Algebraic Geometry, PS 4

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1 Lecture 7

Exercise 1: Let X be a projective d subspace in $\mathbb{P}_k^n$. By definition, X is given by $n-d$ linearly independent linear equations. By a projective change of coordinates, X is isomorphic to the projective subspace of $\mathbb{P}_k^n$ given by the linear equations

$$\{T_0 = 0,\ T_1 = 0,\ \dots,\ T_{n-d} = 0\}.$$

Then projection onto the last $d+1$ coordinates gives an isomorphism of X with $\mathbb{P}_k^d$, as required.
Exercise 2: Consider the *Segre* embedding $s(1,1)_K(\mathbb{P}_k^1(K) \times \mathbb{P}_k^1(K)) \to \mathbb{P}_k^3(K)$ for any K. It is defined by

$$s(1,1)_K([X_0, X_1], [Y_0, Y_1]) = [X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1] = [Z_0, Z_1, Z_2, Z_3].$$

Let Q be the hypersurface in $\mathbb{P}_k^3$ given by

$$Z_0Z_3 - Z_1Z_2 = 0 \tag{1}$$

It is clear that the image of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ under $s(1,1)_K$ is contained in Q since any point in the image satisfies 1. We show that the two are in fact isomorphic. Define $f : Q \to \mathbb{P}_k^1 \times \mathbb{P}_k^1$ by

$$f([Z_0, Z_1, Z_2, Z_3]) = \begin{cases} (1, Z_2/Z_0), (1, Z_1/Z_0) & \text{if } Z_0 \neq 0 \\ (1, Z_3/Z_1), (Z_0/Z_1, 1) & \text{if } Z_1 \neq 0 \\ (Z_0/Z_2, 1), (1, Z_3/Z_2) & \text{if } Z_2 \neq 0 \\ (Z_1/Z_3, 1), (Z_2/Z_3, 1) & \text{if } Z_3 \neq 0 \end{cases}.$$

It is not difficult to check that f is well defined: one need only observe that on the patches of $\mathbb{P}_k^3$ where more than one Z_i is nonzero the different definitions of f all agree. For example, if $Z_0, Z_1 \neq 0$

we have

$$\begin{aligned}
[1, Z_2/Z_0], [1, Z_1/Z_0] &= [Z_0, Z_2], [Z_0, Z_1] \\
&= [Z_0 Z_1, Z_2 Z_1], [Z_0, Z_1] \\
&= [Z_0 Z_1, Z_0 Z_3], [Z_0, Z_1] \\
&= [Z_1, Z_3], [Z_0, Z_1] \\
&= [1, Z_3/Z_1], [Z_0/Z_1, 1].
\end{aligned}$$

The other compatibility computations are identical, and we omit them. Observe that

$$s(1,1)_K \circ f = \mathrm{id}_f$$

and

$$f \circ s(1,1)_K = \mathrm{id}_{\mathbb{P}^1_k \times \mathbb{P}^1_k}.$$

Since both $f, s(1,1)_K$ are morphisms, this shows that they are isomorphisms and $\mathbb{P}^1_k \times \mathbb{P}^1_k \simeq Q$. Now suppose that k is an algebraically closed field and let

$$F(T_0, T_1, T_2, T_3) = (T_0, T_1, T_2, T_3) \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

be a symmetric, nondegenerate quadratic form (that is, $a_{ij} = a_{ji}$ and the matrix (a_{ij}) is nonsingular). Since k is algebraically closed and not of characteristic 2, we know that every symmetric, nonsingular matrix is equivalent (by elementary row and column operations) to the identity matrix. Moreover, by a projective change of coordinates (projective automorphism) on the form F, we can perform any row and column operations on the matrix (a_{ij}). It follows that the hypersurface defined by F is (via an appropriate projective change of coordinates) isomorphic to Q, which is given by the symmetric, nonsingular matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}.$$

