p-adic Hodge Theory, MATH 847 Spring 2011

Homework Problems

1. Let I be a directed set and $\{G_i\}_{i\in I}$ an inverse system of finite groups with projection maps $\phi_{ij} : G_i \to G_j$ for all $i, j \in I$ satisfying $j \leq i$. Give each G_i the discrete topology and denote by π the product $\pi := \prod_{i\in I} G_i$ endowed with the product topology. Define

$$G := \varprojlim_{i\in I} G_i := \{(g_i)_{i\in I} \mid \phi_{ij}(g_i) = g_j \text{ for all } j \leq i\} \subseteq \pi$$

   (a) Show that G is a closed subset of π.

   (b) Give G the subspace topology. Show that G is compact and totally disconnected for this topology.

   (c) Prove that the natural projection maps $\phi_i : G \to G_i$ are continuous, and that the (open) subgroups $K_i := \ker \phi_i$ for a basis of open neighborhoods of the identity.

   (d) Show that a subgroup of G is open if and only if it is closed and of finite index.

2. Let $I \subseteq \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the inertia subgroup and $W \subseteq I$ the wild inertia subgroup. Show that there is a non-canonical isomorphism of topological groups

$$I/W \simeq \prod_{\ell \neq p} \mathbf{Z}_\ell.$$

3. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{Q}_p)$ be a continuous representation. Show that for all $\ell \neq p$, the image under ρ of any wild inertia group W_ℓ at ℓ is finite. Is the same necessarily true of the image of any I_ℓ?

4. Let F be a finite extension of $\mathbf{Q}_\ell$, and suppose $\rho : G_F \to \mathrm{GL}_n(\mathbf{Q}_p)$ is a continuous representation. Show that $\overline{F}^{\ker(\rho)}$ is infinitely (wildly) ramified if and only if the image of (wild) inertia under ρ is infinite.

5. Do Exercise 1.4.3 in the notes.

6. Let K be a p-adic field. Show that the image of the p-adic cyclotomic character $\chi : G_K \to \mathbf{Z}_p^\times$ is closed.

7. Show that the two definitions of *continuous representation* given in Definition 1.2.1 of the notes really are equivalent.

8. Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C})$ be a continuous representation.

   (a) Prove that up to conjugation by an element of $\mathrm{GL}_n(\mathbf{C})$, the representation ρ factors through $\mathrm{GL}_n(K)$ for some field K of finite degree over $\mathbf{Q}$. (You may use the fact that any compact, totally disconnected subgroup of $\mathrm{GL}_n(\mathbf{C})$ is finite).

   (b) Prove that we may take K above to be an abelian extension of $\mathbf{Q}$.

   (c) For a prime p, is it the case that any continuous $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C}_p)$ must factor through $\mathrm{GL}_n(K)$ for some $K/\mathbf{Q}_p$ of finite degree?

9. Let Γ be a profinite group and R a complete discrete valuation ring with fraction field K that is a p-adic field. We suppose that Γ acts on R via continuous automorphisms (and hence also on K). Recall that if V is a finite dimensional vector space over K, an R-lattice in V is a finite free R-submodule Λ of V with the property that $\Lambda \otimes_R K \simeq V$. Show that any V with semilinear Γ action (i.e. $g(\alpha v) = g(\alpha)g(v)$ for all $\alpha \in K$ and $v \in V$) admits a Γ-stable R-lattice Λ as follows:

   (a) Choose any R-lattice $\Lambda_0 \subseteq V$. By choosing bases, show that $\mathrm{Aut}_R(\Lambda_0)$ is an open subgroup of $\mathrm{Aut}_K(V)$.

   (b) Conclude that the preimage Γ_0 of $\mathrm{Aut}_R(\Lambda_0)$ in Γ under the representation $\rho : \Gamma \to \mathrm{Aut}_K(V)$ is of finite index in Γ.

   (c) Letting $\{\gamma_i\}$ be any finite set of coset representatives for Γ/Γ_0, show that the sum (taken inside V)
   $$\Sigma_i \rho(\gamma_i)\Lambda_0$$
   is a Γ-stable R-lattice in V.

