Algebraic Geometry, PS 1

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Exercise 1. Recall that for any field k, two systems of algebraic equations $S, S^{'}$ define the same algebraic variety if and only if $(S) = (S^{'})$. Let (S) be the $k[T_1, \ldots, T_n] := k[T]$ ideal generated by the elementary symmetric polynomials S_i of degree at most n in the T_i, and let $(S^{'})$ be the ideal generated by the set of polynomials

$$S^{'} = \left\{ X_i := \sum_{j=1}^{n} T_j^i : 1 \leq i \leq n \right\}.$$

It is an elementary result from algebra that (S) consists of *all* symmetric polynomials in the T_i, (see, for example [1, p. 550]) and therefore that $(S^{'}) \subset (S)$. The formula

$$j!S_j = X_1^j - X_j + \sum_{i=1}^{\lfloor \frac{j}{2} \rfloor} \binom{j}{i} (X_j - X_i X_{j-i})$$

may be proved by induction and shows that $j!S_j \in (S^{'})$ for each $1 \leq j \leq n$. Now if $\mathrm{char}(k) \not| n!$ then $j!$ is invertible in k for each $1 \leq j \leq n$ and therefore $S_j \in (S^{'})$. Moreover, the representation of S_j given above is unique so we see that $(S) = (S^{'})$ if and only if $\mathrm{char}(k) > n$.

Exercise 2. Consider the systems

$$\begin{aligned} S &= \left\{ T_1^2 + T_2, T_1 \right\} \\ S^{'} &= \left\{ T_2^2 T_1^2 + T_2^3 + T_2 + T_1 T_2, T_2 T_1^2 + T_2^2 + T_1 \right\} \end{aligned}$$

of algebraic equations over $\mathbb{Q}$. The system $S^{'}$ is equal to

$$\left\{ T_2(T_2 T_1^2 + T_2^2 + T_1) + T_2, T_2 T_1^2 + T_2^2 + T_1 \right\},$$

which is equivalent to the system

$$\left\{ T_2, T_2 T_1^2 + T_2^2 + T_1 \right\},$$

which is in turn equivalent to

$$\{T_2, T_1\}.$$

On the other hand, S is equivalent to

$$\{T_2, T_1\},$$

since if T_1 and $T_1^2 + T_2$ vanish on any variety V then so does T_2 and conversely (because we are working over a field). Thus, S and S' define the same $\mathbb{Q}$ varieties.

Exercise 3. Let $X \subset \mathbb{A}_k^n, Y \subset \mathbb{A}_k^m$ be two affine algebraic k varieties. Let X, Y be generated by the systems $S = \{f_1, \ldots, f_r\} \subset k[T_1, \ldots, T_n]$ and $S = \{g_1, \ldots, g_s\} \subset K[U_1, \ldots, U_m]$. Let

$$I = \{f_1, \ldots, f_r, g_1, \ldots g_s\} \subset k[T_1, \ldots, T_n, U_1, \ldots, U_m].$$

We then define the cartesian product of X, Y as

$$X \times Y(K) := \mathrm{Sol}(I; K).$$

Clearly, $X \times Y(K) = X(K) \times Y(K)$ for any k algebra K (we just concatenate solutions).

Exercise 4. Let us show that an arbitrary intersection of algebraic varieties is again an algebraic variety and that any finite union of algebraic varieties is an algebraic variety.

1. Let S be a finite index set and let V_s for $s \in S$ be algebraic k sets and let $I_s = I(V_s)$. Define $I = \prod_{s \in S} I_s$. Then we claim that

   $$V(I) = \bigcup_{s \in S} V_s.$$

   To see this, observe that if $x \in V_s$ for some s then every $f \in I_s$ vanishes at x, so any finite sum of products of elements of all the I_r vanishes at x (since one of the factors *must* be in I_s); i.e. $x \in V(I)$. Conversely, let $x \in V(I)$ and suppose that $x \notin V_s$ for all s. Then for each s there exists f_s such that $f_s(x) \neq 0$, whence $f = \prod_{s \in S} f_s(x) \neq 0$, but $f \in I$, a contradiction.

