p-adic Hodge Theory, MATH 726 Fall 2008

Assignment 3

1. Let K be a p-adic field and set $B_{\mathrm{dR}}^{\mathrm{naive}} := \mathbf{C}_K((t))$, equipped with the $\mathbf{C}_K$-semilinear G_K-action defined by $g.t^n := \chi^n(g)t^n$ where $\chi : G_K \to \mathbf{Z}_p^\times$ is the p-adic cyclotomic character. Give $B_{\mathrm{dR}}^{\mathrm{naive}}$ the t-adic filtration, so it becomes a filtered $\mathbf{C}_K$-vector space with semilinear G_K-action. We define
$$D_{\mathrm{dR}}^{\mathrm{naive}} : \mathrm{Rep}_{\mathbf{Q}_p}(G_K) \to \mathrm{Fil}_K$$
by $D_{\mathrm{dR}}^{\mathrm{naive}}(V) := (V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}^{\mathrm{naive}})^{G_K}$ with filtration induced by the filtration on $B_{\mathrm{dR}}^{\mathrm{naive}}$, and we call $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ "naively de Rham" if $\dim_K D_{\mathrm{dR}}^{\mathrm{naive}}(V) = \dim_{\mathbf{Q}_p}(V)$. Prove that V is naively de Rham if and only if it is Hodge-Tate.

2. Let K be a 2-adic field, and consider any choice of $\epsilon = (1, \zeta_2, \zeta_4, \zeta_8, \ldots) \in R_K$, with $\{\zeta_{2^i}\}$ a collection of compatible primitive 2^i th roots of 1 in $\mathscr{O}_{\mathbf{C}_K}$. Show that $[\epsilon] - 1 \in W(R)$ is a generator of the principal ideal $\ker\theta$. Bonus: Show that the corresponding statement is *false* for $p > 2$.

3. Do Exercise 4.4.8 in the notes.

4. Suppose $V \in \mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ is 1-dimensional. Show that V is Hodge-Tate if and only if it is de Rham.

5. Prove that the Frobenius automorphism of $W(R)[1/p]$ does not preserve $\ker\theta_K$, and so does not naturally extend to B_{dR}^+.

6. Prove $W(R) \cap (\ker\theta_K)^j = (\ker\theta)^j$ for all $j \geq 1$.

   The next two problems are taken from Berger's article "An introduction to the theory of p-adic representations".

7. Let K be a p-adic field, fix $q \in K$ with $|q| < 1$ and set $E_q := \overline{K}^\times / q^{\mathbf{Z}}$, considered as a G_K-module through the action on $\overline{K}^\times$. We saw on Assignment 2, problem 4 that $V_p(E_q) := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_r E_q[p^r]$ is 2-dimensional $\mathbf{Q}_p$-representation of G_K, and that
$$e := (\epsilon^{(r)})_{r\geq 0} \quad \text{and} \quad f := (q^{(r)})_{r \geq 0}$$
give a basis of $V_p(E_q)$ where $\epsilon^0 = 1$, $\epsilon^{(1)} \neq 1$, $q^{(0)} = q$ and for all $r \geq 1$, we have $(\epsilon^{(r+1)})^p = \epsilon^{(r)}$ and $(q^{(r+1)})^p = q^{(r)}$. Denote by $\underline{\epsilon}$ and $\underline{q}$ the elements of R defined by the p-power compatible sequences $(\epsilon^{(r)})$ and $(q^{(r)})$.
   (a) Show that $g.e = \chi(g)e$ and $g.f = f + c(g)e$ for some $c(g) \in \mathbf{Z}_p$ depending on g.
   (b) Show that the series $\sum_{n\geq 1}(-1)^{n+1}\frac{([\underline{q}]/q-1)^n}{n}$ for $\log(\frac{1}{q}[\underline{q}])$ makes sense and converges in B_{dR}^+. We define
   $$u := \log_p(q) + \log(\frac{1}{q}[\underline{q}]).$$
   Morally, $u = \log([\underline{q}])$.
   (c) Let $t = \log([\underline{\epsilon}]) \in B_{\mathrm{dR}}$. Show that $g.t = \chi(g)t$ and $g.u = u + c(g)t$ for $c(g)$ as in (1).
   (d) Prove that $V_p(E_q)$ is de Rham. Hint: all you have to show is that the K-vector space $(B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V_p(E_q))^{G_K}$ has dimension 2. Do this by using u and t to appropriately modify the B_{dR}-basis $1 \otimes e$ and $1 \otimes f$ of $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V_p(E_q)$ to be G_K-invariant.

8. We can generalize exercise (7). Let V be any extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ in $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$. Prove that V is de Rham as follows:
   (a) Let $\widehat{K^\times}$ be the projective limit $\varprojlim_n (K^\times/(K^\times)^{p^n})$ with transition maps the natural projection maps. Fix a choice $(\epsilon^{(n)})$ of a compatible system of p-power roots of unity in $\overline{K}$ and Consider the map $\delta : \widehat{K^\times} \to H^1_{\mathrm{cont}}(G_K, \mathbf{Z}_p(1))$ defined as follows: for $q = q^{(0)}$ in $\widehat{K^\times}$, choose a sequence $(q^{(n)})_{n\geq 0}$ in $\overline{K}$ with $(q^{(n+1)})^p = q^{(n)}$ for all n and let $\delta(q)$ be the cocycle c determined by $g(q^{(n)}) = (q^{(n)}) \cdot (\epsilon^{(n)})^{c(g)}$. Show that any two choices of $(q^{(n)})$ give cohomologous cycles, so δ is well-defined.
   (b) Prove that δ induces an isomorphism $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \widehat{K^\times} \simeq H^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1))$.
   (c) Look over your work on Assignment 2, problem 3 and convince yourself that $H^1_{\mathrm{cont}}(G_K, \mathbf{Q}_p(1))$ classifies isomorphism classes of G_K-extensions of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$. Conclude that we can choose a basis $\{e, f\}$ of V such that $g.e = \chi(g)e$ and $g.f = f + c(g)e$ where $c(g)$ is the cocycle corresponding to $q \in \mathbf{Q}_p \otimes \widehat{K^\times}$ as above.
   (d) Defining $u =$ "$\log([\underline{q}])$" as above, show that we can appropriately modify the basis $\{1 \otimes e, 1 \otimes f\}$ of $B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ so as to be G_K-invariant. Conclude that V is de Rham.