10. Do Exercise 2.5.1 in the notes.

11. Let K be a p-adic field and $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Define the dual of W by $W^* := \mathrm{Hom}_{\mathbf{C}_K-\mathrm{lin}}(W, \mathbf{C}_K)$ with G_K-action given by $g.\varphi(w) := g\varphi(g^{-1}w)$ (i.e. W^* as a $\mathbf{C}_K$-vector space is the usual $\mathbf{C}_K$-linear dual of W). Verify that indeed $W^* \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and that $W^{**} \simeq W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Show that W^* is Hodge-Tate if and only if W is. Hint: you may want to use the "concrete" characterization of Hodge-Tate representations given in class.

12. It may be helpful to know a little Galois cohomology for this exercise. I recommend looking at Tate's article `http://modular.math.washington.edu/Tables/Notes/tate-pcmi.html` or Serre's book.

    Let $\eta : G_K \to \mathbf{Z}_p^\times$ be any continuous character. Fix an extension
    $$0 \longrightarrow \mathbf{C}_K(\eta) \longrightarrow W \longrightarrow \mathbf{C}_K \longrightarrow 0 \tag{1}$$
    in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.

   (a) By choosing a $\mathbf{C}_K$-linear vector space splitting of this exact sequence, show that we may identify W with $\mathbf{C}_K(\eta) \oplus \mathbf{C}_K$ with $g \in G_K$-acting via
   $$g(v, \alpha) = (g.v + g\alpha \cdot \tau(g), g\alpha)$$
   where $\tau : G_K \to \mathbf{C}_K(\eta)$ is a function satisfying $\tau(hg) = \eta(g)\tau(h) + \tau(g)$, i.e. τ is a 1-cocycle.

   (b) Prove that τ is continuous, and that making a different choice of splitting alters τ by a coboundary.

   (c) Show that the association $W \rightsquigarrow \tau$ induces a bijection between isomorphism classes of extensions of $\mathbf{C}_K$ by $\mathbf{C}_K(\eta)$ and the set $H^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta))$. If you feel energetic, show that this is even an isomorphism of abelian groups, where we add two extensions by taking their Baer sum.

   (d) Deduce from the Ax-Sen-Tate theorem that if $\eta(I_K)$ is infinite, then (1) splits (as an extension in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$!) and that this splitting is *unique*.

13. Let K be a p-adic field and fix $q \in K$ with $|q| < 1$. Then $q^{\mathbf{Z}} := \{q^n \mid n \in \mathbf{Z}\}$ is a discrete subgroup (lattice) of $\overline{K}^{\times}$. Consider the quotient $E_q := \overline{K}^{\times}/q^{\mathbf{Z}}$; this abelian group admits a natural structure of G_K-module through the action on $\overline{K}^{\times}$. For each $r \geq 0$, let $E_q[p^r]$ be the subgroup of E_q consisting of p^r-torsion elements.

   (a) Let ζ be a primitive p^r-th root of unity and choose a p^r-th root ξ of q in $\overline{K}^{\times}$. Show that the natural map $i_{\zeta,q} : (\mathbf{Z}/p^r\mathbf{Z})^2 \to E_q[p^r]$ induced by

   $$(m, n) \mapsto \xi^n \zeta^m \in \overline{K}^{\times}$$

   is an isomorphism of abelian groups. What happens to $\iota_{\zeta,\xi}$ if we change our choices of ζ and ξ?

   (b) Define $T_p(E_q) := \varprojlim_r E_q[p^r]$ by using the natural multiplication by p maps $E_q[p^{r+1}] \to E_q p^r]$. Show that $T_p(E_q)$ is a free $\mathbf{Z}_p$-module of rank 2 and gives a continuous 2-dimensional representation $\rho_{E_q} : G_K \to \mathrm{GL}_2(\mathbf{Z}_p)$.

   (c) Set $V_p(E_q) := T_p(E_q) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Using (a), show that the natural maps $\mathbf{Z}/p^r\mathbf{Z} \to (\mathbf{Z}/p^r\mathbf{Z})^2$ and $(\mathbf{Z}/p^r\mathbf{Z})^2 \to \mathbf{Z}/p^r\mathbf{Z}$ given by $m \mapsto (m, 0)$ and $(m, n) \mapsto n$ realize $V_p(E_q)$ as an extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$, i.e. that we have a canonical exact sequence of continuous G_K-modules

   $$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow V_p(E_q) \longrightarrow \mathbf{Q}_p \longrightarrow 0\,. \tag{2}$$

   (d) Prove that $V_p(E_q)$ is Hodge-Tate. Hint: Use Problem (3).

