

Thank organizers for opp to speak!

§ Intro

$K/\mathbb{Q}_p$ a p -adic field
 $R =$ ring of integers $\mathcal{O}_K$, uniformizer π , $S = \text{Spec } R$
 $k = R/\pi R$, residue field.

X/K a smooth, proper algebraic variety.

For each $i \geq 0$, have the alg. de Rham cohomology

$H_{\text{dR}}^i(X/K) =$ finite dim'l K -vector space

equipped with the Hodge Filtration: describing filtration by K -subspaces.

Ex $X =$ curve, $H_{\text{dR}}^i = 0$ for $i > 2$ and the Hodge filtration of H_{dR}^1 is given by

(*)
$$0 \rightarrow H^0(X, \Omega_{X/K}^1) \rightarrow H_{\text{dR}}^1(X/K) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0$$

Problem: Describe R -lattices in $H_{\text{dR}}^i(X/K)$.

Why do we expect to be able to do this?

- 1) If $\mathcal{X}/R$ is a smooth proper model of X/K then $H_{\text{dR}}^i(\mathcal{X}/R)$ gives such a lattice. (alt: H_{cris}^i)
- 2) If $\mathcal{X}/R$ is a semi-stable proper model, can (alt: $H_{\text{log-cris}}^i$) use the de Rham cohomology with log-poles along singular locus.

3) p -adic Hodge theory: $H_{\text{dR}}^i(X/K) \simeq (H_{\text{et}}^i(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_K}$

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Can find G_K -stable lattices in $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)$
as G_K is compact.

Why do we care about lattices?

- 1) Detect when a cohomology class is divisible by p .
- 2) Reduction mod p
- 3) Technically powerful tool.
- 4) Can work with "completed" dR-coh.

ex: Study p -adic modular forms.

Questions: • What to do if X does not have
a smooth/^{ss} and proper model?

- In previous constructions, how does the lattice depend on the model?
- Functoriality: If $f: X \rightarrow Y$ is a K -morphism, do we get a map on lattices?
- Is there a "canonical" lattice?

My talk: give fairly complete answers to these q's
when X = curve or abelian variety.

§ Curves

Fix a smooth, proper, and geom. connected curve X over K . We'll construct a "canonical" lattices in $H_{\text{dR}}^i(X/K)$ by using a certain class of models of X :

Def'n An admissible model $\mathcal{X}_R$ of X/K is a flat ^{normal} and proper $S = \text{Spec } R$ -scheme ~~of~~ of pure relative dim 1 (rel. curve) such that

- 1) $\mathcal{X}_K \cong X$
- 2) $\mathcal{X}$ is cohomologically flat
- 3) $\mathcal{X}$ ~~unsmooth~~ has rational singularities

Rmks: • By def, $\mathcal{X} \xrightarrow{f} S$ is coh flat if $\left[f_* \mathcal{O}_{\mathcal{X}} \right]$ comes w/ base change on S . Equivalently: $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a free R -module.

– Mild hypothesis; eg. $\text{Pic}_{\mathcal{X}/S} = \text{alg. space}$ iff $\mathcal{X}$ is coh. flat.

– Thm (Raynaud): If the gcd of the geom. multiplicities of the components of $\mathcal{X}_K$ is prime to p , ^(in part) or if $\mathcal{X}$ admits a section

~~$\mathcal{X}$~~ $n =$ geom. mult. of X_K .
ptn

- $x \in \mathcal{X}$ a point of codim 2 is a rat'l sing iff there exists a resolution of singularities $\mathcal{X}' \rightarrow \mathcal{X}$ ($\mathcal{X}'$ regular) with $(R^1 \pi_* \mathcal{O}_{\mathcal{X}'})_x = 0$

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Ex: • X = normal + semistable is admissible

• X = regular + coh. flat is admissible

→ - regular + existence of a section

• modular
curves of
any p -level
(3 section)

$X_0(Np^r)$ has an adm. model over $\mathbb{Z}_p[\zeta_p]$
if $p > 3$, or if $p=2,3$ and $N \geq 5$
(thanks to $A = \text{reg. coc.}$ $\Rightarrow A^G$ has uniting if
 $\#G$ is prime is a unit in A)

known
reg. model

* Cor: X/K with a section has an adm. model.

