Book problems §6.1:

7. Prove that subgroups and quotient groups of nilpotent groups are nilpotent (your proof should work for infinite groups). Give an example of a group G which possesses a normal subgroup H such that H and G/H are nilpotent but G is not nilpotent.

Solution: Let G be nilpotent, so that for some $n > 0$ we have $G^n = 1$ where $G^0 = G$ and $G^{i+1} = [G^i, G]$. Let $H \leq G$ be a subgroup of G. Then we claim that $H^i \leq G^i$ for all i. Indeed, $H \leq G$ by assumption and we may assume inductively that $H^i \leq G^i$. Then since

$$\{xyx^{-1}y^{-1} : x \in H^i, y \in H\} \subset \{xyx^{-1}y^{-1} : x \in G^i, y \in G\},$$

it follows that $H^{i+1} \leq G^{i+1}$ so that if $G^n = 1$ then also $H^n = 1$, i.e. H is nilpotent. Now let $\varphi : G \to K$ be any surjective group homomorphism. We assert that $\varphi(G^i) = K^i$ as sets. Indeed, suppose inductively that $\varphi(G^i) = K^i$. Observe that

$$\begin{aligned}\{xyx^{-1}y^{-1} : x \in K^i, y \in K\} &= \{xyx^{-1}y^{-1} : x\varphi(G^i), y \in \varphi(G)\}\\ &= \{\varphi(hgh^{-1}g^{-1}) : h \in G^i, g \in G\}\\ &= \varphi(G^{i+1})\end{aligned}$$

since φ is a homomorphism. Hence if $G^n = 1$ then $K^n = \varphi(G^n) = 1$. Therefore, any homomorphic image of G is nilpotent. If $H \lhd G$, we may take φ to be the natural map $G \to G/H$, so that G/H is nilpotent.

Let $G = \mathfrak{S}_3$ and $H = \langle(123)\rangle \simeq \mathbf{Z}/3$. Then clearly $H \lhd G$ and $G/H \simeq \mathbf{Z}/2$. Since $H, G/H$ are abelian, they are both nilpotent (the commutator subgroup of an abelian group is trivial). However, G has trivial center so that $Z_n(G) = 1$ for all n whence G is *not* nilpotent. Many groups G with this property may be constructed via the semidirect product. For example, let $K = \mathbf{Z}/(p^n)$ for any odd prime p and $n \geq 1$. Let $G = \mathrm{Aut}(K) \ltimes_\varphi K$ with $\varphi : \mathrm{Aut}(K) \to \mathrm{Aut}(K)$ an isomorphism. We see that $K, G/K$ are nilpotent (since they are both abelian) but since φ is an isomorphism, the center of G is trivial (indeed, if for any $g \neq 1$ we can find some $h \in \mathrm{Aut}(K)$ with $h(g) \neq g$, i.e. $hgh^{-1} \neq g$) so that $Z_n(G) = 1$ for all n, hence G is not nilpotent.

12. Find the upper and lower central series for $\mathfrak{A}_4$ and $\mathfrak{S}_4$.

Solution: Let $G = \mathfrak{S}_4$ and $H = \mathfrak{A}_4$. We showed on HW 3, §5.4, #4 that $G^1 = [G, G] = H$ and $H^1 = [H, H] = V_4$, the Klein four group. Now $[H, H] \leq [H, G] = G^2 \lhd G^1$. But we also showed on HW 3 that the only normal subgroups of H containing V_4 are V_4 and H. Observe, however, that $(23)(124)(23)(142) = (234)$, so that G^2 contains a 3-cycle, and hence properly contains V_4. It follows that $G^2 = H$ and hence that $G^n = H$ for all $n \geq 1$. Now observe that $(123)(12)(34)(132)(12)(34) = (13)(24)$ and $(132)(13)(24)(123)(13)(24) = (12)(34)$. Since $H^2 = [\mathfrak{A}_4, V_4]$ is a subgroup of V_4 containing *two* 2×2-cycles, it must be all of V_4, so that $H^n = V_4$ for all $n \geq 1$.

