrow A'$. Indeed, viewing the multiplication on L by an element $b \in B$ as an A-linear map, the determinant yields a homogeneous polynomial law of degree n from B to A and, hence, a map of $\mathrm{TS}^n_A(B)$

to A. Furthermore, one can show that $\det_L$ is a morphism of A-algebras; cf. [SGA 4_{III}], Exp. XVII, 6.3.1.

If $f: X \longrightarrow S$ is affine and if $\mathscr{L}$ is an $\mathcal{O}_X$-module such that $f_*\mathscr{L}$ is locally free over S of rank n, one can construct a morphism

$$\sigma_{\mathscr{L}}: S \longrightarrow (X/S)^{(n)}$$

by gluing the local morphisms constructed above.

Now let $f: X \longrightarrow S$ be quasi-projective and consider an element $D \in \mathrm{Hilb}^n_{X/S}(T)$ for an S-scheme T, i.e., a subscheme D of $X \times_S T$ which is finite and locally free of rank n over T. Then $(f_T)_*\mathcal{O}_D$ is a locally free $\mathcal{O}_T$-module of rank n. So the above construction gives rise to a section

$$\sigma_{\mathcal{O}_D}: T \longrightarrow (D/T)^{(n)} \longrightarrow (X/S)^{(n)} .$$

Thus we get a canonical morphism

$$\sigma: \mathrm{Hilb}^n_{X/S} \longrightarrow (X/S)^{(n)} .$$

On the other hand, if $f: X \longrightarrow S$ is a separated smooth curve, each section s of f gives rise to a relative Cartier divisor $s(S)$ of X over S of degree 1. Namely, due to 2.2/7 the vanishing ideal of $\sigma(S)$ is locally principal. So we get a morphism

$$X^n \longrightarrow \mathrm{Hilb}^n_{X/S} , \qquad (s_1, \ldots, s_n) \longmapsto \sum s_i(S) ,$$

from the n-fold product of X over S to the Hilbert functor which is symmetric. Hence it factors through $(X/S)^{(n)}$. Note that, in this case, $\mathrm{Hilb}^n_{X/S}$ coincides with the subfunctor of $\mathrm{Div}^n_{X/S}$ consisting of all divisors with proper support. So it induces a morphism

$$\alpha: (X/S)^{(n)} \longrightarrow \mathrm{Hilb}^n_{X/S} .$$

Proposition 3 ([SGA 4_{III}], Exp. XVII, 6.3.9). *If $X \longrightarrow S$ is a smooth and quasi-projective morphism of relative dimension* 1, *then, for each $n \in \mathbb{N}$, the canonical morphisms*

$$\sigma: \mathrm{Hilb}^n_{X/S} \longrightarrow (X/S)^{(n)} \qquad \textit{and} \qquad \alpha: (X/S)^{(n)} \longrightarrow \mathrm{Hilb}^n_{X/S}$$

are isomorphisms and inverse to each other.

Now let us consider the case where $f: X \longrightarrow S$ is a *faithfully flat projective curve of genus g whose geometric fibres are reduced and connected.* Denote by X' the smooth locus of X. Note that X' is S-dense in X and that, moreover, the canonical map

$$(X'/S)^{(g)} \longrightarrow (X/S)^{(g)}$$

is an open immersion with S-dense image, as one can easily verify by using the fact that $(X/S)^{(g)}$ commutes with any base change. Since X is proper over S, the functor $\mathrm{Hilb}^g_{X'/S}$ is an open subfunctor of $\mathrm{Hilb}^g_{X/S}$, and since X' is smooth over S, it is already an open subfunctor of $\mathrm{Div}^g_{X/S}$; cf. 8.2/6. Furthermore, since X is proper and flat over S, the functor $\mathrm{Div}^g_{X/S}$ is a subfunctor of $\mathrm{Hilb}^g_{X/S}$. Hence, we have a commutative diagram of canonical maps

$$\begin{array}{ccc} \mathrm{Hilb}^g_{X'/S} & \xrightarrow{\sim} & (X'/S)^{(g)} \\ \downarrow & & \downarrow \\ \mathrm{Div}^g_{X/S} & \longrightarrow & (X/S)^{(g)} \end{array} .$$