Since Q is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ by the above, we see that every hypersurface given by an equation of the form F is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$. The projection maps $p_i : Q \to \mathbb{P}^1_k$ for $i = 1, 2$ are given as the first and second components (respectively) of our function f above.
Exercise 3: Recall that Ver^n_1 is the image of $\mathbb{P}^1_k$ under the map

$$v_{1,n}([X_0, X_1]) = [X_0^n, X_0^{n-1} X_1, \ldots, X_1^n] := [Z_0, Z_1, \ldots, Z_n] \in \mathbb{P}^n_k.$$

Since $Z_i = X_0^{n-i} X_1^i$, we observe that any point of Ver_1^n satisfies $Z_i Z_j = Z_k Z_l$ for $i + j = k + l$. Moreover, since $v_{1,n}$ is a morphism, we know that the image Ver_1^n is closed in $\mathbb{P}^n$. Now let C be the affine curve given by

$$\left\{Z_i - Z_1^i = 0\right\}_{i=2}^n.$$

Observe that for any point $(1, Z_1, Z_2, \dots, Z_n)$ of C we have

$$Z_i Z_j = Z_1^{i+j},$$

which only depends on $i + j$. Therefore, C is contained (via the usual inclusion of affine space into projective space) in Ver_1^n since the defining ideal of Ver_1^n vanishes on C. Observe that C is not a closed subset of $\mathbb{P}_k^n$. Indeed, any point of C satisfied the equations

$$Z_i Z_j = Z_k Z_l$$

for $i + j = k + l$. However, $(0, 0, \dots, 1)$ also satisfies these equations but is not a point of C. On the other hand, any point of $\mathrm{Ver}_1^n(K)$ with $Z_0 \neq 0$ has the form

$$(1, t, t^2, \dots, t^n)$$

for $t \in K$, while there is only one point with $Z_0 = 0$, the point $(0, 0, \dots, 1)$. It is clear that any affine point of $\mathrm{Ver}_1^n(K)$ is also a point of C. It follows that $\mathrm{Ver}_1^n(K) = C \cup (0, 0, \dots, 1)$. Since C is not closed in $\mathbb{P}_k^n$ and Ver_1^n is, it follows that the projective closure of C is Ver_1^n.

Exercise 6:

1. Let M be the set of $(n+1) \times (n+1)$ rank 1 matrices, up to proportionality. We know that the image S of $\mathbb{P}_k^n(K)'' \times \mathbb{P}_k^n(K)''$ under the Segre embedding is isomorphic to $\mathbb{P}_k^n(K)'' \times \mathbb{P}_k^n(K)''$. Any point of S has the form
$$(X_i Y_j)_{i,j=0}^n$$
with $(X_0, \dots, X_n), (Y_0, \dots, Y_n) \in \mathbb{P}_k^n$. Therefore, we define
$$f : S \to M$$
by
$$f(X_0 Y_0, \dots, X_n Y_n) = (X_i Y_j).$$
We must check that matrix $(X_i Y_J)$ has rank 1. Certainly, the rank of $(X_i Y_j)$ is at least 1 since not all $X_i Y_j$ are 0 (this would imply that $X_i = 0, Y_i$ for all i, which is impossible since $(X_0, \dots, X_n), (Y_0, \dots, Y_n)$ are points of projective space). However, $(X_i Y_j)$ has at most rank 1 since the determinant of any 2×2 minor is
$$X_i Y_j \cdot X_(i + r) Y_(j + s) - X_(i + r) Y_j \cdot X_i Y_(j + s) = 0,$$

for any $1 \le r \le n-i$ and $1 \le s \le n-j$. Therefore, the matrix (X_iY_j) indeed has rank 1. Our map f is thus well defined. It is clear that f is injective, for $X_iY_j = X_i'Y_j'$ for all i,j implies that $X_i = X_i', Y_j = Y_j'$. Surjectivity is similarly clear: any 2×2 minor of a rank 1 $(n+1) \times (n+1)$ matrix has determinant 0, so that the entries of our matrix (when considered as a lexographically ordered tuple) satisfy the defining equations of S and hence gives a point of S.

