Algebraic Geometry, PS 2

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Exercise 1. Let X be the subvariety of $\mathbb{A}_k^2$ defined by the equation $T_2^2 - T_1^2 - T_1^3$ and let $f : \mathbb{A}_k^1 \to X$ be the morphism given by

$$T_1 \mapsto T^2 - 1, T_2 \mapsto T(T^2 - 1).$$

We then have a map $f^* : \mathcal{O}(X) \to k[T]$ defined as above on the generators T_1, T_2 of $\mathcal{O}(X)$. Observe that

$$g(T) = f^*(p(T_1, T_2)) = p(T^2 - 1, T(T^2 - 1)) \in k[T]$$

satisfies $g(1) = g(-1) = p(0,0) \in k$. Conversely, suppose that $g(T)$ satisfies $g(1) = g(-1)$ and $\mathrm{char}(k) \neq 2$. Then the polynomial $G(T) = g(T) - g(1)$ is divisible by $T \pm 1$ and hence by $T^2 - 1$. Now observe that $T^{2n} = ((T^2 - 1) + 1)^n$ is a polynomial in $(T^2 - 1)$ and $T^{2n+1} = T \cdot T^{2n}$ is T times a polynomial in $T^2 - 1$. Thus, any polynomial may be written as a k linear combination of $(T^2 - 1)^i$ and $T(T^2 - 1)^j$ with $i, j \geq 0$. It thus follows that $g(T) \in f^*(\mathcal{O}(X))$. Thus $f^*(\mathcal{O}(X))$ is the subring of $k[T]$ consisting of polynomials $g(T)$ such that $g(1) = g(-1)$.

Now suppose that $\mathrm{char}(k) = 2$. Then $T^2 - 1 = (T - 1)^2$ and if $g(T)$ is in the image of f^* then $g(T) - g(1)$ has a double root at $T = 1$ (since it is divisible by $(T - 1)^2$ and therefore $g'(1) = 0$. Conversely, suppose that $g'(1) = 0$ for $g \in k[T]$. Then $g'(T)$ is a polynomial in $T - 1$ with no constant term. It follows that $g(T)$ is a polynomial in $T - 1$ with no linear term. We can therefore write $g(T) = (T - 1)^2 h(T) + g(0)$. Now observe that $T^{2n} = ((T - 1)^2 + 1)^n$ is a polynomial in $(T - 1)^2$ and $T^{2j+1} = T((T - 1)^2 + 1)^j$ is a k linear combination of terms of the form $(T - 1)^{2i}$ and $T(T - 1)^{2j}$ for $i, j \geq 0$. Therefore, h is a k linear combination of such terms so that $g(T)$ is in the image of $\mathcal{O}(X)$ under f^*.

When $\mathrm{char}(K) = 2$, the variety X is isomorphic to the variety Y given by the equation $U_1^2 - U_2^3 = 0$. To see this, define

$$f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$$

by

$$f^*(U_1) = T_2 - T_1, f^*(U_2) = T_1.$$

This is clearly well defined since $f(U_1^2 - U_2^3) = T_2^2 - T_1^2 - T_1^3$. Moreover, the morphism g given by

$$g^*(T_1) = U_2, g^*(T_2) = U_1 + U_2$$

is inverse to f^* (i.e. $f^*g^* : \mathcal{O}(X) \to \mathcal{O}(X) = \mathrm{id}_{\mathcal{O}(X)}$ and $g^*f^* : \mathcal{O}(Y) \to \mathcal{O}(Y) = \mathrm{id}_{\mathcal{O}(Y)}$). It therefore follows that $\mathcal{O}(X), \mathcal{O}(Y)$ are isomorphic and hence that X, Y are isomorphic.

Exercise 2. Let Y be the variety given by $T_1T_2 - 1 = 0$. If Y were isomorphic to $\mathbb{A}_k^1$ then we would have an isomorphism

$$f^* : k[T_1, T_2]/(T_1, T_2) \longrightarrow k[U].$$

So suppose that $f^*(T_1) = g(U)$ and $f^*(T_2) = h(U)$. Then $f^*(T_1T_2) = gh = f^*(1) = 1$. Therefore g, h are invertible elements of $k[U]$ and hence are both elements of k. It follows that the image of f^* is contained in k and hence f^* is not an isomorphism of $\mathcal{O}(Y)$ with $\mathcal{O}(\mathbb{A}_k^1)$.