The S-scheme $(X'/S)^{(g)}$ is smooth. Indeed, by étale localization it is enough to treat the case $X' = \mathbb{A}^1_S$. But then the smoothness of $(X'/S)^{(g)}$ follows from the theorem on symmetric functions. Now, let $D \subset X \times_S (X'/S)^{(g)}$ be the effective relative Cartier divisor of degree g which is induced by the map $(X'/S)^{(g)} \longrightarrow \mathrm{Div}^g_{X/S}$. We will refer to D as the universal Cartier divisor of degree g. Let $W \subset (X'/S)^{(g)}$ be the subscheme of all points $w \in (X'/S)^{(g)}$ such that $H^1(X_w, \mathcal{O}_{X_w}(D_w))$ vanishes; so

$$W = \{w \in (X'/S)^{(g)}; H^1(X_w, \mathcal{O}_{X_w}(D_w)) = 0\} .$$

Then, due to the semicontinuity theorem [EGA III_2], 7.7.5, W is an open subscheme of $(X'/S)^{(g)}$, and the following lemma shows that $W \longrightarrow S$ is surjective.

Lemma 4. *Let X be a proper curve over a separably closed field K. Assume that X is geometrically reduced and connected. Then there exists an effective Cartier divisor D_0 of degree $g = \dim_K H^1(X, \mathcal{O}_X)$ on X whose support is contained in the smooth locus of X and which satisfies $H^0(X, \mathcal{O}_X(D_0)) = K$ and $H^1(X, \mathcal{O}_X(D_0)) = 0$.*

In particular, keeping the notations of above, the map $W \longrightarrow S$ is surjective.

Proof. The Riemann-Roch theorem implies $H^0(X, \mathcal{O}_X(D_0)) = K$ if $H^1(X, \mathcal{O}_X(D_0)) = 0$. So it suffices to show the existence of an effective Cartier divisor D_0 of degree g satisfying $H^1(X, \mathcal{O}_X(D_0)) = 0$. Let ω be a dualizing sheaf on X; cf. [FGA], n°149, Sect. 6, Thm. 3 bis. Then, for any Cartier divisor E of X, there is a canonical isomorphism

$$H^1(X, \mathcal{O}_X(E)) \xrightarrow{\sim} H^0(X, \omega(-E))^{\vee} ,$$

where $\omega(-E)$ is the $\mathcal{O}_X$-module $\omega \otimes \mathcal{O}_X(-E)$. In particular, $\dim_K H^0(X, \omega) = g$. Proceeding by induction, we will show that there exist points $x_1, \ldots, x_g$ of the smooth locus of X such that

$$\dim_K H^0(X, \omega(-x_1 - \ldots - x_i)) = g - i , \qquad \text{for } i = 1, \ldots, g .$$

Since the $\mathcal{O}_X$-module ω has no embedded components, the support of a non-zero section of ω cannot consist of finitely many points. So one can choose a rational point x_{i+1} of the smooth locus of X such that there is an element of $H^0(X, \omega(-x_1 - \ldots - x_i))$ which does not vanish at x_{i+1}. Then,

$$D_0 = x_1 + \ldots + x_g$$

is an effective Cartier divisor as required. $\square$

Due to [EGA III_2], 7.9.9, the direct image $(f_W)_*\mathcal{O}_{X \times_S W}(D)$ is locally free of rank 1, and the canonical morphism

$$((f_W)_*\mathcal{O}_{X \times_S W}(D))_w \otimes_{\mathcal{O}_{W,w}} k(w) \xrightarrow{\sim} H^0(X_w, \mathcal{O}_{X_w}(D_w))$$

is bijective; cf. Mumford [3], Sect. 5, Cor. 3.

The universal Cartier divisor D gives rise to a canonical map

$$\rho : W \longrightarrow \mathrm{Pic}^{(g)}_{X/S}$$

where $\mathrm{Pic}^{(g)}_{X/S}$ is the open subfunctor of $\mathrm{Pic}_{X/S}$ consisting of line bundles of (total) degree g; cf. Section. 9.1. Next we want to prove that ρ is an open immersion.

Lemma 5. *Keeping the notations of above, the canonical map*

$$\rho : W \longrightarrow \mathrm{Pic}^{(g)}_{X/S}$$

is an open immersion.