Now the center of $\mathfrak{S}_4$ is trivial since all elements of a given cycle type are conjugate (so if σ, τ have any given cycle type then there exists $g \in \mathfrak{S}_4$ with $g\sigma g^{-1} = \tau$; since there are at least two distinct elements of any given cycle type, σ cannot be in $Z(\mathfrak{S}_4)$). It follows that $Z_i(G) = \{1\}$ for all $i \geq 1$. Similarly, $Z(H) \lhd H$ and so $Z(H) = V_4$ or $\{1\}$ since H is not abelian. But we have seen that $[H, V_4] = V_4$ so that $Z(H) \neq V_4$; hence $Z(H) = \{1\}$ whence $Z_i(H) = \{1\}$ for all $i \geq 1$.

13. Find the upper and lower central series for $\mathfrak{S}_n$ and $\mathfrak{A}_n$ for $n \geq 5$.

Solution: Recall that $\mathfrak{A}_n$ is simple for $n \geq 5$ and that we showed on HW 3, §5.4, #5 that $[\mathfrak{S}_n, \mathfrak{S}_n] = \mathfrak{A}_n$, and since $[\mathfrak{A}_n, \mathfrak{A}_n] \lhd \mathfrak{A}_n$ is nontrivial ($\mathfrak{A}_n$ is not abelian) we have $[\mathfrak{A}_n, \mathfrak{A}_n] = \mathfrak{A}_n$. Since

$$\mathfrak{A}_n = [\mathfrak{S}_n, \mathfrak{S}_n] \supseteq [\mathfrak{S}_n, \mathfrak{A}_n] \supseteq [\mathfrak{A}_n, \mathfrak{A}_n] = \mathfrak{A}_n,$$

we have equality throughout and the lower central series for $\mathfrak{S}_n, \mathfrak{A}_n$ are

$$\mathfrak{S}_n^k = \mathfrak{A}_n$$
$$\mathfrak{A}_n^k = \mathfrak{A}_n$$

for all $k \geq 1$ and $n \geq 5$.

Certainly $Z(\mathfrak{S}_n) \cap \mathfrak{A}_n = Z(\mathfrak{A}_n) \lhd \mathfrak{A}_n$. It follows that $Z(\mathfrak{A}_n) = \{1\}$ and since the image of $Z(\mathfrak{S}_n)$ in $\mathfrak{S}_n/[\mathfrak{S}_n, \mathfrak{S}_n]$ is trivial, we see that $Z(\mathfrak{S}_n) \subseteq \mathfrak{A}_n$ (since $\mathfrak{A}_n$ is the kernel of the natural map $\mathfrak{S}_n \to \mathfrak{S}_n/[\mathfrak{S}_n, \mathfrak{S}_n]$). Therefore, $Z(\mathfrak{S}_n) = \{1\}$. Thus, the upper central series for $\mathfrak{S}_n$ and $\mathfrak{A}_n$ are

$$Z_k(\mathfrak{S}_n) = \{1\}$$
$$Z_k(\mathfrak{A}_n) = \{1\}$$

for all $k \geq 1$.

1. Compute the upper and lower central series for $G = U_3(\mathbf{R})$, the group of strictly upper triangular invertible matrices in $\mathrm{GL}_3(\mathbf{R})$.

Solution: Let

$$g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Then straightforward computation gives

$$ghg^{-1}h^{-1} = \begin{pmatrix} 1 & 0 & za - xc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that $g \in Z(G)$ if and only if for all x, z we have $za - xc = 0$, i.e., $a = c = 0$. It follows that $Z(G)$ consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which form an abelian group B isomorphic to $\mathbf{R}$ under addition. But it is also evident from these calculations that $[G, G] = B$ since every element of B is a commutator and B is already a group. Now since $G^1 = [G, G] = Z_1(G) = Z(G)$, we have $G^2 = [G, G^1] = [G, Z(G)] = \{1\}$ since every commutator is trivial. Finally, since $Z_1(G) = [G, G]$, we know that $G/Z_1(G)$ is abelian and hence $Z_2(G)/Z_1(G) = Z(G/Z_1(G)) = G/Z_1(G)$ so that $Z_2(G) = G$. Thus, the upper and lower central series of G are given, respectively, by

$$Z_0(G) = 1 \qquad Z_1(G) = B \qquad Z_2(G) = Z$$
$$G^0 = G \qquad G^1 = B \qquad G^2 = \{1\}.$$

