# $p$-adic Hodge Theory, MATH 726 Fall 2008

Assignment 4

1. Let $K$ be a $p$-adic field. This exercise gives an alternative way of seeing that $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_K) \to \mathrm{Fil}_K$ is not full.

   (a) Let $V, V' \in \mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_K)$. Prove that $D_{\mathrm{dR}}(V)$ and $D_{\mathrm{dR}}(V')$ are isomorphic in $\mathrm{Fil}_K$ if and only if $V$ and $V'$ have the same Hodge-Tate numbers; i.e. if and only if they have the same Hodge-Tate weights and for each Hodge-Tate weight $i$, the multiplicities $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V))$ and $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V'))$ are equal.

   (b) Show that there exists a non-split extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ in $\mathrm{Rep}_{\mathbf{Q}_p}^{\mathrm{dR}}(G_K)$. Hint: Think back to previous assignments.

   (c) Show that $D_{\mathrm{dR}}$ can not be full.

2. Let $F$ be a field. Do one (or more) of the following:

   (a) For objects $D, D'$ of $\mathrm{Fil}_F$, show that the canonical $F$-linear isomorphism $D \otimes_F {D'}^* \simeq \mathrm{Hom}_F(D', D)$ is an isomorphism in $\mathrm{Fil}_F$, where the tensor product is given its usual tensor-product filtration and $\mathrm{Hom}_F(D', D)$ is given the filtration $\mathrm{Fil}^i \mathrm{Hom}_F(D', D) := \mathrm{Hom}_{\mathrm{Fil}_F}(D', D[i])$.

   (b) Show that the canonical $F$-linear isomorphisms

   $$\det(D^*) \simeq \det(D)^* \qquad \text{and} \qquad \det(D \otimes D') \simeq \det(D)^{\dim_F D'} \otimes \det(D')^{\dim_F D}$$

   are isomorphisms in $\mathrm{Fil}_F$.

   (c) Prove that for a short exact sequence in $\mathrm{Fil}_F$

   $$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

   the canonical $F$-linear isomorphism $\det(D') \otimes \det(D'') \simeq \det(D)$ is an isomorphism in $\mathrm{Fil}_F$.

3. Let $n$ be a positive integer and $K$ a $p$-adic field. Show that if $V$ is any extension

   $$0 \longrightarrow \mathbf{Q}_p(n) \longrightarrow V \longrightarrow \mathbf{Q}_p \longrightarrow 0$$

   in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$, then $V$ is de Rham. Hint: adapt the argument of Example 6.3.5 in the notes.

4. Let $D$ be a $K_0$-vector space with a $\sigma$-semilinear endomorphism $\phi : D \to D$. If $D$ has finite $K_0$ dimension, show that $\phi$ is injective if and only if it is bijective. Give a counterexample to this with $D$ of infinite dimension.

5. Let $D$ be an isocrystal over $K_0$. Prove that $t_N(D) = t_N(\det D)$. Hint: first show that if $D(\alpha)$ and $D(\beta)$ are isoclinic of slopes $\alpha$ and $\beta$ respectively, then $D(\alpha) \otimes_{K_0} D(\beta)$ is isoclinic of slope $\alpha + \beta$. Then work with a basis for $D$ adapted to the isoclinic decomposition of $D$ as guaranteed by Lemma 7.2.7.

6. Let $D$ be a filtered $(\varphi, N)$-module over $K$. Prove that $D$ is weakly admissible if and only if $D^*$ is.

7. Let $h : M' \to M$ be a bijective morphism in $\mathrm{Fil}_K$. Show that $h$ is an isomorphism in $\mathrm{Fil}_K$ if and only if $\det(h) : \det(M') \to \det(M)$ is an isomorphism.