Proof. First of all let us show that ρ is a monomorphism. So, let L_1 and L_2 be elements of $W(T)$ for an S-scheme T giving rise to the same element in $\mathrm{Pic}^{(g)}_{X/S}(T)$. Let us denote by L_1 (resp. L_2) the associated divisors of $X \times_S T$, too. Due to 8.1/3, we may assume that the associated line bundles $\mathcal{O}_{X_T}(L_1)$ and $\mathcal{O}_{X_T}(L_2)$ are isomorphic. Since the direct images $(f_T)_*\mathcal{O}_{X_T}(L_i)$ are locally free of rank 1, it follows that L_1 and L_2 are equal and, hence, that ρ is a monomorphism. Now we prove that ρ is relatively representable by an open immersion. It has to be shown that, for any S-scheme T and for any morphism $\lambda : T \longrightarrow \mathrm{Pic}^{(g)}_{X/S}$, the induced morphism

$$\rho_T : W \times_{\mathrm{Pic}^{(g)}_{X/S}} T \longrightarrow T$$

is an open immersion. Since it suffices to check this after étale surjective base change, we may assume that the morphism λ is induced by a line bundle $\mathcal{L}$ on $X \times_S T$. The image of ρ_T is contained in the subset T' of T consisting of all points $t \in T$ satisfying $H^1(X_t, \mathcal{L}_t) = 0$. Since T' is open in T by [EGA III$_2$], 7.7.5, we may replace T by T'. In this case, $H^0(X_t, \mathcal{L}_t)$ is a $k(t)$-vector space of rank 1 for each $t \in T$. Moreover $(f_T)_*\mathcal{L}$ is locally free of rank 1 and a local generator of $(f_T)_*\mathcal{L}$ gives rise to a generator of $H^0(X_t, \mathcal{L}_t)$ on any fibre X_t. Therefore, a local generator of $(f_T)_*\mathcal{L}$ is uniquely determined up to a unit of the base. Hence, the local generators of $(f_T)_*\mathcal{L}$ give rise to a closed subscheme L of $X \times_S T$ whose defining ideal is locally generated by one element. Due to 8.2/6, there exists a largest open subscheme T'' of T such that the restriction of L to $X \times_S T''$ is an effective relative Cartier divisor. It is clear that ρ_T factors through T''. So we may replace T by T'' and we may assume that L is an effective relative Cartier divisor. Thus we can view λ as a section of $\mathrm{Div}^g_{X/S}$ and, hence, of $(X/S)^{(g)}$. Since W is an open subscheme of $(X/S)^{(g)}$, the map ρ_T can be represented by the open immersion of the inverse image $\lambda^{-1}(W)$ into T. □

Lemma 6. *Keeping the notations of above, there exist a surjective étale extension $S' \longrightarrow S$, an open subscheme W' of $W \times_S S'$ with geometrically connected fibres, and a section $\varepsilon' : S' \longrightarrow W'$ such that*

$$W' \longrightarrow \mathrm{Pic}^0_{X \times_S S'/S'}, \qquad w' \longmapsto \rho(w') - \rho \circ \varepsilon' \circ p'(w)$$

is an open immersion, where $p' : W' \longrightarrow S'$ is the structural morphism.

Proof. If there is a section $\varepsilon : S \longrightarrow W$, we can assume that the geometric fibres of W are connected after replacing W by an open subscheme; cf. [EGA IV$_3$], 15.6.5. Then we can transform the morphism

$$\rho : W \longrightarrow \mathrm{Pic}^{(g)}_{X/S}$$

by a translation into an open immersion

$$W \longrightarrow \mathrm{Pic}_{X/S}, \qquad w \longmapsto \rho(w) - \rho \circ \varepsilon \circ p(w),$$

where $p : W \longrightarrow S$ is the structural morphism. Since the fibres of W over S are geometrically connected, the image of the above map is contained in $\mathrm{Pic}^0_{X/S}$. In the general case, one can perform a surjective étale extension $S' \longrightarrow S$ in order to get a section $S' \longrightarrow W$, because $W \longrightarrow S$ is smooth and surjective. Since the g-fold symmetric product $(X/S)^{(g)}$ commutes with the extension $S' \longrightarrow S$, one is reduced to the case discussed before. □