   (e) Prove that (2) is non-split as an extension of representations of G_K, even if we extend scalars to $\overline{K}$.

14. Let K be a p-adic field with finite residue field $\mathbf{F}_q$. Pick $\alpha \in \mathrm{GL}_n(\mathbf{C}_K)$ and consider the uramified Galois representation defined by

$$G_K \twoheadrightarrow G_{\mathbf{F}_q} \simeq \widehat{\mathbf{Z}} \longrightarrow \mathrm{GL}_n(\mathbf{C}_K)$$

defined by sending $1 \in \widehat{\mathbf{Z}}$ to α. Show that this is a continuous representation if and only if all eigenvalues of the matrix α have absolute value 1. Use this to give an example of a continuous, n-dimensional G_K-representation with $\mathbf{C}_K$ coefficients that does not factor through $\mathrm{GL}_n(L)$ for any algebraic extension L/K.

15. Let K be a p-adic field containing μ_p and let $\chi : G_K \to \mathbf{Z}_p^{\times}$ be the cyclotomoc character.

   (a) Show that χ has image in $1 + p\mathbf{Z}_p$.

   (b) For any $s \in \mathbf{Z}_p$, show that the character χ^s of G_K defined by the composition of χ with the map $1 + p\mathbf{Z}_p \to 1 + p\mathbf{Z}_p$ given by $x \mapsto x^s$ makes sense and is continuous.

   (c) Prove that χ^s is Hodge-Tate if and only if $s \in \mathbf{Z}$.

16. Fix a p-adic field and let η be a nontrivial finite order continuous character $\eta : G_K \to \mathbf{Q}_p^{\times}$.

   (a) Show that η factors through the natural inclusion $\mathbf{Z}_p^{\times} \hookrightarrow \mathbf{Q}_p^{\times}$.

   (b) Prove that there are no nonzero G_K-homomorphisms $K \to K(\eta)$.

(c) Suppose that L/K is finite Galois and the restriction of η to G_L is trivial. Show that there exists a nonzero homomorphism $L \to L(\eta)$ of L-modules with semilinear G_K-action, and hence that these two G_K-modules are isomorphic.

17. Do Exercise 3.4.1 in the notes.

18. Fix a field E of characteristic p and let (M, φ_M) be an étale φ-module over E. Define $M^\vee$ to be the E-linear dual of M and let $\varphi_{M^\vee}$ be the map

$$M^\vee \longrightarrow (\varphi_E^*(M))^\vee \longrightarrow M^\vee \tag{3}$$

where the first map takes a linear functional ℓ on M to the linear functional on $\varphi_E^*(M) := M \otimes_{E,\varphi_E} E$ given by $m \otimes e \mapsto \varphi_E(\ell(m))e$, and the second map is the E-linear dual of the inverse of the E-linear isomorphism $\varphi_E^*(M) \to M$ given by the linearization of φ_M. Prove that $\varphi_{M^\vee}$ is semilinear over φ_E, and that its linearization is an isomorphism. Hint: show that the linearization of first map in (3) is the canonical isomorphism

$$\varphi_E^*(M^\vee) = \mathrm{Hom}_E(M, E) \otimes_{E,\varphi} E \simeq \mathrm{Hom}_E(M, E_\varphi) \simeq \mathrm{Hom}_{\varphi-\mathrm{sl}}(M, E) \simeq \mathrm{Hom}_E(\varphi_E^*(M), E) = \varphi_E^*(M)^\vee$$

where E_φ denotes E as an E-module via φ_E, and $\mathrm{Hom}_{\varphi-\mathrm{sl}}$ is the E-module of φ_E-semilinear E-module homomorphisms.

19. Let M be any étale φ-module over $\mathscr{O}_{\mathscr{E}}$. Show that $\mathbf{V}_{\mathscr{E}}(M)$ is continuous as a G_E-representation.