The remaining exercises in this assignment develop some basic notions in representation theory. We fix throughout a group G and a field F. An (linear) *representation* of G on a vector space V over F is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Concretely, we're "representing" elements of G by endomorphisms of a vector space, but it must be stressed that ρ might *not* be injective. The case of infinite G (e.g., Lie groups) and infinite-dimensinal V (e.g., function spaces) are extremely fundamental, but intuition is developed by first understanding finite G and $\dim V < \infty$. We'll often refer to the data of (V, ρ) as a *representation space of* G. The most classical example is $G = \mathfrak{S}_n$ with its representation on F^n through permutations of the standard basis. Also, when $\dim V = 1$ then a representation of G on V is just a group homomorphism $\chi : G \to F^\times$.

2. We say that a representation space (V, ρ) is *irreducible* if $V \neq 0$ and there do not exist any G-stable subspaces aside from 0 and V.

(i) Prove that the natural representation of $\mathrm{GL}(V)$ on V is irreducible when $0 < \dim V < \infty$.

Solution: Suppose that we had a nontrivial $\mathrm{GL}(V)$ stable subspace $W \subset V$. Let $w \in W$ be any nonzero vector. Then since $\mathrm{GL}(V)$ acts transitively on the one dimensional subspaces of V, for any nonzero $v \in V$ there exists $g \in \mathrm{GL}(V)$ such that $gw = v$. It follows that $W = V$ and hence V is irreducible.

(*ii*) Consider the permutation representation of $G = \mathfrak{S}_3$ on F^3. If $3! = 6 \neq 0$ in F, show that the hyperplane $H : x_1 + x_2 + x_3 = 0$ and the line spanned by $(1,1,1)$ are complementary G-stable subspaces on which G acts irreducibly. If $2 = 0$ in F, show the same claims. If $3 = 0$ in F (e.g., $F = \mathbf{F}_3$), show that there is *no* line in F^3 complementary to H which is G-stable, and that the action of G on H is also *not* irreducible (find a G-stable line in H).

Solution: Clearly H is stable under the action of $\mathfrak{S}_3$ (if $x_1+x_2+x_3 = 0$ then for any $\sigma \in \mathfrak{S}_3$ we also have $x_{\sigma(1)}+x_{\sigma(2)}+x_{\sigma(3)} = x_1+x_2+x_3 = 0$). It is similarly obvious that the line spanned by $(1,1,1,)$ is $\mathfrak{S}_3$-stable. Since the line is one-dimensional, it must be irreducible. Suppose we have a $\mathfrak{S}_3$ stable line in H spanned by (x, y, z). Then it is not difficult to see that we must have $x = y = z$. Since our purported line is in H, we have $3x = 0$, and since $3 \neq 0 \in F$ we have $x = 0$ whence no such stable line exists and H is irreducible. To see that H and the line spanned by $(1,1,1)$ are complementary, observe that any $v = (x, y, z) \in F^3$ may be written in the form $v = (x - \alpha, y - \alpha, z - \alpha) + \alpha(1,1,1)$, where $\alpha = (x + y + z)/3$ and clearly $\alpha(1,1,1)$ is in our line and $(x - \alpha, y - \alpha, z - \alpha) \in H$. Here we are crucially using that $3 \neq 0$ in F. Thus, F^3 splits as $H \oplus L$ where L is spanned by $(1,1,1)$. Observe that this argument also works when $2 = 0 \in F$. The case $3 = 0 \in F$ is a little different. It follows from what we have said above that the line L spanned by $(1,1,1)$ and H are both $\mathfrak{S}_3$-stable (since our argument does not incorporate $\operatorname{char} F$ in any way). However, in this case, L is contained in H since $1 + 1 + 1 = 0$. Thus, H is not irreducible. Now suppose the span of (x, y, z) is another stable line. Then as before, we must have $x = y = z$ so that the *only* $\mathfrak{S}_3$-stable line in F^3 is L. Therefore, there is no line in F^3 complementary to H.