In the following, keep the notations of Lemma 6. Assume $S = S'$ and $W = W'$ and that there is a section $\varepsilon : S \longrightarrow W$. The group law of $\mathrm{Pic}_{X/S}$ induces an S-birational group law on W. We want to describe this S-birational group law on W in terms of divisors. So consider the projections

$$p_i : W \times_S W \longrightarrow W$$

for $i = 1, 2$, and let p be the structural morphism $p : W \longrightarrow S$. Since a morphism from an S-scheme T to W corresponds to an effective relative Cartier divisor of degree g on $X \times_S T$, namely, to the pull-back of the universal divisor D on $X \times_S W$, the projections p_1 and p_2 give rise to divisors D_1 and D_2 on $X \times_S W \times_S W$. Furthermore, let D_0 be the divisor on $X \times_S W \times_S W$ induced by ε. Then consider the locally free sheaf

$$\mathcal{L} = \mathcal{O}_{X \times_S W \times_S W}(D_1 - D_0 + D_2).$$

on $X \times_S W \times_S W$. The pull-back of $\mathcal{L}$ via

$$(\mathrm{id}_W, \varepsilon \circ p) : W \longrightarrow W \times_S W$$

is isomorphic to $\mathcal{O}_{X \times_S W}(D)$. Since the fibres of W are geometrically irreducible, there is a p_1-dense open subscheme W_1 of $W \times_S W$ such that, for each point t of W_1, the restriction $\mathcal{L}_t$ of $\mathcal{L}$ to the fibre $X \times_S t$ satisfies $H^1(X_t, \mathcal{L}_t) = 0$. As before, we conclude that $(f_{W_1})_*\mathcal{L}$ is locally free of rank 1 over W_1 and that, for any $t \in W_1$, a generator of $H^0(X_t, \mathcal{L}_t)$ lifts to a local generator of $(f_{W_1})_*\mathcal{L}$ at t. A local generator of $(f_{W_1})_*\mathcal{L}$ is uniquely determined up to a unit of the base. Hence, the local generators of $(f_{W_1})_*\mathcal{L}$ give rise to a subscheme D_{12} of $X \times_S W_1$ whose defining ideal can locally be generated by one element. Since the pull-back of D_{12} by $(\mathrm{id}_W, \varepsilon \circ p)$ coincides with D which is an effective relative Cartier divisor, we see by Lemma 8.2/6 that there exists a p_1-dense open subscheme V_1 of W_1 such that $D_{12}|_{V_1}$ is an effective relative Cartier divisor of degree g. Since W is an open subfunctor of $\mathrm{Div}^g_{X/S}$, we see, after replacing V_1 by a smaller p_1-dense open subscheme of V_1, that $D_{12}|_{V_1}$ gives rise to a V_1-valued point of W. Proceeding similarly with the other projection, it is easy to show that the mapping

$$W \times_S W \dashrightarrow W, \qquad (D_1, D_2) \longmapsto D_{12}$$

gives rise to a strict S-birational group law; cf. 5.2/1.

In analogy to the classical case where the base S consists of a field, we call the S-group scheme associated to this S-birational group law the *Jacobian* of X over S if it exists. In the case where S consists of a field, it can easily be shown that the existence of the Jacobian implies the representability of $\mathrm{Pic}_{X/S}$; namely the latter is a disjoint sum of "translates" of the Jacobian. Furthermore, it is clear that the Jacobian coincides with $\mathrm{Pic}^0_{X/S}$. So, even for a general base, the Jacobian represents the subfunctor $\mathrm{Pic}^0_{X/S}$ as defined in Section 8.4. For example, if S is a local scheme which is normal and strictly henselian, the results of Section 5.3 can be used to show that the Jacobian is represented by a separated and smooth S-scheme. Summarizing our discussion, we can state the following result.

Theorem 7. *Let S be a normal strictly henselian local scheme and let $f: X \longrightarrow S$ be a flat projective morphism whose geometric fibres are reduced and connected curves. Then the Jacobian of X is a smooth and separated S-scheme. It coincides with $\mathrm{Pic}^0_{X/S}$ as defined in Section 8.4.*

If one admits Theorem 8.3/1, namely that $\mathrm{Pic}_{X/S}$ is an algebraic space over S, one can drop the assumption of S being normal in Theorem 7. Indeed, due to 8.4/2, $\mathrm{Pic}_{X/S}$ is smooth over S, since X is a relative curve. Hence, $\mathrm{Pic}^0_{X/S}$ is represented by an open subspace of $\mathrm{Pic}_{X/S}$. So in order to prove that $\mathrm{Pic}^0_{X/S}$ is a scheme, it suffices to show that $\mathrm{Pic}^0_{X/S}$ can be covered by the translates $\lambda W'$, where W' is the open subscheme of $\mathrm{Pic}^0_{X/S}$ constructed in Lemma 6, and where λ ranges over $W'(S)$. Since W' is smooth and faithfully flat over S, we have enough sections λ to cover $\mathrm{Pic}^0_{X/S}$ by translates $\lambda W'$; cf. 5.3/7. So every point of $\mathrm{Pic}^0_{X/S}$ has a scheme-like neighborhood. Hence $\mathrm{Pic}^0_{X/S}$ is a scheme.