Exercise 3. Let $f : K^2 \to K^2$ be given by $(x, y) \mapsto (x, xy)$. Then the image of f is neither open nor closed and dense in K^2. We first show that it is dense. Clearly, the image is

$$U = \left\{(a, b) \in K^2 : a \neq 0 \quad \text{or} \quad a = b = 0\right\}.$$

Suppose now that $f \in k[T_1, T_2]$ vanishes on U. Then in particular, f vanishes on all points of the line (t, b) for any fixed b and $t \neq 0$. However, on this line, f is a polynomial of the variable t alone with infinitely many roots. It follows that $f \equiv 0$ and thus that $I(\bar{U}) = (0)$, where $\bar{U}$ denotes the closure of U in the Zariski topology. Therefore, the image of f is dense. Since the image is dense and not all of K^2, it cannot be closed. Now suppose that U is open. Then the complement is closed. Let $f(T_1, T_2)$ be any polynomial vanishing on $T_1 = 0, T_2 \neq 0$. Then by the same argument as before, f must also vanish at $(0, 0)$, a contradiction. Hence the image is not open.

Exercise 4. Suppose that $f : \mathbb{A}_k^1 \to A_k^1$ is an isomorphism. then we have an isomorphism

$$f^* : k[T] \longrightarrow k[U].$$

Let g^* be inverse to f^*. Suppose that $f^*(T) = P(U)$ and that $g^*(U) = Q(T)$. Then $Q \circ P = P \circ Q = \mathrm{id}$. It follows that P, Q have. Conversely, and morphism $f^* : k[T] \to k[U]$ given by $f^*(T) = aU + b$ for $a, b \in k$ and $a \neq 0$ is an isomorphism, with inverse given by $g^*(U) = a^{-1}(U - b)$.

Exercise 5. Recall that we have the identification

$$X(K) = \mathrm{Hom}_k\left(\mathcal{O}(X), K\right).$$

Therefore, we want to show that

$$\mathrm{Hom}_k\left(\mathcal{O}(X), K\right) \times \mathrm{Hom}_k\left(\mathcal{O}(Y), K\right) = \mathrm{Hom}_k\left(\mathcal{O}(X) \otimes \mathcal{O}(Y), K\right)$$

(that is, we have a bijection between the two sets). Given any homomorphism

$$(\alpha, \beta) \in \mathrm{Hom}_k\left(\mathcal{O}(X), K\right) \times \mathrm{Hom}_k\left(\mathcal{O}(Y), K\right)$$

we get a bilinear map

$$\alpha\beta : \mathcal{O}(X) \times \mathcal{O}(Y) \longrightarrow K$$

given by $\alpha\beta(x, y) : \alpha(x)\beta(y)$. By the universal property of the tensor product, we get a unique linear map $L : \mathcal{O}(X) \otimes \mathcal{O}(Y) \to K$ such that the usual diagram commutes. To give the correspondence in the reverse direction, let $L \in \operatorname{Hom}_k(\mathcal{O}(X) \otimes \mathcal{O}(Y), K)$. Then by composition with the natural map $N : \mathcal{O}(X) \otimes \mathcal{O}(Y) \to \mathcal{O}(X) \otimes \mathcal{O}(Y)$ we get a bilinear map $B : \mathcal{O}(X) \times \mathcal{O}(Y) \to K$. This gives elements $\alpha = B(*, 1) \in \operatorname{Hom}_k(\mathcal{O}(X), K)$ and $\beta = B(1, *) \in \operatorname{Hom}_k(\mathcal{O}(Y), K)$. It is not difficult to see that $B(x, y) = \alpha(x)\beta(y)$ and hence, (recalling how multiplication is defined in the tensor product) that this is the reverse correspondence required. Thus we have a bijection and we are done.