(*iii*) Changing the ground field can make a big difference. For example, consider the representation ρ of $\mathbf{Z}/3$ on $\mathbf{R}^2$ in which $1 \in \mathbf{Z}/3$ acts by the counterclockwise rotation through an angle of $2\pi/3$. Write down the matrix for this and prove this representation is irreducible. Over $\mathbf{C}^2$, the same matrix defines a representation of $\mathbf{Z}/3$ but use eigenvalues to explicitly determine two G-stable lines and prove these are *the only two* such lines in the representation space.

Solution: From linear algebra we know that the linear transformation of $\mathbf{R}^2$ corresponding to rotation about the origin through θ radians corresponds to the matrix (in the standard basis)

$$r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

so that we have

$$\rho(1) = r(2\pi/3) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

We easily compute the characteristic polynomial $c(x)$ of $\rho(1)$ and find $c(x) = x^2 + x + 1$ has no real roots. (Observe that $c(x) = (x^3 - 1)/(x - 1)$). If there were any stable lines in $\mathbf{R}^2$, then $\rho(1)$ would have a real eigenvalue, which is evidently not the case.

Since the $c(x)$ has distinct eigenvalues in C, it has 1-dimensional eigenspaces over $\mathbf{C}$ and these are the only stable lines (as a stable line is acted upon through a scaling action). Since the $\mathbf{C}^2$ is 2-diml, there are two such lines.

Direct computation shows that the lines spanned by

$$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \qquad\qquad v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

are stable. (Indeed, v_1, v_2 are eigenvectors of $\rho(1)$ with eigenvalues $e^{2\pi i/3}$, $e^{-2\pi i/3}$ respectively).

(*iv*) If (V, ρ) and (V', ρ') are irreducible, prove that any G-compatible linear map $T : V \to V'$ is either zero or an isomorphism (note that $\ker T$ and $\mathrm{im} T$ are G-stable!).

Solution: Let $T : V \to V'$ be any G-compatible linear map. We claim that $\ker T$ and $\operatorname{Im} T$ are G-stable. Indeed, let $v \in \ker T$. Then since T is G-compatible, we have $T(\rho(g)v) = \rho'(g)(Tv) = \rho'(g)(0) = 0$ so that $\rho(g)v \in \ker T$. Similarly, if $v' = Tv$ then $\rho'(g)v' = \rho'(g)Tv = T\rho(g)v \in \operatorname{Im} T$. Hence, $\operatorname{Im} T$ and $\ker T$ are G-stable subspaces of V, V' respectively. Therefore, if V is irreducible then $\ker T = \{0\}$ or V and T is injective or the zero map, while if V' is irreducible then $\operatorname{Im} T = V'$ or $\{0\}$ so that T is either surjective or the zero map. It follows that if V, V' are both irreducible then T is either an isomorphism or the zero map.

(v) If (V, ρ) is irreducible, $\dim V$ is finite, and F is *algebraically closed* (so characteristic polynomials make sense and have roots in F), show that the only G-compatible linear endomorphisms $T : V \to V$ (i.e., $T \circ \rho(g) = \rho(g) \circ T$ for all $g \in G$) are scalar multiplications. Hint: T has some nonzero eigenspace.

Solution: Let $W \subset T$ be any nonzero eigenspace of V with associated eigenvalue $\lambda \in F$ (here is where we use the fact that F is algebraically closed). Then W is the kernel of the linear map $(T - \lambda) : V \to V$. Since $W \neq \{0\}$, $(T - \lambda)$ is not an isomorphism. Since V is irreducible, we have by part (ii) that $T = \lambda$ on all of V; that is, T is scalar multiplication.

3. Suppose (V, ρ) is a finite-dimensional representation space and G is finite with $|G| \neq 0$ in F. If $W \subseteq V$ is a subspace which is stable under the G-action, prove as follows that there exists a complementary subspace W' which is also stable under the G-action (so (V, ρ) is build up as direct sum of W and W' *together with their G-actions*).

(i) Let $\pi : V \to W$ be an arbitrary linear map which restricts to the identity on W (i.e., $\pi(w) = w$ for all $w \in W \subseteq V$). Show that if π is G-equivariant in the sense that $\pi \circ \rho(g) = \rho(g) \circ \pi$ for all $g \in G$, then $\ker \pi$ is G-stable and provides a G-stable complement to W.