Thm (sst red'n). There exists a finite ext'n K' of K
s.t. $X_{K'}$ has a ^{reg} sst model ($\Rightarrow$ admiss. model)

* We do not want to extend K' (crystalline coh,
Galois action, etc). Also, sst. models of $X_0(Np^r)$ $r > 1$
can be very hard to describe (adm. models are not).

Conj: X/K a smooth, proper curve over K . There
exists an adm. model X'/R of X .

plausible: regular model exists, just need section or coh. flat.

* * Main point of adm. curves:

normal = X'
 $\downarrow$ = proper birat'l
 X

$\left\{ \begin{array}{l} X = \text{adm} \Rightarrow X' = \text{adm} \end{array} \right.$

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Thm: X/K geom conn'd, sm + proper, with
adm model $\mathcal{X}$ over R .

Then there exists

$$(*) H(\mathcal{X}) := (0 \rightarrow H^0(\mathcal{X}, \omega_{\mathcal{X}/S}) \rightarrow H^1(\mathcal{X}) \rightarrow H^1(\mathcal{X}, \omega_{\mathcal{X}}) \rightarrow 0)$$

a s.e.s. of free R -modules such that

$$1) (*) \otimes_R K \text{ recovers } H(X) = (0 \rightarrow H^0(X, \Omega_{X/K}^1) \rightarrow H_{dR}^1(X/K) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0)$$

i.e. $(*)$ gives an integral str on Hodge fil.

2) $(*)$ is canonically indep of chosen ^{adm} model $\mathcal{X}$.

3) Any finite $f: X \rightarrow Y$ over K induces pullback + trace

maps

$$\begin{array}{c} H(X) \\ f_* \downarrow \uparrow f^* \\ H(Y) \end{array} \left[\begin{array}{c} 0 \rightarrow H^0(X, \Omega_{X/K}^1) \rightarrow H_{dR}^1(X/K) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0 \\ f_* \downarrow \uparrow f^* \quad f_* \downarrow \uparrow f^* \quad f_* \downarrow \uparrow f^* \\ 0 \rightarrow H^0(Y, \Omega_{Y/K}^1) \rightarrow H_{dR}^1(Y/K) \rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow 0 \end{array} \right]$$

and f^*, f_* respect the integral str's provided by
adm. models of $\mathcal{X}, \mathcal{Y}$.

4) The cup-product auto-duality of

$$H(X) \left[0 \rightarrow H^0(X, \Omega_{X/K}^1) \rightarrow H_{dR}^1(X/K) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0 \right]$$

restricts to an auto-duality of $(*)$, and

f_*, f^* are adjoint wrt this duality

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Cor Eigenvalues of Hecke ops acting on forms of wt. α , level N are algebraic integers
(No need to construct Hecke ops integrally!) omit

Construction

$\mathcal{X}$ = adm. model of X ; $\mathcal{X}$ = normal + rel dim 1, so $\mathcal{X}$ is CM by Serre.

$\Rightarrow$ Rel. dualizing cplx. of $\mathcal{X} \xrightarrow{f} S$, $f^! \mathcal{O}_S$, ~~is~~ consists of a single sheaf concentrated in some degree

$\omega_{\mathcal{X}/S}$ = flat, coherent, formation compatible w/ arb. base change on S & étale loc. on $\mathcal{X}$.

$$\begin{array}{ll} \mathcal{X} \xrightarrow{f} S & \text{smooth, } \omega_{\mathcal{X}/S} = \Omega^1_{\mathcal{X}/S} \\ \mathcal{X} \xrightarrow{f} S & \text{sst, } \omega_{\mathcal{X}/S} = \Omega^1_{\mathcal{X}/S}(\log \mathcal{X}_K) \end{array}$$

Main technical tool: Grothendieck duality:

$\mathcal{X}$ is coh. flat $\Leftrightarrow R^i f_* \mathcal{O}_{\mathcal{X}}$ loc. free for all i

$$\Rightarrow R^j f_* \mathcal{O}_{\mathcal{X}} \otimes R^{1-j} f_* \omega_{\mathcal{X}/S} \xrightarrow{\cup} R^j f_* \omega_{\mathcal{X}/S} \xrightarrow{\text{Trace}} \mathcal{O}_S$$

induces isoms

$$R^{1-j} f_* \omega_{\mathcal{X}/S} \simeq (R^j f_* \mathcal{O}_{\mathcal{X}})^\vee$$

for $j=0,1$.