20. Let $E = \mathbf{F}_p$, so $G_E \simeq \widehat{\mathbf{Z}}$. Let $\rho : G_E \to \mathrm{Aut}_{\mathbf{F}_p}(V)$ be a continuous representation on a d-dimensional $\mathbf{F}_p$-vector space V, and let $(M, \varphi_M) = \mathbf{D}_E(V)$ be the associated étale φ-module over E. Since $E = \mathbf{F}_p$, we canonically have $\varphi^*(M) = M$ so that the linearlization φ_M^{lin} of φ_M is an $\mathbf{F}_p$-linear endomorphism of the d-dimensional $\mathbf{F}_p$-vector space M. Identifying G_E with $\widehat{Z}$ show that $\det(\rho(1))$ is the inverse of $\det(\varphi_M^{\mathrm{lin}})$.

21. Fix a pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ as in the notes and let (M, φ_M) be a φ-module over $\mathscr{O}_{\mathscr{E}}$; i.e. a finitely generated $\mathscr{O}_{\mathscr{E}}$-module with a φ-semilinear endomorphism $\varphi_M : M \to M$. Show that φ_M is étale if and only if φ_M mod p is étale. Hint: first show that M and $\varphi^*(M)$ are abstractly isomorphic as $\mathscr{O}_E$-modules—i.e. that they have the same rank and invariant factors. Conclude that φ_M is an isomorphism if and only if it is surjective, and show that surjectivity may be checked modulo p.

22. Let M be a finitely generated module over a complete discrete valuation ring R of characteristic zero with uniformizer p. Suppose that G is a monoid acting on R by ring endomorphisms and on M by semilinear module endomorphisms. Show that for each n, G acts on M/p^nM and that $\varprojlim_n (M/p^n)^G = M^G$.

23. Prove that $\mathbf{V}_{\mathscr{E}}(\mathscr{E}/\mathscr{O}_{\mathscr{E}}) = \mathbf{Q}_p/\mathbf{Z}_p$.

24. Do Exercise 3.4.3 in the notes.

25. Let $\mathscr{O}_{\mathscr{E}}$ and $\mathscr{O}_{\mathscr{E}'}$ be two complete discrete valuation rings, each with fraction field of characteristic zero and uniformizer p. Let φ and φ' be endomorphisms of $\mathscr{E}$ and $\mathscr{E}'$ which lift the p-power Frobenius map modulo p, and suppose that $f : \mathscr{O}_{\mathscr{E}} \to \mathscr{O}_{\mathscr{E}'}$ is a local homomorphism which intertwines φ and φ'. If the induced map on residue fields $E \to E'$ is finite and *purely inseparable*, show that the base change map $\Phi M_{\mathscr{O}_{\mathscr{E}}}^{\text{ét}} \to \Phi M_{\mathscr{O}_{\mathscr{E}'}}^{\text{ét}}$ given by sending (M, φ_M) to $(M \otimes_{\mathscr{O}_{\mathscr{E}}} \mathscr{O}_{\mathscr{E}'}, \varphi_M \otimes \varphi)$ is an equivalence of categories.

26. Do Exercise 4.5.1 in the notes.

27. Let K be a p-adic field and set $B_{\mathrm{dR}}^{\mathrm{naive}} := \mathbf{C}_K((t))$, equipped with the $\mathbf{C}_K$-semilinear G_K-action defined by $g.t^n := \chi^n(g)t^n$ where $\chi : G_K \to \mathbf{Z}_p^\times$ is the p-adic cyclotomic character. Give $B_{\mathrm{dR}}^{\mathrm{naive}}$ the t-adic filtration, so it becomes a filtered $\mathbf{C}_K$-vector space with semilinear G_K-action. We define

$$D_{\mathrm{dR}}^{\mathrm{naive}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$$

by $D_{\mathrm{dR}}^{\mathrm{naive}}(V) := (V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^{\mathrm{naive}})^{G_K}$ with filtration induced by the filtration on $B_{\mathrm{dR}}^{\mathrm{naive}}$, and we call $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ "naively de Rham" if $\dim_K D_{\mathrm{dR}}^{\mathrm{naive}}(V) = \dim_{\mathbf{Q}_p}(V)$. Prove that V is naively de Rham if and only if it is Hodge-Tate.