If the geometric fibres of X over S are irreducible and reduced, and if there is a section $\sigma: S \longrightarrow X$ contained in the smooth locus of X, one can construct the whole Picard scheme $\mathrm{Pic}_{X/S}$ from $\mathrm{Pic}^0_{X/S}$ by translations. Namely,

$$\mathrm{Pic}_{X/S} = \coprod_{n \in \mathbb{Z}} (\mathrm{Pic}^0_{X/S} + n \cdot [\sigma(S)]),$$

where $[\sigma(S)]$ is the element of $\mathrm{Pic}_{X/S}$ associated to the Cartier divisor $\sigma(S)$; due to 2.2/7 the vanishing ideal of $\sigma(S)$ is an effective relative Cartier divisor of degree 1. It is not hard to show directly that the right-hand side represents the relative Picard functor in this case. So, for a normal and strictly henselian base, one obtains a different approach to the representability of $\mathrm{Pic}_{X/S}$ in the case of a flat projective curve X over S whose geometric fibres are reduced and irreducible.

In the case where the base S consists of a field, one has to perform a finite separable field extension $S' \longrightarrow S$ in order to get enough sections. Then the preceding construction yields the representability of $\mathrm{Pic}^0_{X/S} \times_S S'$ over the base S' and the representability over the given base is reduced to a problem of descent. If S consists of a field, this problem is not a serious one and can be overcome easily as was demonstrated by Serre and Weil. In Section 9.4, we will dicuss the representability of $\mathrm{Pic}^0_{X/S}$ by a separated S-scheme in the case of a more general base.

9.4 Construction via Algebraic Spaces

In the following, let $f: X \longrightarrow S$ be a proper and flat curve which is locally of finite presentation over the scheme S. So far we have discussed the case where the geometric fibres of X are reduced and connected. Now we want to study more general cases. Due to the general result 8.3/1, we know that $\mathrm{Pic}_{X/S}$ is an algebraic space if f is cohomologically flat in dimension zero. Recall that f is said to be cohomologically flat in dimension zero if, for every S-scheme S', the canonical morphism

$$(f_*\mathcal{O}_X) \otimes \mathcal{O}_{S'} \xrightarrow{\sim} f'_*\mathcal{O}_{X'}$$

is an isomorphism, where $X' = X \times_S S'$. For example, the condition is satisfied if the geometric fibres of X/S are reduced; cf. [EGA III_2], 7.8.6. The cohomological flatness of f is closely related to the condition that the arithmetic genus of the fibres of X is locally constant on S.

Indeed, if f is cohomologically flat in dimension zero, $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module by 8.1/7 and $\dim_{k(s)} H^0(X_s, \mathcal{O}_{X_s})$ is locally constant on S. Moreover, since the Euler-Poincaré characteristic of the fibres of X is locally constant on S by [EGA III_2], 7.9.4, the dimension $\dim_{k(s)} H^1(X_s, \mathcal{O}_{X_s})$ must be locally constant on S. Conversely, if the arithmetic genus of the fibres of X is locally constant on S, the same arguments as above show that $\dim_{k(s)} H^0(X_s, \mathcal{O}_{X_s})$ is locally constant on S. Then it follows from [EGA III_2], 7.8.4 that f is cohomologically flat in dimension zero at least if S is reduced.

If X is cohomologically flat over S in dimension zero, $\mathrm{Pic}_{X/S}$ is an algebraic space over S, but, in general, we cannot expect $\mathrm{Pic}_{X/S}$ to be a scheme, as Mumford's example shows; cf. Section 8.2. Since $\mathrm{Pic}_{X/S}$ is smooth over S by 8.4/2, $\mathrm{Pic}^0_{X/S}$ is represented by an open subspace of $\mathrm{Pic}_{X/S}$ which may be a scheme, even if $\mathrm{Pic}_{X/S}$ is not. The main task of this section will be to present conditions under which $\mathrm{Pic}^0_{X/S}$ is a scheme. We remind the reader that by Theorem 9.3/7 this is the case if the fibres of X are not too bad and if X admits many sections over S. Now let us first state the main results on the representability of $\mathrm{Pic}_{X/S}$ and of $\mathrm{Pic}^0_{X/S}$ in the case of relative curves, afterwards we will sketch their proofs.