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If $i: X \hookrightarrow \mathcal{X}$, get $\omega_{\mathcal{X}/S} \xrightarrow{\text{due to } S\text{-flatness}} i_* i^* \omega_{\mathcal{X}/S} \cong \Omega_{X/K}$

so

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & \xrightarrow{\quad ? \quad} & \omega_{\mathcal{X}/S} \\ \downarrow \text{ } \mathcal{X} = \text{flat} & & \downarrow \\ i_* \mathcal{O}_X & \xrightarrow{id} & i_* \Omega'_{X/K} \end{array} \quad \} = \text{dR-cplx of } X/K$$

Prop $i_* d$ carries $\mathcal{O}_{\mathcal{X}}$ into $\omega_{\mathcal{X}/S}$ (in fact, $\omega_{\mathcal{X}/S} = \text{quotient of } \Omega'_{\mathcal{X}/S}$)

"pf": First assume $\exists \mathcal{X} \xrightarrow{\pi} \mathcal{C}$ w/ $\mathcal{C}/S$ smooth curve.

Duality thm gives

$$\pi_* \omega_{\mathcal{X}/S} \cong \text{Hom}_{\mathcal{O}_{\mathcal{C}}}(\pi_* \mathcal{O}_{\mathcal{X}}, \Omega'_{\mathcal{C}/S})$$

" $d\mathcal{M} \mapsto (t \mapsto \text{Tr}(td)\mathcal{M})$ " on gen. fiber

Using this explicit description of $\omega_{\mathcal{X}/S}$ and the differential characterization of the different, show that d carries $\mathcal{O}_{\mathcal{X}}$ into $\omega_{\mathcal{X}/S}$.

In general, can find π étale locally; conclude using $\omega_{\mathcal{X}/S}$ comes w/ étale loc.

Get sub complex

$$\omega_{\mathcal{X}/S}^\bullet := (\mathcal{O}_{\mathcal{X}} \xrightarrow{d} \omega_{\mathcal{X}/S})$$

of dR-cplx of X/K . Define $H^i(\mathcal{X}) := H^i(\mathcal{O}_{\mathcal{X}} \xrightarrow{d} \omega_{\mathcal{X}/S})$

OMT

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4/5 min

Using filtration "bête" have spectral seq.

$$E_1^{m,n} = H^m(X, \omega_{X/S}^n) \Rightarrow H^{m+n}(X)$$

Claim: This degenerates at E_1

pf: Must show $d: H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \omega_{X/S})$

is zero for all $i(0)$. Since the corr. sp. seq.

for $X_k = X$ degenerates at E_1 , we know $d_k = 0$,

so $\text{Im}(d)$ is torsion. But $H^i(X, \omega_{X/S})$

is $\mathbb{K}$ -dual to $H^{1-i}(X, \mathcal{O}_X)$, = torsion free for $i=1$

by $X = \text{flat}$, for $i=0$ by $X = \text{coh. flat}$. $\square$

Get "Hodge filtration"

$$0 \rightarrow H^0(X, \omega_{X/S}) \rightarrow H^1(X) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0$$

that recovers Hodge fil of $H_{dR}^1(X_K/K)$ after $\otimes K$
as $\omega_{X/S}^\circ \otimes K \simeq \Omega_{X/K}^\circ$

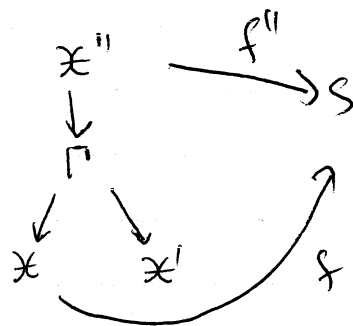
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- Why does this integral str not depend on choice of adm. model?

$\mathcal{X}, \mathcal{X}'$ models of X .

$\Gamma \subseteq \mathcal{X} \times_S \mathcal{X}'$ the closure in $\mathcal{X} \times_S \mathcal{X}'$ of $\Gamma_k = \text{graph of } \mathcal{X}_k \xrightarrow{\sim} \mathcal{X}'_k$
Get maps $\Gamma \xrightarrow{p_1} \mathcal{X}, \Gamma \xrightarrow{p_2} \mathcal{X}'$.