2. We know that any point of $\mathrm{Ver}_n^2(K)''$ can be written

$$(Z_{ij}) := (X_iX_j)$$

for $i,j = 0 \ldots n$ and $(X_0, \ldots, X_n) \in \mathbb{P}_k^n(K)''$. We therefore define

$$f : \mathrm{Ver}_n^2(K)'' \to M$$

(where M is the set of rank 1 symmetric square matrices with coefficients in K of size $m = \binom{n+2}{2}$ up to proportionality) by

$$f(Z_{ij}) = (Z_{ij})$$

(where we clearly mean the matrix with entries Z_{ij} on the right and the point Z_{ij} on the left). This is well defined since both the point (Z_{ij}) and the matrix (Z_{ij}) are defined up to proportionality and the symmetric matrix (Z_{ij}) has rank 1 since it certainly does not have rank 0 (since not all entries are 0) but we have

$$Z_{ij}Z_{(i+r)(j+s)} - Z_{(i+r)j}Z_{i(j+s)} = X_iX_jX_{i+r}X_{j+s} - X_{i+r}X_jX_iX_{j+s} = 0$$

so that every 2×2 minor has determinant 0. As in part a), it is clear that f is injective. Surjectivity follows similarly since any 2×2 minor of a rank 1 symmetric matrix must have determinant 0, that is, any such matrix, when considered as a lexographically ordered tuple (that is, $(a_{ij}) \mapsto (a_ij)$ with $0 \le i \le j \le n$) satisfies the defining equations of $\mathrm{Ver}_n^2(K)''$.

2 Lecture 8

Exercise 1: Let $S = \{(a,b,c) \in \mathbb{P}^2(K) : a \neq 0, b \neq 0\} \cup \{(1,0,0)\}$. If $S = Z_1 \backslash Z_2$ for some projective algebraic sets Z_1, Z_2, then for any copy of affine space $\mathbb{A}^2(K) \subset \mathbb{P}^2(K)$ we have $S \cap \mathbb{A}^2(K) = Z_1 \cap \mathbb{A}^2(K) \backslash Z_2 \cap \mathbb{A}^2(K)$, and $Z_i \cap \mathbb{A}^2(K)$ is a closed (in the subspace topology) set in $\mathbb{A}^2(K)$. Therefore, it is enough to show that $S \cap \mathbb{A}^2(K)$ is not of the form $Z_1 \backslash Z_2$ for affine algebraic sets Z_1, Z_2. We choose the copy of $\mathbb{A}^2(K)$ corresponding to $a = 1$. Then $T := S \cap \mathbb{A}^2(K) = \{(b,c) \in \mathbb{A}^2(K) : b \neq 0\} \cup (0,0)$. Since (by a previous problem set) T is dense in $\mathbb{A}^2(K)$, we must have $Z_1 = \mathbb{A}^2(K)$. But this requires that $Z_2 = \{(b,c) \in \mathbb{A}^2(K) : b = 0\} \setminus (0,0)$, which is not closed since any polynomial vanishing on the line $b = 0$ must also vanish at the origin (a polynomial restricted to a line is a polynomial in one variable and hence has finitely many roots unless it is identically 0 on the line). Thus, T is not locally closed, and it follows that S is not quasi projective.

Exercise 4: The union of quasi projective algebraic sets is not necessarily quasi projective. For example, the sets $S_1 = \{(a, b, c) \in \mathbb{P}^2(K) : a \neq 0, b \neq 0\}$ and $S_2 = (1, 0, 0)$ are both quasi projective, but their union is, by problem 1, not quasi projective. Any finite intersection of quasi projective sets is again quasi projective. For if S_1, S_2 are quasi projective, we may write $S_i = Z_{1i} \setminus Z_{2i}$ for closed sets Z_{ij} and $i, j = 1, 2$. It follows that

$$S_1 \cap S_2 = \{Z_{11} \setminus Z_{21}\} \cap \{Z_{12} \setminus Z_{22}\} = (Z_{11} \cap Z_{21}) \setminus (Z_{21} \cup Z_{22})$$

is quasi projective since the intersection and union of closed (algebraic) sets is again closed. However, an arbitrary intersection of quasi projective varieties is not necessarily quasi projective. For example,

$$\bigcap_{x \in \mathbb{A}^1(K) \setminus 0} \{(b, c) \in \mathbb{A}^2(K) : c \neq x\}$$

is an arbitrary intersection of quasi projective sets that is (by problem 1) not quasi projective.