2. Now let S be an arbitrary index set and let V_s, I_s be as above. Set $I = \sum_{s \in S} I_s$. Now we claim that

   $$\bigcap_{s \in S} V_s = V(I).$$

   Let $x \in V_s$ for all s. Then I_S vanishes at x for each s, so that I also vanishes at x; that is, $x \in V(I)$. Conversely, since $I_s \subset \sum_{s \in S} I_s$, we see that $V(I) \subset V(I_s)$ (since the V operator is inclusion reversing) and hence that $V(I) \subset \bigcap V_s$.

Exercise 5. Let

$$S, S^{'} \subset k[T_1, \ldots, T_n]$$

be two systems of linear equations and suppose that $\mathrm{Sol}((S);k) = \mathrm{Sol}((S');k)$. Since S is linear, we can identify S with a subset of k^n via

$$\sum_{i=1}^{n} a_i T_i \mapsto (a_1, \dots, a_n).$$

Indeed, we can in fact view S as a subspace of k^n (as a vector space) since if $f(c_1, \dots, c_n) = 0$ for some $f \in S$ and $c = (c_1, \dots, c_n) \in k^n$ then $f(\lambda c_1, \dots, \lambda c_n) = 0$ also since f is linear. It is not difficult to see that under this identification, $\mathrm{Sol}((S);k)$ is just the orthogonal complement of S in k^n with respect to the usual inner product. Now observe that any space is determined uniquely by its orthogonal complement, so that the *vector spaces* S, S' must be equal. This implies that the *ideals* $(S), (S')$ are equal. The converse of this result is trivial.

Exercise 6. We show that if a ring A is Noetherian then so is the polynomial ring $A[X]$. For suppose that $A[X]$ is not Noetherian. Then there exists a nonstabilizing chain of ideals $I_1 \subset I_2 \subset \ldots \subset A[X]$. Let $f_1 \in I$ be a polynomial of minimal degree in X. Define f_i for $i \geq 2$ inductively to be a polynomial of minimal degree in $I \setminus (f_1, f_2, \dots, f_{i-1})$. Let f_i have degree $d(i)$ in X and leading coefficient a_i. Then we claim that $(a_1) \subset (a_1, a_2) \subset \ldots \subset A$ is a nonstabilizing chain of ideals. For suppose that it stabilizes: then there exists some n such that $(a_1, \dots a_n) = (a_1, \dots, a_m)$ for all $m \geq n$, so that we can write

$$a_{n+1} = \sum_{i=1}^{n} b_i a_i$$

where $b_i \in A$. Then the polynomial $f_{n+1} - \sum_{i=1}^{n} b_i X^{d(n)-d(i)} f_i$ is in $I \setminus (f_1, \dots, f_n)$ (since $f_{n+1} \notin (f_1, \dots, f_n)$), but is of lower degree than f_{n+1}. This is a contradiction and our chain of ideals in A does not terminate. The contrapositive then shows that if A is Noetherian, so is $A[X]$, and by induction, this implies that $A[T_1, \dots, T_n]$ is Noetherian.

Exercise 7. Let $A = k[T_1, T_2]/(T_1^2 - T_2^3)$. Let K denote the field of fractions of A. Then $T_1/T_2 \in K \setminus A$ and

$$\left(\frac{T_1}{T_2}\right)^2 = \frac{T_1^2}{T_2^2} = \frac{T_2^3}{T_2^2} = T_2.$$

Therefore, T_1/T_2 is a root of the polynomial $X^2 - T_2$. It is therefore integral over A.

Exercise 8. Let X, Y be two affine algebraic sets in K^n. Suppose that $f \in I(X) \cap I(Y)$. Then f vanishes on X and on Y so that f vanishes on $X \cup Y$ and hence $f \in I(X \cup Y)$. Conversely, if $f \in I(X \cup Y)$ then f vanishes on X and on Y and therefore $f \in I(X)$ and $f \in I(Y)$. In short, this shows that

$$I(X \cup Y) = I(X) \cap I(Y).$$

We do not always have $I(X) \cap I(Y) = I(X)I(Y)$. For example, let $Y = X = \mathrm{Sol}(T_1; K)$. Then $I(X) = I(Y) = (T_1)$ (since this is a prime ideal it is already radical) and hence $I(X) \cap I(Y) = (T_1)$. On the other hand, $I(X)I(Y) = (T_1^2)$.