Theorem 1 (Deligne [1], Prop. 4.3). *Let $X \longrightarrow S$ be a semi-stable curve which is locally of finite presentation. Then $\mathrm{Pic}_{X/S}$ is a smooth algebraic space over S. The identity component $\mathrm{Pic}^0_{X/S}$ is a smooth separated S-scheme and there is a canonical S-ample line bundle $\mathcal{L}(X/S)$ on $\mathrm{Pic}^0_{X/S}$. Furthermore, $\mathrm{Pic}^0_{X/S}$ is semi-abelian.*

If the base scheme S is the spectrum of a discrete valuation ring, one can prove the representability of $\mathrm{Pic}_{X/S}$ by an algebraic space and the representability of $\mathrm{Pic}^0_{X/S}$ by a separated S-scheme under far weaker assumptions on the fibres of X than in Theorem 1.

Theorem 2 (Raynaud [6], Thm. 8.2.1). *Let S be the spectrum of a discrete valuation ring. Let $f: X \longrightarrow S$ be a proper flat curve such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and let X be normal. If the greatest common divisor of the geometric multiplicities of the irreducible* *components of X_s in X_s is 1 where s is the closed point of S, then*

(a) $\mathrm{Pic}_{X/S}$ *is an algebraic space over S,*

(b) $\mathrm{Pic}^0_{X/S}$ *is represented by a separated S-scheme.*

Corollary 3. *Let S be the spectrum of a discrete valuation ring. Let $f: X \longrightarrow S$ be a proper flat curve with connected generic fibre. Assume that X is regular and that there is a rational point on the generic fibre of X. Then $\mathrm{Pic}_{X/S}$ is an algebraic space over S and $\mathrm{Pic}^0_{X/S}$ is a separated S-scheme.*

Corollary 3 is easily deduced from Theorem 2. Indeed, due to the valuative criterion of properness, the given rational point on the generic fibre extends to a section σ of X over S. Due to 3.1/2, the image of σ is contained in the smooth locus of X. So there exists an irreducible component of the special fibre X_s of X having geometric multiplicity 1 in X_s. Therefore Theorem 2 applies and the assertion is clear. □

Now let us turn to the proofs. For the proof of Theorem 1, we need further information on $\mathrm{Pic}^0_{X/S}$ in the case of smooth relative curves.

Proposition 4. *Let $f: X \longrightarrow S$ be a proper smooth morphism of schemes whose geometric fibres are connected curves. Then $\mathrm{Pic}^0_{X/S}$ is an abelian S-scheme and there is a canonical S-ample rigidified line bundle $\mathscr{L}(X/S)$ on $\mathrm{Pic}^0_{X/S}$.*

The construction of $\mathscr{L}(X/S)$ is canonical in such a way that, for any base change $S' \longrightarrow S$, there is a canonical isomorphism of rigidified line bundles

$$\mathscr{L}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \xrightarrow{\sim} \mathscr{L}(X'/S') ,$$

where X' denotes the S'-scheme $X \times_S S'$. One will use this fact to show the representability of $\mathrm{Pic}^0_{X/S}$ by an S-scheme in the more general case of semi-stable curves.

Proof of Proposition 4. In order to keep notations simple, let us write P instead of $\mathrm{Pic}^0_{X/S}$ in the following. Due to 6.1/7, it suffices to prove the assertion after étale surjective base change $S' \longrightarrow S$. So we may assume that $X \longrightarrow S$ is projective; cf. 9.3/2. Then $\mathrm{Pic}_{X/S}$ is a separated smooth S-scheme by 9.3/1 and the identity component P is quasi-projective over S. Since P is proper over S by 8.4/3, it is even projective over S. So it remains to explain the construction of the canonical S-ample sheaf $\mathscr{L}(X/S)$ on P.