Exercise 5: Let us find the irreducible components of the projective subset S of $\mathbb{P}^3(K)$given by $T_2T_0 - T_1^2 = 0,\ T_1T_3 - T_2^2 = 0$. Suppose first that $T_0 \neq 0$. Then if $T_1 = 0$, we must also have $T_2 = 0$, while there are no restrictions on T_3. This gives the set of points $(1, 0, 0, t)$ for $t \in K$. If $T_1 \neq 0$, then we have $T_2 = T_1^2$ and $T_1T_3 = T_1^4$, so we get the points $(1, t, t^2, t^3)$ for $t \in K$. If $T_0 = 0$, we get the point $(0, 0, 0, 1)$. It is clear from this that S is the union of two irreducible curves given by

$$\begin{aligned} C_1 :& \{(1, 0, 0, t) : t \in K\} \cup \{(0, 0, 0, 1)\} \\ C_2 :& \{(1, t, t^2, t^3) : t \in K\} \cup \{(0, 0, 0, 1)\}, \end{aligned}$$

or given as projective curves by the equations

$$\begin{aligned} &\{Z_1 = 0,\ Z_2 = 0\} \\ &\{Z_1^2 - Z_2Z_0 = 0,\ Z_1^3 - Z_3Z_0^2 = 0,\ Z_2^3 - Z_0Z_3^2\}. \end{aligned}$$

Exercise 8: We first show that a regular map is continuous. To do this, we must show that the inverse image of a closed set is closed. We know that a regular map is given locally by polynomials. Let $f : X \to Y$ be a regular map of quasi projective varieties and let C be a closed set in Y given by the polynomials g_s. Let U_i be a finite covering of X by open sets. Then $f|_{U_i}$ is just a polynomial map, for each i, so set $f|_{U_i} = f_i$. Set $V_i = f^{-1}(C) \cap U_i$. Then it is clear that V_i is the quasi projective algebraic set defined by $g_s \circ f_i$. For these polynomials certainly vanish on V_i. On the other hand, if they vanished at some point not in V_i then g_s would vanish at a point not in C, a contradiction. This shows that V_i is closed. The union of all V_i is $f^{-1}(C)$, and since this is a finite union, we see that f is continuous.

Now let us show that the composition of regular maps is regular. Let $f : X \to Y$ and $g : Y \to Z$ be regular maps of quasi projective varieties. Let U_i, W_j be finite coverings of X, Y by open sets such that $f|_{U_i} := f_i$, $g|_{W_j} := g_j$ are polynomial maps. Set

$$V_{ij} = f^{-1}(W_j) \cap U_i.$$

Since f is continuous and W_j is open, we see that V_{ij} gives an open cover of X. Moreover, $f|_{V_{ij}} = f_i$ and $g|_{W_j \cap f_i(U_i)} = g_j$. It follows that

$$g \circ f|_{V_ij} = f_i \circ g_j$$

is the composition of two polynomial maps and hence given by polynomials. It follows that $g \circ f$ is regular.

3 Lecture 9

Exercise 1: We show that the set of all degree m homogenous polynomials that split into the product of a degree k and a degree $m - k$ homogenous polynomial is closed. It will then follow that the set of degree m reducible homogenous polynomials is closed since it is a finite union of closed sets. Under the Veronese embedding, v_n^m hypersurfaces of degree m in $\mathbb{P}^n$ correspond to points of $\mathbb{P}^{N_m}$, where $N_m = \binom{n+m}{m} - 1$, since such a hypersurface is just a linear subspace (two homogenous degree m polynomials give the same hypersurface if and only if they are proportional) of $\mathbb{A}^{N_m}$. Multiplication of polynomials gives a regular map (given by appropriate equations in the coefficients)