Exercise 6. To show that the correspondence $k \to O(n, K)$ is an abstract affine algebraic k variety, it is enough to construct a finitely generated k algebra A such that for each K there is a bijection $X(K) \leftrightarrow \operatorname{Hom}_k(A, K)$. We therefore let

$$A = k[T_{11}, \dots, T_{nn}]/I,$$

where I is the ideal generated by

$$\sum_{k=1}^{n} T_{ik}T_{jk} - \delta_{ij}$$

for $1 \le i, j \le n$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Any $\varphi \in \operatorname{Hom}_k(A, K)$ is given by its values on generators, so let $\varphi(T_{ij}) = b_{ij} \in K$. Then the required bijection is given by

$$\varphi \mapsto (b_{ij}) = M \in O(n, K).$$

It is clear that this satisfies all the required properties of an abstract affine algebraic variety.

Exercise 7. Let $S_1 = \mathbb{Z} \subset \mathbb{C}$. Consider the map $f : S_1 \to S_1$ given by

$$f(x) = \begin{cases} x & \text{if } x \ne 0, 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x = 0 \end{cases}.$$

Now the Zariski topology on $\mathbb{Z} \subset \mathbb{C}$ is just that the closed sets are the finite sets. Clearly the preimage of any closed (=finite) set of S_1 under f is again a (finite=) closed set, so that f is continuous in the Zariski topology. On the other hand, f is not regular: if it were given by a polynomial then $f - x$ is a polynomial map from $S_1 \subset \mathbb{C}$ to itself with infinitely many zeroes, and is therefore identically zero, which is a contradiction.

Exercise 8. Suppose that k is a field with $\mathrm{char}(k) \neq 2$. Let X be the affine algebraic K set defined by the equations $T_1^2 + T_2^2 + T_3^2 = 0, T_1^2 - T_2^2 - t_3^2 + 1 = 0$. It is not difficult to see that this system of equations is equivalent to the system

$$T_1^2 + 1/2, T_2^2 + T_3^2 - 1/2.$$

If k does not contain a square root of $-1/2$ then this system has no solutions in k and hence is empty. Otherwise,

$$X = U_1 \cup U_2,$$

where U_i is the variety given by the system of equations

$$\left(T_1 + (-1)^i \sqrt{-\frac{1}{2}}, T_2^2 + T_3^2 - \frac{1}{2}\right),$$

each of which is obviously irreducible.

Exercise 9. It is not difficult to see that the system of equations

$$T_2^2 - T_1T_3 = 0, T_1^2 - T_2^3$$

defining a variety X has one solution set

$$T_1 = T_2 = 0, T_3 \in \mathbb{A}_k^1.$$

Let V_1 be the corresponding variety. Then V_1 is clearly one irreducible component of X. Now we may suppose that $T_1, T_2 \neq 0$ so that our system of equations is equivalent to

$$T_1T_2^2 - T_2^3T_3 = 0, T_1^2 - T_2^3 = 0,$$

which is equivalent to

$$T_1 - T_2T_3 = 0, T_1^2 - T_2^3 = 0.$$

Since we are working over a field, this is equivalent to

$$T_2 = T_3^2, T_1 = T_3^3.$$

Thus, if we let V_2 be the variety corresponding to

$$T_2 = T_3^2, T_1 = T_3^3, T + 3 \in \mathbb{A}_k^1,$$

we see that V_2 is irreducible and that

$$X = V_1 \cup V_2.$$

Moreover, it is easy to see that V_1, V_2 are birationally isomorphic to $\mathbb{A}^1_k$. One simply takes the maps

$$\begin{aligned}\phi_1 :& R(V_1) \longrightarrow k(T)\\ \phi_2 :& R(V_2) \longrightarrow k(T)\end{aligned}$$

given by

$$\begin{aligned}\phi_1(T_3) = T, \phi_1(T_2) = \phi_1(T_1) = 0,\\ \phi_2(T_3) = T, \phi_2(T_2) = T^2, \phi_2(T_1) = T^3.\end{aligned}$$

It is clear that these maps are isomorphisms.