Solution: Since $\pi : V \to W$ is a linear, G-compatible map (this is what it means to be G-equivariant), $\ker \pi$ is a G-stable subspace of V by our solution to Problem 2 (iv). We claim that $V = W \oplus \ker \pi$. Indeed, let $v \in v$. Then $\pi(v) \in W$, and $v - \pi(v) \in \ker \pi$ since $\pi(v - \pi(v)) = \pi(v) - \pi^2(v) = 0$ since π is the identity on W. Thus $v = \pi(v) + (v - \pi(v))$, as required.

(ii) In general, we won't be so lucky, so just pick *any* linear $\pi : V \to W$ which lifts the identity on W (make this by using bases to make a complementary subspace away from which we're projecting). Now we average! Using that W is G-stabe and $|G| \neq 0$ in F, it *makes* sense to form the F-linear map

$$\pi' = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1}).$$

Using the fraction out front and that π lifts the identity on the G-stable W, prove that π' restricts to the identity on W and *is G-equivariant*. Deduce that $\ker \pi'$ provides the desired *G-stable* complement. This averaging trick is due to Maschke.

Solution: Let $\pi : V \to W$ be any linear map with $\pi|_W = \mathrm{id}$. Observe that for $w \in W$ we have

$$\begin{aligned}
\pi'(w) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1})(w) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(g)\rho(g^{-1})w
\end{aligned}$$

since W is G-stable and $\pi|_W = \mathrm{id}$,

$$\begin{aligned}
&= \frac{1}{|G|} \sum_{g \in G} w \\
&= w,
\end{aligned}$$

so that π' restricts to the identity on W. We now claim that π' is G-equivariant. Indeed, for any $h \in G$ we have

$$\begin{aligned}\pi' \circ \rho(h) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)\circ\pi\circ\rho(g^{-1})\rho(h)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho(h)\rho(h^{-1}g)\rho(g^{-1}h)\end{aligned}$$

since $\rho : G \to \mathrm{GL}(V)$ is a homomorphism,

$$= \rho(h) \circ \frac{1}{|G|}\sum_{g\in G}\rho(g)\circ\pi\circ\rho(g^{-1})$$

since left multiplication by $h^{-1} \in G$ is an automorphism of G,

$$= \rho(h) \circ \pi,$$

so that π' is G-equivariant. By part (i), we see that $\ker \pi'$ provides the desired G-stable complement of W.

(iii) Consider the permutation representation of $\mathfrak{S}_n$ on F^n, with W the hyperplane $\sum x_j = 0$ and $n! \neq 0$ in F. Find an $\mathfrak{S}_n$-stable complement to W.

Solution: Clearly the line L in F^n spanned by $v' = (1, 1, 1, \ldots, 1)$ is stable under the action of $\mathfrak{S}_n$. Moreover, since $n! \neq 0$ in F, for any $v = (x_1, x_2, \ldots, x_n) \in F^n$ we let $\alpha = (x_1 + x_2 + \cdots + x_n)/n$ and define $w = v - \alpha v'$. Then $w \in W$ and $v = w + \alpha v'$ so that $W \oplus L = F^n$ is a direct sum decomposition of F^n into $\mathfrak{S}_n$-stable subspaces. Since L is one dimensional, it is irreducible.

4. A very important representation of a finite group G is its *left regular representation.* Let $V = F[G] \stackrel{\text{def}}{=} \oplus F e_g$ be a vector space whose basis is indexed by the elements of G, and define $\rho_{\text{reg}} : G \to \mathrm{GL}(V)$ by $\rho_{\text{reg}}(g) : e_{g'} \mapsto e_{gg'}$. This "$F$-linearizes" the left multiplication action of G on itself as a set. It's a big space (think of $G = \mathfrak{S}_n$)!

(i) If (V, ρ) is a nonzero finite-dimensional F-linear representation of G and $|G| \neq 0$ in F, use induction on dimension and Exercise 3 to show that V is a direct sum of G-stable subspaces V_i on which G acts *irreducibly.* Thus, to describe all finite-dimensional representations of G up to isomorphism it is "enough" in such cases to describe the irreducible ones (of course, the real art is to actually locate the V_i's explicitly in interesting situations).

