

1 x Thank Organizers

§1. Intro

R - discrete val'n ring $(\mathbb{Z}_p, \mathbb{C}[[t]], \mathbb{F}_p[[t]], \dots)$

$K = \text{Frac}(R)$ $(\mathbb{Q}_p, \mathbb{C}((t)), \mathbb{F}_p((t)), \dots)$

$k = R/\mathfrak{m}_R$ $(\mathbb{F}_p, \mathbb{C}, \mathbb{F}_p, \dots)$

X/K - smooth + proper variety

$H_{\text{dR}}^i(X/K)$ - alg. dR-coh.

- These are f.d. K -v.s. equipped w/ descending filtration: Hodge fil.
- Encode lots of important geometric + arithmetic information

Problem: "Geometrically describe" R -lattices in $H_{\text{dR}}^i(X/K)$.

Ex: • $\mathcal{X}/R$ a smooth, proper R -sch, $\mathcal{X}_K = X$.

Then $H_{\text{dR}}^i(\mathcal{X}/R)$ is an R -lattice in $H_{\text{dR}}^i(X/K)$

- More generally, $\mathcal{X}/R$ a sst. model of X ; $H_{\text{log dR}}^i(\mathcal{X}/R)$ gives an R -lattice

Questions: 1) What if X does not have a sst. model / R ?

2) functionality?

3) What happens if we change the model?

This talk: answer these questions for curves/ab-varieties.

§2. Curves

X/K — smooth + proper curve

We'll construct "a" "canonical" $\mathbb{Z}$ -lattice in $H_{\text{dR}}^1(X/K)$ using certain models of X over $\mathbb{Z}$.

Def'n: An admissible model of X over $\mathbb{Z}$ is a proper, flat, normal $\mathbb{Z}$ -scheme $\mathcal{X}$ of pure rel. dim 1. S.t.

- 1) $\mathcal{X}_K \cong X$
- 2) $\mathcal{X}$ is cohomologically flat ($\Leftrightarrow H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \text{free } \mathbb{Z}\text{-mod}$)
- 3) $\mathcal{X}$ has rat'l. singularities ($\exists$ regular $\tilde{\mathcal{X}}$ and proper bivate $\tilde{\mathcal{X}} \xrightarrow{\pi} \mathcal{X}$ s.t. $R_{\pi}^1 \mathcal{O}_{\tilde{\mathcal{X}}} = 0$)

Rems • Thm (Raynaud): Let $n = \text{geom. mult. of } \mathcal{X}_K$. Then
 $p \nmid n \Rightarrow \mathcal{X}$ is coh. flat.
 • $\mathcal{X}(\mathbb{Z}) \neq \emptyset \Rightarrow \mathcal{X}$ is coh flat.

- Ex 1) $\mathcal{X} = \text{normal} + \text{sst}$ is an adm. model of $\mathcal{X}_K$
 2) $\mathcal{X} = \text{regular} + \text{coh flat}$ is an adm model of $\mathcal{X}_K$
 3) Any X/K with a section has an adm. model.
 $\Rightarrow$ Any X has an adm. model after unr. base change.

Conj Any X/K has an adm. model over $\mathbb{Z}$. (False; see Raynaud's examples of non-coh flat curves)

Rems: By sst. red'n thm, X/K has a sst model after finite ext'n of K . We do not want to extend K ! (Galois action, crystalline coh, etc)

Also, sst. model hard to describe, ext'n is v. ramified!!

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Get complex $\omega_{X/R}^\bullet := (\mathcal{O}_X \rightarrow \omega_{X/R})$

Defn $H^i(X) := H^i(\omega_{X/R}^\bullet)$

Have "Hodge $\rightarrow dR$ " sp seq. $E_1^{p,q} = H^p(X, \omega_{X/R}^q) \Rightarrow H^{p+q}(X)$

Claim $d: H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \omega_{X/R})$ is zero

pf By Hodge th, $d_K = 0$. Hence $\text{Im}(d)$ is torsion.

But $H^i(X, \omega_{X/R}) \simeq H^{-i}(X, \mathcal{O}_{X/R})^\vee$ by GD, hence torsion free ($X = \text{flat} + \text{coh flat}$).

Get:

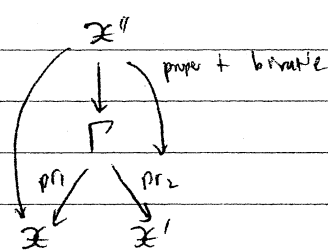
$$H(X) := 0 \rightarrow H^0(X, \omega_{X/R}) \rightarrow H^1(X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0$$

and $H(X) \otimes_R K \simeq H(X)$ since $\omega_{X/R}^\bullet \otimes_R K \simeq \Omega_{X/K}^\bullet$

Why is $H(X)$ canonically indep of X ?