Exercise 9. Let $I \subset k[T_1, T_2]$ be generated by $T_1^2T_2, T_1T_2^3$ and suppose that $f \in \mathrm{rad}(I)$. Then $f^N = aT_1^2T_2 + bT_1T_2^3$ for some $N \in \mathbb{N}$ and $a, b \in k[T_1, T_2]$. This shows that f^N is divisible by T_1T_2 and therefore that f is divisible by T_1T_2 (since T_1 and T_2 are both irreducible, clearly). Hence $f \in (T_1T_2)$ and therefore that $\mathrm{rad}(I) \subset (T_1T_2)$. On the other hand, $(T_1T_2)^2 = T_2(T_1^2T_2) \in I$ so that $(T_1T_2) \subset \mathrm{rad}(I)$ and therefore $\mathrm{rad}(I) = (T_1T_2)$.

Exercise 10. We claim that the Zariski topology on $\mathbb{A}^n(K)$ is not Hausdorff. For let $x, y \in \mathbb{A}^n(K)$ be distinct points. Then if the Zariski topology were Hausdorff, we could find varieties V, U such that $x \in \mathbb{A}^n(K) \setminus V$ and $y \in \mathbb{A}^n(K) \setminus U$ and

$$\mathbb{A}^n(K) \setminus V \cap \mathbb{A}^n(K) \setminus U = \mathbb{A}^n(K) \setminus (V \cup U) = \emptyset,$$

or equivalently, $V \cup U = \mathbb{A}^n(K)$. But observe that $\mathbb{A}^n(K)$ is an irreducible variety and therefore we must have $V = \mathbb{A}^n(K)$ or $U = \mathbb{A}^n(K)$ and consequently we cannot separate x, y. The Zariski topology on $\mathbb{A}^n(K)$ is, however separable: Let x, y be distinct points of $\mathbb{A}^n(K)$. Then the variety $V = \mathrm{Sol}(T - x; K)$ contains x and not y; consequently $O = \mathbb{A}^n(K) \setminus V$ is an open set containing y and not x. The same is not true in general for the Zariski topology on $\mathbb{A}^n_k(K)$ is $k \neq K$. As an example, let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2})$. Let $x = \sqrt{2}$ and $y = -\sqrt{2}$. Then any algebraic $\mathbb{Q}$ set containing x must consist of polynomials all divisible by $T_2 - 2$ and therefore contains y also. I conjecture that if $x \notin G_{K/k}y$ then x, y can be separated in the Zariski topology, where $G_{K/k}$ denotes the galois group of K/k, which has a natural action on K^n and hence on any variety.

Exercise 11. Observe that any solution in K^n to the system of algebraic equations

$$T_1^3 = T_2^3 = T_1T_2(T_1 + T_2) = 0$$

over any field K must have $T_1 = T_2 = 0$ (this is already implied by the first two equations over a field). Any polynomial $f \not\equiv 0$ vanishing at $T_1 = T_2 = 0$ is certainly in the ideal (T_1, T_2): if some term of f is not divisible by either T_1 or T_2 then there exists some point of K^n where f does not vanish. Conversely, any polynomial in this ideal vanishes at $T_1 = T_2 = 0$. Hence, $I(V) = (T_1, T_2) \subset k[T_1, \ldots T_n]$.

Exercise 12. Observe that the set $S = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ has a limit point in $\mathbb{C}^2$. The closure of S in the Zariski $\mathbb{C}^2$ topology is contained in some closed set, that is, some variety V. Now let $f \in I(V)$. Then f is a polynomial in z_1, z_2 that vanishes on a subset of $\mathbb{C}^2$ having a limit point in $\mathbb{C}^2$. From several complex variables we know that any holomorphic function that vanishes on a set having a limit point vanishes identically. Therefore, $f \equiv 0$ and $I(V) = (0)$ so that $V = C^2$. This shows that S is dense in $\mathbb{C}^2$.

References

[1] Artin, M. *Algebra.* Prentice Hall, N.J. 1991.