Solution: We induct on $\dim V$. For $\dim V = 1$, the assertion is obvious, so suppose that $\dim V > 1$. Let $W \subset V$ be a nonzero G-stable subspace of V of minimal dimension. Then certainly W is irreducible. If $W = V$ then we are done. Otherwise, using Problem 3 (ii), (which applies since $|G| \neq 0$ in F so that we can construct the projection π') we construct a G-stable subspace $V' \subset V$ with $W \oplus V' = V$. Since $W \neq \{0\}$, we must have $\dim W \geq 1$ and hence $\dim V' < \dim V$ so by induction $V' = V_2 \oplus \ldots \oplus V_n$ is a direct sum of irreducible G-stable subspaces. Thus, $V \simeq W \oplus V_2 \oplus \ldots \oplus V_n$ is a direct sum of irreducible G-stable subspaces.

(ii) Show that if (V, ρ) is a nonzero representation of G, then by choosing a nonzero $v_0 \in V$ the natural map $\pi_{v_0} : F[G] \to V$ defined by $e_g \mapsto \rho(g)(v_0)$ is a map of representation spaces (i.e., π_{v_0} is linear and commutes with the G-actions: $\pi_{v_0} \circ \rho_{\text{reg}}(g) = \rho(g) \circ \pi_{v_0}$ for all $g \in G$). Prove that the image of π_{v_0} is nonzero and G-stable, and conclude that if (V, ρ) is *irreducible* then π_{v_0} is *surjective* (in particular, $\dim V \leq |G|$ is finite!). Consequently, all irreducible G-representations are quotients of the left regular representation (over F).

Solution: That $\pi_{v_0} : F[G] \to V$ is linear follows immediately from its definition (we have defined it on a basis and extended linearly to all of $F[G]$). Moreover, we have

$$\pi_{v_0} \circ \rho_{\text{reg}}(g)(e_{g'}) = \pi_{v_0}(e_{gg'}) = \rho(gg')(v_0)$$

while

$$\rho(g) \circ \pi_{v_0}(e_{g'}) = \rho(g)(\rho(g')(v_0)) = \rho(gg')(v_0).$$

Since π_{v_0}, $\rho(g)$, and $\rho_{\mathrm{reg}}(g)$ are linear, and we have checked that $\pi_{v_0} \circ \rho_{\mathrm{reg}}(g) = \rho(g) \circ \pi_{v_0}$ on a basis of $F[G]$, it follows that this relation holds on all of $F[G]$. Now our proof of Problem 2 (*iv*) shows that $\mathrm{Im}\,\pi_{v_0}$ is G-stable. Moreover, $\mathrm{Im}\,\pi_{v_0}$ is nonzero since $v_0 \neq 0$ and $\rho(g) \in \mathrm{GL}(V)$. Therefore, if V is irreducible, π_{v_0} must be surjective. It follows that $\dim V \leq |G|$ for any irreducible representation V of G. Moreover, since π_{v_0} is surjective and G-equivariant, we have $F[G]/\ker \pi_{v_0} \simeq V$, where the isomorphism of vector spaces preserves the G action on each–that is, every irreducible representation of G is a quotient the left regular representation.

(*iii*) Prove that any finite-dimensional representation space (V, ρ) of G admits a rising chain of G-stable subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

such that each V_i a G-stable subspace and V_i/V_{i-1} (for $1 \leq i \leq n$) with its natural G-action is *irreducible*. We call this a *length n filtration* of V by G-stable subspaces. By a Jordan-Hölder style of argument, prove that any two such filtrations with irreducible successive quotients have the same length and give rise to the same collection of irreducible successive quotient representations (up to isomorphism), perhaps in different orderings.