$\Gamma_K = \text{gph of } X_K \simeq X'_K$
in $(X \times X')_K$

$\Gamma = \text{closure of } \Gamma_K$
in $X \times X'$



$X'' = \text{normalization of } \Gamma$

As X is adm model, so is X'' . So may assume given $X' \rightarrow X$. Get

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(X, \omega_{X/R}) & \rightarrow & H^1(X) & \rightarrow & H^1(X, \mathcal{O}_X) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H^0(X', \omega_{X'/R}) & \rightarrow & H^1(X') & \rightarrow & H^1(X', \mathcal{O}_{X'}) \rightarrow 0 \end{array}$$

RH = isom by X has RS. $\Rightarrow$ middle is isom by five term.
LH = isom by GD.

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Functionality: Given fm $f: X \rightarrow Y$, must find adm models $\tilde{X}, \tilde{Y}$ and $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ extending f .

- For pullback, just use "graph trick"
- For trace, need $\tilde{f}$ = finite flat (then use GD)
→ Must use different methods! Gmshen-Ramond

Rem: Can not do this w/ regular models!

§. Ab. vars.

A/K - ab. var.

$H_{\text{dR}}^n(A/K) \cong \wedge^n H_{\text{dR}}^1(A/K)$, so just treat H_{dR}^1

Have Hodge fil:

$$H(A) := (0 \rightarrow H^0(A, \Omega_{A/K}^1) \rightarrow H_{\text{dR}}^1(A/K) \rightarrow H^1(A, \mathcal{O}_A) \rightarrow 0)$$

Q: How to equip $H(A)$ w/ canonical integral str?

Let $\mathcal{A}/R = \text{Hér}(A)$. If $\mathcal{A}$ is proper, can use $H_{\text{dR}}^1(\mathcal{A}/R)$; In general, no good!

Must reinterpret $H(A)$ (after Lurie, Rosenlicht):

Let $E_A^* = \text{univ ext}^n \text{ of } A^* \text{ by a vect.-gp.}$

$$(*) \quad \begin{array}{ccccccc} 0 & \rightarrow & \omega_A & \rightarrow & E_A^* & \rightarrow & A^* \rightarrow 0 \\ & & \downarrow \exists! & & \downarrow \exists! & & \parallel \\ 0 & \rightarrow & \mathcal{V} & \rightarrow & E & \rightarrow & A^* \rightarrow 0 \end{array} \quad \leftarrow \text{exact seq of smooth groups.}$$

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Then $\exists$ canonical isom of k -v.s.

$$0 \rightarrow \text{Lie } \omega_A \rightarrow \text{Lie } E_{A^*} \rightarrow \text{Lie } A^* \rightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \rightarrow H^0(U, \Omega_{A/k}^1) \rightarrow H_{\text{dR}}^1(A/k) \rightarrow H^1(U, \mathcal{O}_A) \rightarrow 0 = H^1(A)$$

Strategy to equip $H^1(A)$ w/ integral str: extend (*) to exact seq of smooth gps over $\mathbb{R}$.

Problem In general, $A^* := \text{Mer}(A^*)$ has no universal ext'n by a vect. gp!

Instead, must use:

Thm (Artin, Mazur, Messing, Milne): The functor

$$\text{Extng}(\mathcal{A}^*, G_m): (\text{Sch}/S)_{\text{fppf}} \rightarrow (\text{Ab gps})$$

$$T \mapsto \left\{ \begin{array}{l} \text{ext'ns} \\ 0 \rightarrow G_{m,T} \rightarrow E \rightarrow A_T^* \rightarrow 0 \\ \uparrow \quad \uparrow \\ \text{+ rigidification} \quad \text{Inf}_T^1(\mathcal{O}_T) \end{array} \right\} / \text{isom}$$

is represented by a smooth gp. sch, and there is an ex. seq. of smooth gps

$$0 \rightarrow \omega_A \rightarrow \text{Extng}(A, G_m) \rightarrow A^{*,0} \rightarrow 0$$

← idem. comp

whose generic fiber ex. seq is (*).

Cor The exact seq. of free $\mathbb{R}$ -mod

$$H^1(\mathcal{A}) = 0 \rightarrow \text{Lie } \omega_A \rightarrow \text{Lie } \text{Extng}(A, G_m) \rightarrow \text{Lie } A^{*,0} \rightarrow 0$$

is a canonical integral str on $H^1(A)$.