p-adic Hodge Theory, MATH 726 Fall 2008

Assignment 2

1. Let Γ be a profinite group and R a complete discrete valuation ring with fraction field K that is a p-adic field. We suppose that Γ acts on R via continuous automorphisms (and hence also on K). Recall that if V is a finite dimensional vector space over K, an R-lattice in V is a finite free R-submodule Λ of V with the property that $\Lambda \otimes_R K \simeq V$. Show that any V with semilinear Γ action (i.e. $g(\alpha v) = g(\alpha)g(v)$ for all $\alpha \in K$ and $v \in V$) admits a Γ-stable R-lattice Λ as follows:
   (a) Choose any R-lattice $\Lambda_0 \subseteq V$. By choosing bases, show that $\mathrm{Aut}_R(\Lambda_0)$ is an open subgroup of $\mathrm{Aut}_K(V)$.
   (b) Conclude that the preimage Γ_0 of $\mathrm{Aut}_R(\Lambda_0)$ in Γ under the representation $\rho : \Gamma \to \mathrm{Aut}_K(V)$ is of finite index in Γ.
   (c) Letting $\{\gamma_i\}$ be any finite set of coset representatives for Γ/Γ_0, show that the sum (taken inside V)
   $$\Sigma_i \rho(\gamma_i)\Lambda_0$$
   is a Γ-stable R-lattice in V.

2. Let K be a p-adic field and $W \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Define the dual of W by $W^* := \mathrm{Hom}_{\mathbf{C}_K-\mathrm{lin}}(W, \mathbf{C}_K)$ with G_K-action given by $g.\varphi(w) := g\varphi(g^{-1}w)$ (i.e. W^* as a $\mathbf{C}_K$-vector space is the usual $\mathbf{C}_K$-linear dual of W). Verify that indeed $W^* \in \mathrm{Rep}_{\mathbf{C}_K}(G_K)$ and that $W^{**} \simeq W$ in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$. Show that W^* is Hodge-Tate if and only if W is. Hint: you may want to use the "concrete" characterization of Hodge-Tate representations given in class.

3. It may be helpful to know a little Galois cohomology for this exercise. I recommend looking at Tate's article http://modular.math.washington.edu/Tables/Notes/tate-pcmi.html or Serre's book.

   Let $\eta : G_K \to \mathbf{Z}_p^\times$ be any continuous character. Fix an extension
   $$0 \longrightarrow \mathbf{C}_K(\eta) \longrightarrow W \longrightarrow \mathbf{C}_K \longrightarrow 0 \tag{1}$$
   in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$.
   (a) By choosing a $\mathbf{C}_K$-linear vector space splitting of this exact sequence, show that we may identify W with $\mathbf{C}_K(\eta) \oplus \mathbf{C}_K$ with $g \in G_K$-acting via
   $$g(v, \alpha) = (g.v + g\alpha \cdot \tau(g), g\alpha)$$
   where $\tau : G_K \to \mathbf{C}_K(\eta)$ is a function satisfying $\tau(hg) = \eta(g)\tau(h) + \tau(g)$, i.e. τ is a 1-cocycle.
   (b) Prove that τ is continuous, and that making a different choice of splitting alters τ by a coboundary.
   (c) Show that the association $W \rightsquigarrow \tau$ induces a bijection between isomorphism classes of extensions of $\mathbf{C}_K$ by $\mathbf{C}_K(\eta)$ and the set $H^1_{\mathrm{cont}}(G_K, \mathbf{C}_K(\eta))$. If you feel energetic, show that this is even an isomorphism of abelian groups, where we add two extensions by taking their Baer sum.
   (d) Deduce from the Ax-Sen-Tate theorem that if $\eta(I_K)$ is infinite, then (1) splits (as an extension in $\mathrm{Rep}_{\mathbf{C}_K}(G_K)$!) and that this splitting is *unique*.