28. Let K be a 2-adic field, and consider any choice of $\epsilon = (1, \zeta_2, \zeta_4, \zeta_8, \ldots) \in R_K$, with $\{\zeta_{2^i}\}$ a collection of compatible primitive 2^i th roots of 1 in $\mathscr{O}_{\mathbf{C}_K}$. Show that $[\epsilon] - 1 \in W(R)$ is a generator of the principal ideal $\ker\theta$. Bonus: Show that the corresponding statement is *false* for $p > 2$.

29. Do Exercise 4.5.2 in the notes.

30. Suppose $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is 1-dimensional. Show that V is Hodge-Tate if and only if it is de Rham (cf. Example 6.3.9 of the notes).

31. Prove that the Frobenius automorphism of $W(R)[1/p]$ does not preserve $\ker\theta_K$, and so does not naturally extend to B_{dR}^+.

32. Prove $W(R) \cap (\ker\theta_K)^j = (\ker\theta)^j$ for all $j \geq 1$.

33. Do Exercise 4.5.3.

The next two problems are taken from Berger's article "An introduction to the theory of p-adic representations".

34. Let K be a p-adic field, fix $q \in K$ with $|q| < 1$ and set $E_q := \overline{K}^\times / q^{\mathbf{Z}}$, considered as a G_K-module through the action on $\overline{K}^\times$. We saw on Assignment 2, problem 4 that $V_p(E_q) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_r E_q[p^r]$ is 2-dimensional $\mathbf{Q}_p$-representation of G_K, and that

$$e := (\epsilon^{(r)})_{r\geq 0} \quad \text{and} \quad f := (q^{(r)})_{r \geq 0}$$

give a basis of $V_p(E_q)$ where $\epsilon^0 = 1$, $\epsilon^{(1)} \neq 1$, $q^{(0)} = q$ and for all $r \geq 1$, we have $(\epsilon^{(r+1)})^p = \epsilon^{(r)}$ and $(q^{(r+1)})^p = q^{(r)}$. Denote by $\underline{\epsilon}$ and $\underline{q}$ the elements of R defined by the p-power compatible sequences $(\epsilon^{(r)})$ and $(q^{(r)})$.

(a) Show that $g.e = \chi(g)e$ and $g.f = f + c(g)e$ for some $c(g) \in \mathbf{Z}_p$ depending on g.

(b) Show that the series $\sum_{n\geq 1}(-1)^{n+1}\frac{([\underline{q}]/q-1)^n}{n}$ for $\log(\frac{1}{q}[\underline{q}])$ makes sense and converges in B_{dR}^+. We define

$$u := \log_p(q) + \log(\frac{1}{q}[\underline{q}]).$$

Morally, $u = \log([\underline{q}])$.

(c) Let $t = \log([\underline{\epsilon}]) \in B_{\mathrm{dR}}$. Show that $g.t = \chi(g)t$ and $g.u = u + c(g)t$ for $c(g)$ as in (1).

(d) Prove that $V_p(E_q)$ is de Rham. Hint: all you have to show is that the K-vector space $(B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V_p(E_q))^{G_K}$ has dimension 2. Do this by using u and t to appropriately modify the B_{dR}-basis $1 \otimes e$ and $1 \otimes f$ of $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V_p(E_q)$ to be G_K-invariant.

35. We can generalize exercise (7). Let V be any extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. Prove that V is de Rham as follows:

(a) Let $\widehat{K^\times}$ be the projective limit $\varprojlim_n (K^\times/(K^\times)^{p^n})$ with transition maps the natural projection maps. Fix a choice $(\epsilon^{(n)})$ of a compatible system of p-power roots of unity in $\overline{K}$ and Consider the map $\delta : \widehat{K^\times} \to H^1_{\mathrm{cont}}(G_K, \mathbf{Z}_p(1))$ defined as follows: for $q = q^{(0)}$ in $\widehat{K^\times}$, choose a sequence $(q^{(n)})_{n\geq 0}$ in $\overline{K}$ with $(q^{(n+1)})^p = q^{(n)}$ for all n and let $\delta(q)$ be the cocycle c determined by $g(q^{(n)}) = (q^{(n)}) \cdot (\epsilon^{(n)})^{c(g)}$. Show that any two choices of $(q^{(n)})$ give cohomologous cycles, so δ is well-defined.

(b) Prove that δ induces an isomorphism $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \widehat{K^\times} \simeq H^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1))$.