Solution: We proceed by induction on $\dim V$. Suppose that for all $r < n$ and W of dimension r there exists a rising chain of G-stable subspaces with irreducible succesive quotients, and let $\dim V = n$. If V is irreducible, then we are done, so suppose that V is reducible. Let V_1 be any nonzero G-stable irreducible subspace of minimal dimension. Then $\dim V/V_1 < \dim V$ so we may apply our induction hypothesis to conclude that there exists a chain of G-stable spaces

$$\{0\} = W_1 \subseteq W_2 \subseteq \ldots \subseteq W_m = V/V_1$$

with irreducible successive quotients. Now let V_i be the preimage under the natural map $V \to V/V_1$ of W_i (since certainly V_1 is the preimage of $W_1 = \{0\}$, our notation is consistent). Observe that $V_i \supseteq V_1$ and that V_i must be G-stable since V_i/V_1, V_1 are both G-stable This gives a chain

$$\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_m = V$$

of G-stable spaces with successive quotients V_i/V_{i-1} which maps injectively to W_i/W_{i-1} via the map $V \to V/V_1$ (which is evidently G-compatible). Since W_i/W_{i-1} is irreducible by assumption, it follows from Problem 2 (*iv*) that either $V_i/V_{i-1} = \{0\}$ or $V_i/V_{i-1} \simeq W_i/W_{i-1}$ is irreducible. In the former case, we may delete the term V_{i-1} from the chain.

Notice that we have not used that $|G| \neq 0$ in F anywhere. Indeed, one might be tempted to use part (*i*) to say that $V \simeq \oplus_{i=1}^{n} W_i$ with each W_i G-stable and irreducible, and then define $V_k = \oplus_{i=1}^{k} W_i$. This argument only works given that we have such a direct sum decomposition to begin with, which ultimately depended on the existence of a G-equivariant projection $V \to W_i$. In general, it is false that every finite dimensional representation V decomposes as a direct sum of irreducible G-stable subspaces W_i. In other language, V is not in general a semisimple G-module.

We now show that any two such filtrations with irreducible successive quotients have the same length and give rise to the same collection of irreducible successive quotient representations. Let

$$\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_r = V \tag{1}$$

$$\{0\} = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_s = V \tag{2}$$

be any two such filtrations, and suppose that V_i/V_{i-1}, V_i/V_{i-1} are irreducible. We will proceed by induction on $\min(r, s)$. If $\min(r, s) = 1$ then V is irreducible, and we must have $r = s$ and both quotients are isomorphic to V. Therefore, suppose that the result holds for all r, s with $\min(r, s) < r_0$. Now let V admit two filtrations of the form (1), (2) with $r = r_0$ and $s = s_0$ where we assume without loss of generality that $r_0 \leq s_0$.

Observe that the natural maps $V_{r-1}/(V_{r-1} \cap W_{s-1}) \to V/W_{s-1}$ and $W_{s-1}/(V_{r-1} \cap W_{s-1}) \to V/V_{r-1}$ are injections (if an element of V_{r-1} dies in the quotient V/W_{s-1} then it must be in W_{s-1} and hence

$V_{r-1} \cap W_{s-1}$). Moreover, we may suppose that $V_{r-1} \cap W_{s-1}$ is a proper subspace of both V_{r-1} and W_{s-1}: if this were not the case then one of W_{s-1}, V_{r-1} would be contained in the other. Suppose, for example, that $V_{r-1} \subseteq W_{s-1}$. Then W_{s-1}/V_{r-1} is a G-stable subspace of V/V_{r-1}. Irreducibility of V/V_{r-1} forces $W_{s-1} = V_{r-1}$. We then have two filtrations of the common space $V_{r-1} = W_{s-1}$. Now V_{r-1} has a filtration of exact length $r - 1 < r = \min(r, s)$, so by our induction hypothesis ($r - 1 < r = \min(r, s)$) we know that *all* filtrations of $V_{r-1} = W_{s-1}$ (with irreducible successive quotients, as always in this discussion) have exact length $r - 1$ and give rise to the same collection of irreducible quotient representations. It follows that the two filtrations of V have the same length and give rise to the same collection of irreducible factors.