4. Let K be a p-adic field and fix $q \in K$ with $|q| < 1$. Then $q^{\mathbf{Z}} := \{q^n \mid n \in \mathbf{Z}\}$ is a discrete subgroup (lattice) of $\overline{K}^\times$. Consider the quotient $E_q := \overline{K}^\times / q^{\mathbf{Z}}$; this abelian group admits a natural structure of G_K-module through the action on $\overline{K}^\times$. For each $r \geq 0$, let $E_q[p^r]$ be the subgroup of E_q consisting of p^r-torsion elements.
   (a) Let ζ be a primitive p^r-th root of unity and choose a p^r-th root ξ of q in $\overline{K}^\times$. Show that the natural map $i_{\zeta,q} : (\mathbf{Z}/p^r\mathbf{Z})^2 \to E_q[p^r]$ induced by
   $$(m, n) \mapsto \xi^n \zeta^m \in \overline{K}^\times$$
   is an isomorphism of abelian groups. What happens to $\iota_{\zeta,\xi}$ if we change our choices of ζ and ξ?
   (b) Define $T_p(E_q) := \varprojlim_r E_q[p^r]$ by using the natural multiplication by p maps $E_q[p^{r+1}] \to E_q p^r]$. Show that $T_p(E_q)$ is a free $\mathbf{Z}_p$-module of rank 2 and gives a continuous 2-dimensional representation $\rho_{E_q} : G_K \to \mathrm{GL}_2(\mathbf{Z}_p)$.
   (c) Set $V_p(E_q) := T_p(E_q) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Using (a), show that the natural maps $\mathbf{Z}/p^r\mathbf{Z} \to (\mathbf{Z}/p^r\mathbf{Z})^2$ and $(\mathbf{Z}/p^r\mathbf{Z})^2 \to \mathbf{Z}/p^r\mathbf{Z}$ given by $m \mapsto (m, 0)$ and $(m, n) \mapsto n$ realize $V_p(E_q)$ as an extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$, i.e. that we have a canonical exact sequence of continuous G_K-modules
   $$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow V_p(E_q) \longrightarrow \mathbf{Q}_p \longrightarrow 0\,. \tag{2}$$

(d) Prove that $V_p(E_q)$ is Hodge-Tate. Hint: Use Problem (3).

(e) Prove that (2) is non-split as an extension of representations of G_K, even if we extend scalars to $\overline{K}$.

5. Let K be a p-adic field with finite residue field $\mathbf{F}_q$. Pick $\alpha \in \mathrm{GL}_n(\mathbf{C}_K)$ and consider the uramified Galois representation defined by

$$G_K \twoheadrightarrow G_{\mathbf{F}_q} \simeq \widehat{\mathbf{Z}} \longrightarrow \mathrm{GL}_n(\mathbf{C}_K)$$

defined by sending $1 \in \widehat{\mathbf{Z}}$ to α. Show that this is a continuous representation if and only if all eigenvalues of the matrix α have absolute value 1. Use this to give an example of a continuous, n-dimensional G_K-representation with $\mathbf{C}_K$ coefficients that does not factor through $\mathrm{GL}_n(L)$ for any algebraic extension L/K.

6. Let K be a p-adic field containing μ_p and let $\chi : G_K \to \mathbf{Z}_p^\times$ be the cyclotomoc character.

   (a) Show that χ has image in $1 + p\mathbf{Z}_p$.

   (b) For any $s \in \mathbf{Z}_p$, show that the character χ^s of G_K defined by the composition of χ with the map $1+p\mathbf{Z}_p \to 1+p\mathbf{Z}_p$ given by $x \mapsto x^s$ makes sense and is continuous.

   (c) Prove that χ^s is Hodge-Tate if and only if $s \in \mathbf{Z}$.

7. Fix a p-adic field and let η be a nontrivial finite order continuous character $\eta : G_K \to \mathbf{Q}_p^\times$.

   (a) Show that η factors through the natural inclusion $\mathbf{Z}_p^\times \hookrightarrow \mathbf{Q}_p^\times$.

   (b) Prove that there are no nonzero G_K-homomorphisms $K \to K(\eta)$.