Let $\mathcal{X}'' = \text{normalization of } \Gamma$,
so $\mathcal{X}''_k \xrightarrow{\sim} \Gamma_k \xrightarrow{\sim} \mathcal{X}_k$, i.e. $\mathcal{X}''$
is a model of X .



Claim: $\mathcal{X}''$ is admissible.

pf: $\mathcal{X}'' \xrightarrow{p_1} \mathcal{X}$ is proper + birat'l. $\Rightarrow \mathcal{X}''$ has
rat'l singularities (standard fact, pf. due to Artin)

Using spectral seq.

$$\text{omit } \left[R^p f_* R^q p_* \mathcal{O}_{\mathcal{X}''} \Rightarrow R^{p+q} f_* \mathcal{O}_{\mathcal{X}''}, \text{ get} \right]$$

$$\text{get } 0 \rightarrow R^1 f_* (p_* \mathcal{O}_{\mathcal{X}''}) \rightarrow R^1 f_* \mathcal{O}_{\mathcal{X}''} \rightarrow f_* R^1 p_* \mathcal{O}_{\mathcal{X}''}$$

Since $\mathcal{X}$ has RS and $\mathcal{X}''$ is normal $\Rightarrow R^1 p_* \mathcal{O}_{\mathcal{X}''} = 0$
as $p = \text{proper birat'l}$ (general RS. fact)

Also, as $p = \text{proper birat'l}$, $p_* \mathcal{O}_{\mathcal{X}''} \xleftarrow{\sim} \mathcal{O}_{\mathcal{X}}$, so get

$$H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{\sim} H^1(\mathcal{X}'', \mathcal{O}_{\mathcal{X}''})$$

$\Rightarrow H^1(\mathcal{X}'', \mathcal{O}_{\mathcal{X}''})$ is torsion free, hence $\mathcal{X}'' = \text{adm.}$

$$\begin{array}{ccccc}
 R^0 f_* \omega_x \otimes R^1 f_* \mathcal{O}_x & \xrightarrow{\cup} & R^1 f_* \omega_x & \xrightarrow{\gamma_f} & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}_S \\
 \uparrow p^* & & \uparrow p^* & & \uparrow p^* \\
 R^0 f'_* \omega_{x'} \otimes R^1 f'_* \mathcal{O}_{x'} & \xrightarrow{\cup} & R^1 f'_* \omega_{x'} & \xrightarrow{\gamma_{f'}} & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}_S
 \end{array}$$

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Thus, enough to show that if $x \xrightarrow{f} x'$ is a morphism of adm models of X (i.e., an isom on gen fibres) then pullback

$$\begin{array}{ccccccc}
 0 \rightarrow H^0(x', \mathbb{Q}_x \omega_{x'/S}) & \rightarrow & H^1(x') & \rightarrow & H^1(x', \mathbb{Q}_{x'}) & \rightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 \rightarrow H^0(x, \omega_{x/S}) & \rightarrow & H^1(x) & \rightarrow & H^1(x, \mathbb{Q}_x) & \rightarrow & 0
 \end{array}$$

use
 1. is inverse
 trace
 up $p_* \omega_x \rightarrow \omega_{x'}$
 thy of trace
 ↓ Artin's
 result on RS

is an isom. RH is isom by ~~RS~~ RS of x . Roughly,
 LH is isom by Graph duality (+RS). So middle
 is too

using that cup product and trace maps are functorial.

Functoriality: For pullback, given $f: X \rightarrow X'$, x, x' adm models must find adm models $\tilde{x}, \tilde{x}'$ and $\tilde{f}: \tilde{x} \rightarrow \tilde{x}'$

recovering f on gen fibres: use graph arguments

* Can't do this w/ regular models!!

For trace, ~~must~~ given finite $f: X \rightarrow X'$, must find adm models $\tilde{x}, \tilde{x}'$ and finite $\tilde{f}: \tilde{x} \rightarrow \tilde{x}'$.

Can't do it w/ closures in graphs! (projection map not finite). Need "flattening theorems" of ~~G~~ Gusein-Raynaud.