Therefore, we may assume that $V' = V_{r-1} \cap W_{s-1}$ is a proper subspace of W_{s-1} and V_{r-1}. Again, V/W_{s-1}, V/V_{r-1} are irreducible and V_{r-1}/V', W_{s-1}/V' are G-stable, so that the injections $V_{r-1}/V' \to V/W_{s-1}$ and $W_{s-1}/V' \to V/V_{r-1}$ are isomorphisms (since V' is a proper subspace of V, we see that $V_{r-1}/V' \neq \{0\}$ and similarly for W_{s-1}/V'). If $V' = \{0\}$ then we see that V_{r-1}, W_{s-1} are irreducible and $V/V_{r-1} \simeq W_{s-1}$ and $V/W_{s-1} \simeq V_{r-1}$ so that $r = s = 2$ and the pairs of successive quotients $\{V/V_1, V_1\}$ and $\{V/W_1, W_1\}$ agree up to isomorphism (more precisely, $W_1 \simeq V/V_1$, $V_1 \simeq V/W_1$). Thus, suppose $V' = W_{s-1} \cap V_{r-1}$ is nonzero.

As before, we use our inductive hypothesis and assume that all filtrations of V_{r-1} have the same length and give rise to the same collection of irreducible quotient representations (up to isomorphism). But V' is a nonzero G-stable proper subspace of V_{r-1} with $V_{r-1}/V' \simeq V/W_{s-1}$ irreducible. Thus, picking *any* filtration for V' (by (*iii*)) we get a filtration for V_{r-1}. We conclude that all filtrations of V' have exact length $r - 2$ and give rise to the same collection of irreducible quotient representations.

However, V' is a proper subspace of W_{s-1} with $W_{s-1}/V' \simeq V/V_{r-1}$ irreducible, so that W_{s-1} has a filtration of exact length $(r - 2) + 1 = r - 1$, namely the filtration passing through V', so again by induction, all filtrations of W_{s-1} have exact length $r - 1$ and give rise to the same collection of irreducible quotient representations Since we have been given a filtration of W_{s-1} of exact length $s - 1$, we conclude that $s - 1 = r - 1$ whence $r = s$ so that any two filtrations of V have the same length.

Now let

$$\{0\} = V'_0 \subseteq V'_1 \subseteq \ldots \subseteq V'_{r-3} \subseteq V'_{r-2} = V'$$

be any filtration of V'. We have seen that any other filtration has the same length and gives rise to the same isomorphism classes and multiplicities of irreducible quotient representations. It follows that the filtration (1) has successive quotient representations

$$R_1 = \{V'_{i+1}/V'_i \text{ for } 0 \leq i \leq r-3,\ V_{r-1}/V',\ V/V_{r-1}\},$$

while the filtration (2) has successive quotient representations

$$R_2 = \{V'_{i+1}/V'_i \text{ for } 0 \leq i \leq r-3,\ W_{r-1}/V',\ V/W_{r-1}\}.$$

However, we showed earlier that in this situation,

$$V/V_{r-1} \simeq W_{s-1}/V' \qquad \text{and} \qquad V/W_{r-1} \simeq V_{r-1}/V'.$$

It follows that the multi-sets R_1, R_2 contain the same list of irreducible quotient representations up to isomorphism. That is, any two filtrations of V have the same length and give rise to the same isomorphism classes and multiplicities of irreducible quotient representations. This completes the proof.

(*iv*) Using (*iii*), deduce that *up to isomorphism* there are only *finitely many* irreducible representations of G on F-vector spaces of finite dimension. The first real theorems in representation theory provide systematic ways to "explicitly" determine these irreducibles, and in real life one certainly wants to realize them in geometric ways (rather than just as "abstract" creatures) if at all possible. The effective version of this finiteness result makes it possible to determine the structure of various molecules and crystals, given enough knowledge about the symmetry. Ask your friends in physical chemistry about this.

Solution: By part (*ii*), every representation of G on a finite dimensional vector space V is a quotient of the left regular representation $F[G]$. Since $\dim F[G] = |G|$ is finite, by part (*iii*) we know that $F[G]$ admits a filtration

$$\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = F[G] \tag{3}$$

of G-stable subspaces with irreducible successive quotients, and that any two such filtrations have the same length and give rise to the same collection of quotient representations. Thus, every irreducible representation of G on a finite dimensional vector space V occurs as one of the successive quotients of (3) and hence there are, up to isomorphism, only finitely many finite dimensional irreducible representations of a finite group G.