Exercise 10. Let X be the variety defined by $T_2^2 - T_1^2 - T_1^3 = 0$ and let $f : A^1_k \to X$ be the morphism given by $T_1 \mapsto T^2 - 1, T_2 \mapsto T(T^2 - 1)$. We claim that f is birational. We must show that it induces an isomorphism of the fields $R(\mathcal{O}(X))$ and $k(T)$. So define $g : k(T) \to R(\mathcal{O}(X))$ by $T \mapsto T_2/T_1$. Then we easily check

$$\begin{aligned}g \circ f(T_1) &= (T_2^2 - T_1^2)/T_1^2 = T_1^3/T_1^2 = T_1\\ g \circ f(T_2) &= T_2(T_2^2 - T_1^2)/T_1^3 = T_2T_1^3/T_1^3 = T_2\\ f \circ g(T) &= f(T_2/T_1) = T(T^2 - 1)/(T^2 - 1) = T,\end{aligned}$$

which shows that f and g are inverses and hence that f is birational.

Exercise 11. Let $F(T_1, \ldots, T_n) = G(T_1, , \ldots, T_n) + H(T_1, , \ldots, T_n)$ be irreducible, where G is homogenous of degree $d - 1$ and H is homogenous of degree d. Let X be the algebraic set $V(F)$ and let

$$\varphi : R(X) \longrightarrow k(U_1, \ldots, U_{n-1})$$

be given by

$$\varphi(T_1) = -\frac{G(1, U_1, \ldots, U_{n-1})}{H(1, U_1, \ldots, U_{n-1})}$$

and

$$\varphi(T_i) = -U_{i-1}\frac{G(1, U_1, \ldots, U_{n-1})}{H(1, U_1, \ldots, U_{n-1})}$$

if $i > 1$.

Writing λ for $-G(1, U_1, \ldots, U_{n-1})/H(1, U_1, \ldots, U_{n-1})$, we find that

$$\begin{aligned}
\varphi(F) &= G(\lambda, \lambda U_1, \ldots, \lambda U_{n-1}) + H(\lambda, \lambda U_1, \ldots, \lambda U_{n-1}) \\
&= \lambda^{d-1} \left(G(1, U_1, \ldots, U_{n-1}) + \lambda H(1, U_1, \ldots, U_{n-1})\right) \\
&= \lambda^{d-1} \left(G(1, U_1, \ldots, U_{n-1}) - G(1, U_1, \ldots, U_{n-1})\right) \\
&= 0,
\end{aligned}$$

so that φ is well defined. Now let $\psi : k(U_1, \ldots, U_{n-1}) \longrightarrow R(X)$ be given by

$$\psi(U_i) = \frac{T_{i+1}}{T_1}.$$

Then it is not difficult to check that $\psi \circ \varphi : R(X) \to R(X)$ and $\varphi \circ \psi : k(U_1, \ldots, U_{n-1})$ are both the respective identity mappings. It follows that φ is an isomorphism and hence that X is rational.

Exercise 12. We now show that the affine algebraic sets given by $T_1^3 + T_2^3 - 1 = 0$ and $Y^2 - X^3/3 + 1/12 = 0$ are birationally isomorphic. Explicitly, let

$$\begin{aligned}
f(T_1) &= \frac{Y + 1/2}{X} \\
f(T_2) &= \frac{-Y + 1/2}{X} \\
g(X) &= \frac{1}{T_1 + T_2} \\
g(Y) &= \frac{1}{2}\frac{T_1 - T_2}{T_1 + T_2}.
\end{aligned}$$

Then one readily checks that f, g give maps

$$\begin{aligned}
f : k[T_1, T_2]/(T_1^3 + T_2^3 - 1) &\longrightarrow k[X, Y]/(Y^2 - X^3/3 + 1/12) \\
g : k[X, Y]/(Y^2 - X^3/3 + 1/12) &\longrightarrow k[T_1, T_2]/(T_1^3 + T_2^3 - 1)
\end{aligned}$$

that are inverse to eachother (this is just a computation: we have, for example, $f(T_1^3 + T_2^3 - 1) = (3Y^2 + 1/4)/X^3 - 1 = 3/X^3(Y^2 - X^3/3 + 1/12)$, as required) and hence f, g are isomorphisms.