It is enough to look at the universal case. So, since the base of the versal deformations of a smooth curve is smooth over $\mathbb{Z}$ (cf. Deligne and Mumford [1], Cor. 1.7), we may assume that S consists of a regular noetherian ring. Due to 8.2/1, the Picard functor $\mathrm{Pic}_{P/S}$ is a separated S-scheme and, due to 8.4/5, the identity component $\mathrm{Pic}^0_{P/S}$ is represented by an abelian S-scheme. Denote it by P^* and call it the dual of P. There is a universal line bundle $\mathscr{P}$ on $P \times_S P^*$, the Poincaré bundle, which is rigidified along the unit sections of P and P^* over S; cf. 8.2/4. For the construction of the canonical S-ample sheaf $\mathscr{L}(X/S)$ on P/S, we need the existence of the canonical isomorphism

$$\varphi : P \xrightarrow{\sim} P^*$$

which is given by the Θ-divisor. To define the Θ-divisor, assume first that $X \longrightarrow S$ has a section $\sigma : S \longrightarrow X$. Then one has a morphism

$$(X/S)^{(g-1)} \longrightarrow P = \mathrm{Pic}^0_{X/S} , \qquad D_T \longmapsto [D_T] - (g-1)[\sigma_T] ,$$

where, for any S-scheme T and for any T-valued point D_T of $(X/S)^{(g-1)}$ (i.e., for any effective Cartier divisor on $X \times_S T$ of degree $g-1$), we denote by $[D_T]$ the element of $\mathrm{Pic}_{X/S}(T)$ corresponding to D_T and where σ_T denotes the relative Cartier divisor of $X \times_S T$ associated to the section $\sigma_T = \sigma \times_S T$. Let W^{g-1} be the schematic image of this morphism; note that it depends on the section σ. It is not hard to see that the induced map

$$(X/S)^{(g-1)} \longrightarrow W^{g-1}$$

is S-birational; cf. Lemma 9.3/5. Furthermore, W^{g-1} is an effective relative Cartier divisor on P, usually denoted by Θ_σ. If one replaces σ by a second section, Θ_σ has to be replaced by a translate. Now let us consider the morphism

$$\varphi_{\Theta_\sigma} : P \longrightarrow P^* , \qquad t \longmapsto \mathcal{O}_{P_T}(\tau_t^*(\Theta_\sigma)) \otimes (\mathcal{O}_{P_T}(\Theta_\sigma))^{-1}$$

where, for an S-scheme T, we denote by P_T the T-scheme $P \times_S T$ and where $\tau_t : P_T \longrightarrow P_T$ is the translation by the T-valued point $t \in P(T)$. This map is independent of the choice of σ; so we can drop the σ. If we do not have a section, we may perform an étale surjective base change in order to get a section and, hence, to obtain φ_Θ. Because φ_Θ is independent of the chosen section, it is already defined over the given base S by descent theory.

In order to check that the above map is an isomorphism, one can assume that the base scheme S consists of an algebraically closed field. In this case, the assertion is classical; cf. Weil [2], n°62, Cor. 2. Now we set

$$\mathscr{L}(\Theta) = m^*\mathcal{O}_P(\Theta) \otimes p_1^*(\mathcal{O}_P(\Theta))^{-1} \otimes p_2^*(\mathcal{O}_P(\Theta))^{-1}$$

where $m : P \times_S P \longrightarrow P$ is the group law of P and where $p_i : P \times_S P \longrightarrow P$ are the projections for $i = 1, 2$. Note that, a priori, this definition depends on the chosen section σ, but that in fact, due to the theorem of the square, $\mathscr{L}(\Theta)$ is independent of σ. Again, by descent theory, it is already defined over S. The morphism φ_Θ gives rise to an isomorphism

$$\mathrm{id}_P \times_S \varphi_\Theta : P \times_S P \xrightarrow{\sim} P \times_S P^*$$

such that there is an isomorphism of rigidified line bundles

$$\mathscr{L}(\Theta) \xrightarrow{\sim} (\mathrm{id}_P \times_S \varphi_\Theta)^* \mathscr{P} .$$

Consider now the pull-back of $\mathscr{P}$ by the map

$$(\mathrm{id}_P, \varphi_\Theta) : P \longrightarrow P \times_S P^*$$

and denote this line bundle on P by

$$\mathscr{L}(X/S) = (\mathrm{id}_P, \varphi_\Theta)^* \mathscr{P} = (\mathrm{id}_P, \mathrm{id}_P)^* \mathscr{L}(\Theta)$$

which is isomorphic to $\mathcal{O}_P(\Theta + (-1)^*\Theta)$. Then $\mathscr{L}(X/S)$ is rigidified along the unit section and one can show that $\mathscr{L}(X/S)$ is S-ample on P. □