Cup-prod. statements etc: check generically.

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§ Abelian varieties A/K an ab. var.

$$H_{\text{dR}}^n(A/K) \simeq \wedge^n H_{\text{dR}}^1(A/K) \quad \leadsto \text{ suffices to understand } H_{\text{dR}}^1$$

Have "Hodge fil"

$$(X) \quad 0 \rightarrow H^0(A, \Omega_{A/K}^1) \rightarrow H_{\text{dR}}^1(A/K) \rightarrow H^1(A, \mathcal{O}_A) \rightarrow 0$$

$\parallel$ $\parallel$
 inv. 1-forms = ω_A $\text{Lie}(A^*)$

How to equip (X) with a canonical integral structure?

$A/R = \text{Nér}(A)$. If A is proper
 (i.e. $A = \text{ab. sch.} \iff A$ has good red'n)
 then

$$0 \rightarrow \omega_A \rightarrow H_{\text{dR}}^1(A/R) \rightarrow \text{Lie}(A^*) \rightarrow 0$$

gives desired integral str.

In general, A is not proper, so $H_{\text{dR}}^1(A/R)$
 is massive!

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Alternate description of Hodge fl: (Rosenlicht, Serre)

E_{A^*} universal ext'n of A^* by a vector gp

$$(*) \quad \begin{array}{ccccccc} 0 & \rightarrow & \omega_{A^*} & \rightarrow & E_{A^*} & \rightarrow & A^* \rightarrow 0 \\ \text{pushout} & & \exists! \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & V & \rightarrow & E & \rightarrow & A^* \rightarrow 0 \end{array}$$

i.e. $\text{Hom}(\omega_{A^*}, V) \xrightarrow{\delta} \text{Ext}(A^*, V)$

is an isom.

Moreover, $\exists$ nat'l isom of ~~Base~~ K -vect spaces

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Lie } \omega_{A^*} & \rightarrow & \text{Lie } E_{A^*} & \rightarrow & \text{Lie } A^* \rightarrow 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \rightarrow & H^0(A, \Omega_{A/K}^1) & \rightarrow & H_{\text{dR}}^1(A/K) & \rightarrow & H^1(A, \mathcal{O}_A) \rightarrow 0 \end{array}$$

Q: Can $(*)$ be extended to a short exact seq. of smooth gp schemes over R ?

$$0 \rightarrow \omega_A \rightarrow E \rightarrow A^* \xrightarrow{=} \text{Nér}(A^*) \rightarrow 0$$

Problem: In general, A^* has no universal extension by a vector gp! ($\text{Ext}(A^*, G_a)$ can have ~~per~~ torsion!).

Thm: ~~Mazur~~ ^{Milne} (Artin, Mazur, Messing): The functor

$$\text{Ext}_{\mathcal{G}}(A^*, G_m) : (\text{Sch}/S)_{\text{fpt}} \rightarrow (\text{Ab. gps})$$

$$T \longmapsto \left\{ \begin{array}{l} \text{isom classes of extensions} \\ 0 \rightarrow G_{m,T} \rightarrow E \rightarrow A_T^* \rightarrow 0 \\ \text{w/ rigidification} \end{array} \right\}$$

$\uparrow$
 $\text{Inf}'_T(e)$

"Baer sum"

is represented by a smooth group scheme. Moreover, there is an exact seq. of smooth groups / S

$$0 \rightarrow \omega_A \rightarrow \text{Ext}_{\mathcal{G}}(A^*, G_m) \rightarrow A^{*,0} \rightarrow 0$$

$\nwarrow$ identity component

whose generic fiber exact seq. is canonically

$$0 \rightarrow \omega_A \rightarrow E_{A^*} \rightarrow A^* \rightarrow 0.$$

Cor. ^(Gross) The exact seq. of free R-modules

$$0 \rightarrow \omega_A \rightarrow \text{Lie Ext}_{\mathcal{G}}(A, G_m) \rightarrow \text{Lie } A^{*,0} \rightarrow 0$$

is a canonical integral str. on

$$0 \rightarrow H^0(A, \Omega_A^1) \rightarrow H^1_{\text{dR}}(A/k) \rightarrow H^1(A, \mathcal{O}_A) \rightarrow 0$$

pf: Néron mapping property + functoriality!