   (c) Suppose that L/K is finite Galois and the restriction of η to G_L is trivial. Show that there exists a nonzero homomorphism $L \to L(\eta)$ of L-modules with semilinear G_K-action, and hence that these two G_K-modules are isomorphic.

8. Fix a field E of characteristic p and let (M, φ_M) be an étale φ-module over E. Define $M^\vee$ to be the E-linear dual of M and let $\varphi_{M^\vee}$ be the map

$$M^\vee \longrightarrow (\varphi_E^*(M))^\vee \longrightarrow M^\vee \tag{3}$$

where the first map takes a linear functional ℓ on M to the linear functional on $\varphi_E^*(M) := M \otimes_{E,\varphi_E} E$ given by $m \otimes e \mapsto \varphi_E(\ell(m))e$, and the second map is the E-linear dual of the inverse of the E-linear isomorphism $\varphi_E^*(M) \to M$ given by the linearization of φ_M. Prove that $\varphi_{M^\vee}$ is semilinear over φ_E, and that its linearization is an isomorphism. Hint: show that the linearization of first map in (3) is the canonical isomorphism

$$\varphi_E^*(M^\vee) = \mathrm{Hom}_E(M, E) \otimes_{E,\varphi} E \simeq \mathrm{Hom}_E(M, E_\varphi) \simeq \mathrm{Hom}_{\varphi-\mathrm{sl}}(M, E) \simeq \mathrm{Hom}_E(\varphi_E^*(M), E) = \varphi_E^*(M)^\vee$$

where E_φ denotes E as an E-module via φ_E, and $\mathrm{Hom}_{\varphi-\mathrm{sl}}$ is the E-module of φ_E-semilinear E-module homomorphisms.

9. Let M be any étale φ-module over $\mathscr{O}_{\mathscr{E}}$. Show that $\mathbf{V}_{\mathscr{E}}(M)$ is continuous as a G_E-representation.

10. Let $E = \mathbf{F}_p$, so $G_E \simeq \widehat{\mathbf{Z}}$. Let $\rho : G_E \to \mathrm{Aut}_{\mathbf{F}_p}(V)$ be a continuous representation on a d-dimensional $\mathbf{F}_p$-vector space V, and let $(M, \varphi_M) = \mathbf{D}_E(V)$ be the associated étale φ-module over E. Since $E = \mathbf{F}_p$, we canonically have $\varphi^*(M) = M$ so that the linearlization φ_M^{lin} of φ_M is an $\mathbf{F}_p$-linear endomorphism of the d-dimensional $\mathbf{F}_p$-vector space M. Identifying G_E with $\widehat{Z}$ show that $\det(\rho(1))$ is the inverse of $\det(\varphi_M^{\mathrm{lin}})$.

11. Fix a pair $(\mathscr{O}_{\mathscr{E}}, \varphi)$ as in the notes and let (M, φ_M) be a φ-module over $\mathscr{O}_{\mathscr{E}}$; i.e. a finitely generated $\mathscr{O}_{\mathscr{E}}$-module with a φ-semilinear endomorphism $\varphi_M : M \to M$. Show that φ_M is étale if and only if $\varphi_M \bmod p$ is étale. Hint: first show that M and $\varphi^*(M)$ are abstractly isomorphic as $\mathscr{O}_E$-modules—i.e. that they have the same rank and invariant factors. Conclude that φ_M is an isomorphism if and only if it is surjective, and show that surjectivity may be checked modulo p.

12. Let M be a finitely generated module over a complete discrete valuation ring R of characteristic zero with uniformizer p. Suppose that G is a monoid acting on R by ring endomorphisms and on M by semilinear module endomorphisms. Show that for each n, G acts on M/p^nM and that $\varprojlim_n (M/p^n)^G = M^G$.

13. Prove that $\mathbf{V}_{\mathscr{E}}(\mathscr{E}/\mathscr{O}_{\mathscr{E}}) = \mathbf{Q}_p/\mathbf{Z}_p$.