(c) Look over your work on Assignment 2, problem 3 and convince yourself that $H^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1))$ classifies isomorphism classes of G_K-extensions of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$. Conclude that we can choose a basis $\{e, f\}$ of V such that $g.e = \chi(g)e$ and $g.f = f + c(g)e$ where $c(g)$ is the cocycle corresponding to $q \in \mathbf{Q}_p \otimes \widehat{K^\times}$ as above.

(d) Defining $u =$ "$\log([\underline{q}])$" as above, show that we can appropriately modify the basis $\{1\otimes e, 1\otimes f\}$ of $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ so as to be G_K-invariant. Conclude that V is de Rham.

36. Let K be a p-adic field. This exercise gives an alternative way of seeing that $D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$ is not full.

(a) Let $V, V' \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$. Prove that $D_{\mathrm{dR}}(V)$ and $D_{\mathrm{dR}}(V')$ are isomorphic in Fil_K if and only if V and V' have the same Hodge-Tate numbers; i.e. if and only if they have the same Hodge-Tate weights and for each Hodge-Tate weight i, the multiplicities $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V))$ and $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V'))$ are equal.

(b) Show that there exists a non-split extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ in $\mathrm{Rep}^{\mathrm{dR}}_{\mathbf{Q}_p}(G_K)$. Hint: Think back to previous assignments.

(c) Show that D_{dR} can not be full.

37. Let F be a field. Do one (or more) of the following:

(a) For objects D, D' of Fil_F, show that the canonical F-linear isomorphism $D \otimes_F D'^* \simeq \mathrm{Hom}_F(D', D)$ is an isomorphism in Fil_F, where the tensor product is given its usual tensor-product filtration and $\mathrm{Hom}_F(D', D)$ is given the filtration $\mathrm{Fil}^i \mathrm{Hom}_F(D', D) := \mathrm{Hom}_{\mathrm{Fil}_F}(D', D[i])$.

(b) Show that the canonical F-linear isomorphisms

$$\det(D^*) \simeq \det(D)^* \qquad \text{and} \qquad \det(D \otimes D') \simeq \det(D)^{\dim_F D'} \otimes \det(D')^{\dim_F D}$$

are isomorphisms in Fil_F.

(c) Prove that for a short exact sequence in Fil_F

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

the canonical F-linear isomorphism $\det(D') \otimes \det(D'') \simeq \det(D)$ is an isomorphism in Fil_F.

38. Let n, m be positive integers and K a p-adic field. Show that if V is any extension

$$0 \longrightarrow \mathbf{Q}_p(n) \longrightarrow V \longrightarrow \mathbf{Q}_p(m) \longrightarrow 0$$

in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, then V is always Hodge-Tate, and is de Rham if and only if $n > m$. Hint: Use Corollary 6.3.4 and Examples 6.3.5–6.3.6 in the notes. Make sure you understand Example 6.3.6!

39. Do Exercise 7.4.1 in the notes.

40. Do Exercise 7.4.2 in the notes.

41. Do Exercise 7.4.3 in the notes.

42. Do Exercise 7.4.5 in the notes.

43. Do Exercise 7.4.7 in the notes.

44. Do Exercise 7.4.9 in the notes.

45. Let D be a K_0-vector space with a σ-semilinear endomorphism $\phi : D \to D$. If D has finite K_0 dimension, show that ϕ is injective if and only if it is bijective. Give a counterexample to this with D of infinite dimension.

46. Let D be an isocrystal over K_0. Prove that $t_N(D) = t_N(\det D)$. Hint: first show that if $D(\alpha)$ and $D(\beta)$ are isoclinic of slopes α and β respectively, then $D(\alpha) \otimes_{K_0} D(\beta)$ is isoclinic of slope $\alpha + \beta$. Then work with a basis for D adapted to the isoclinic decomposition of D as guaranteed by Lemma 7.2.7.

47. Let D be a filtered (φ, N)-module over K. Prove that D is weakly admissible if and only if D^* is.

48. Let $h : M' \to M$ be a bijective morphism in Fil_K. Show that h is an isomorphism in Fil_K if and only if $\det(h) : \det(M') \to \det(M)$ is an isomorphism.

49. Do Exercise 8.4.1 of the notes.

50. Do Exercise 9.4.1 of the notes.

51. Do Exercise 9.4.2 of the notes.