$$f : \mathbb{P}^{N_k} \times \mathbb{P}^{N_{m-k}} \to \mathbb{P}^{N_m}$$

where $N_i = \binom{n+i}{i} - 1$ as above. Clearly, the set of reducible homogenous degree m polynomials that split as a product of a degree $m - k$ and a degree k homogenous polynomial is the image of f. By the Segre embedding, we have seen that $\mathbb{P}^{N_k} \times \mathbb{P}^{N_{m-k}}$ is a projective variety. By Theorem 1 of Lecture 9, the image of f is closed since f is regular and $\mathbb{P}^{N_k} \times \mathbb{P}^{N_{m-k}}$ is a projective variety and we are done.

Now let $n = d = 2$. We show that a degree 2 homogenous polynomial

$$F = \sum a_{ij} T_i T_j$$

in three variables T_0, T_1, T_2 is reducible if and only if $\det(a_{ij}) = 0$. Since we are working over an algebraically closed field, we may apply a projective change of coordinates to put F in "diagonal form". It is not difficult to see that F is irreducible if and only if the new, diagonalized form is irreducible. But there are only three possibilities for this form: T_i^2, $T_i^2 + T_j^2$, $T_0^2 + T_1^2 + T_2^2$. Observe that since we are working over an algebraically closed field, the first two forms are reducible. Indeed, this shows that a quadratic form F is reducible if and only if it is equivalent to one of the first two forms above (since $T_0^2 + T_1^2 + T_2^2$ is clearly not reducible). But F is equivalent to one of these forms if and only if the matrix (a_{ij}) does not have maximal rank; that is, if and only if the determinant of (a_{ij}) is 0. It follows that the projective variety of degree two reducible polynomials in three variables is given by

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} = 0.$$

Exercise 2: Let $x \in K^2$. If $K^2 \setminus \{x\}$ were an affine algebraic set, it would be given as the zero set of a system of equations, say $F_s = 0$. But any polynomial that vanishes on infinitely many points of a line vanishes on the entire line. We thus consider the restriction of the F_s to any line L passing through x. This shows that all the F_s vanish at x, a contradiction. If $K^2 \setminus \{x\}$ were a projective algebraic set then it's image under a regular map would be closed. But the inclusion $K^2 \setminus \{x\} \hookrightarrow \mathbb{A}^2(K)$ is certainly regular, but since $\mathbb{A}^2$ is irreducible, $K^2 \setminus \{x\}$ is not closed, a contradiction. Now let $y \in \mathbb{P}^n(K)$. Similarly, if $S = \mathbb{P}^n(K) \setminus \{y\}$ were a projective algebraic set, it would be closed. It would follow that $\mathbb{P}^n(K) = S \cup \{y\}$ is the union of two closed sets, a contradiction since we know that $\mathbb{P}^n(K)$ is irreducible. It is false that $\mathbb{P}^n(K) \setminus \{y\}$ is not an affine set. For example, let $n = 1, y = (0,1)$. Then $\mathbb{P}^1(K) \setminus \{y\} \simeq \mathbb{A}^1(K)$.

Exercise 4: Let $X \subset \mathbb{P}^n$ be a connected projective algebraic set and let Y be a projective set defined by $F = 0$, where F is a homogenous polynomial of degree d. Suppose that $X \cap Y = \emptyset$ and let v_n^d be the Veronese map. Then $v_n^d(X) \cap v_n^d(Y) = \emptyset$ since v_n^d is injective (and moreover $v_n^d(X)$ is isomorphic to X). Since F is of degree d, we see that $v_n^d(Y)$ is given by a linear equation. By a projective change of coordinates, we may assume that $v_n^d(Y)$ is given by $T_0 = 0$ and that $v_n^d(X) \cap v_n^d(Y) = \emptyset$. But then it follows that $v_n^d(X)$ is contained in an affine subset of $\mathbb{P}^N$. By Corollary 2 of Lecture 9, we conclude that $v_n^d(X)$ is a single point, and hence, since v_n^d is injective, that X is a single point.