Math 594. Solutions 3

Book problems §5.1:

14. Let $G = A_1 \times A_2 \times \cdots \times A_n$ and for each i let $B_i \lhd A_i$. Prove that $B_1 \times B_2 \times \cdots \times B_n \lhd G$ and that

$$(A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \simeq (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n).$$

Solution: Define

$$\phi : G \longrightarrow (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n)$$

by $(a_1, a_2, \ldots, a_n) \mapsto (a_1B_1, a_2B_2, \ldots, a_nB_n)$. Since for each i we have $B_i \lhd A_i$, it follows that ϕ is a homomorphism, and is obviously surjective. Moreover, it is clear that $\ker(\phi) = B_1 \times B_2 \times \ldots \times B_n$ so that $B_1 \times B_2 \times \ldots \times B_n \lhd G$ and we have

$$(A_1 \times A_2 \times \cdots \times A_n)/(B_1 \times B_2 \times \cdots \times B_n) \simeq (A_1/B_1) \times (A_2/B_2) \times \cdots \times (A_n/B_n),$$

as required.

Book problems §5.4:

4. Find the commutator subgroups of $\mathfrak{S}_4$ and $\mathfrak{A}_4$.

Solution: First we claim that the only normal subgroups of $\mathfrak{A}_4$ are $\mathfrak{A}_4, V_4$, and $\{1\}$, where V_4 is the klein four group. This follows easily from the fact that any normal subgroup must be a union of conjugacy classes, and the fact that $\mathfrak{A}_4$ has conjugacy classes of sizes $1, 4, 4, 3$ corresponding to the identity, two classes of 3-cycles, and the class of 2×2 cycles.

Let $G = \mathfrak{S}_4, H = \mathfrak{A}_4$ and let G' be the commutator subgroup of G. Since $H \lhd G$ and $G/H \simeq C_2$ is abelian, $G' \leqslant H$. Moreover, since $G' \lhd G$, we have trivially $G' \lhd H$. Now G' cannot be trivial since G is non-abelian. Thus $G' = H$ or V_4. Observe that $V_4 \lhd G$ (since it consists of all 2×2 cycles in G and elements of G are conjugate if and only if they have the same cycle type). Pick any copy of $\mathfrak{S}_3 \leqslant G$. Then $\mathfrak{S}_3 \cap V_4 = \{1\}$ so that (since $V_4 \lhd G$), we see that $\mathfrak{S}_3V_4$ is a subgroup of G of size 24 and therefore coincides with G. The Second Isomorphism Theorem gives

$$G/V_4 = \mathfrak{S}_3V_4/V_4 \simeq \mathfrak{S}_3/(V_4 \cap \mathfrak{S}_3) \simeq \mathfrak{S}_3$$

so that G/V_4 is not abelian. It follows that $G' = H$. On the other hand, $H/V_4 \simeq C_3$ (since, for example, it clearly has order 3 and there is only one group with this order) is abelian so that if H' is the commutator subgroup of H, we must have $H' \leqslant V_4$. Again, $H' \lhd H$, so that by our above remarks, if $H' \neq V_4$ then $H' = \{1\}$, which cannot be since H is non-abelian. Thus $H' = V_4$.

5. Prove that $\mathfrak{A}_n$ is the commutator subgroup of $\mathfrak{S}_n$ for all $n \geq 5$.

Solution: For $n \geq 5$, let $G = \mathfrak{S}_n, H = \mathfrak{A}_n$ and let G' be the commutator subgroup of G. Since $G/H \simeq C_2$ is abelian, $G' \leqslant H$. On the other hand, $G' \lhd G$ and so trivially $G' \lhd H$. But H is simple since $n \geq 5$ so that $G' = H$ or $\{1\}$. Since G is non-abelian, it follows that $G' = H$.

15. If A, B are normal subgroups of a group G with G/A and G/B abelian, prove that $G/(A \cap B)$ is abelian.

Solution: Let G' be the commutator subgroup of G. Recall that G/G' is the maximal abelian quotient of G in the sense that if $H \lhd G$, then G/H is abelian if and only if $G' \leqslant H$. Thus, since $G/A, G/B$ are abelian, we have $G' \leqslant A$ and $G' \leqslant B$. Thus, $G' \leqslant A \cap B$ so that $G/(A \cap B)$ is abelian.

Alternatively, define $\phi : G \to (G/A) \times (G/B)$ by $\phi(g) = (gA, gB)$. It is not difficult to see that ϕ is a group homomorphism. Moreover, $g \mapsto (A, B)$ if and only if $g \in (A \cap B)$, so that $\ker \phi = (A \cap B)$ and ϕ descends to an injective group homomorphism $\phi : G/(A \cap B) \to (G/A) \times (G/B)$ so that $G/(A \cap B)$ is isomorphic to a subgroup of $(G/A) \times (G/B)$. But $(G/A) \times (G/B)$ is abelian since G/A and G/B are, and any subgroup of an abelian group is abelian. Therefore, $G/(A \cap B)$ is abelian.

1. If H, K are normal subgroups of a group G with $H \cap K = \{1\}$ and $H.K = G$, show $hk = kh$ for all $k \in K$ and $h \in H$. Conclude that $H \times K \simeq G$.

Solution: Observe that for any $k \in K$ and $h \in H$, we have $khk^{-1}h^{-1} = (khk^{-1})h^{-1} = h'h^{-1} = k(hkh^{-1}) = kk'$ since K, H are normal in G. Thus, $khk^{-1}h^{-1} \in K \cap H$ so that $khk^{-1}h^{-1} = 1$ whence

$kh = hk$ for all $k \in K$ and $h \in H$. Let $\phi : H \times K \to HK$ be given by $(h,k) \mapsto hk$. We claim that ϕ is a homomorphism, where we consider $H \times K$ as a group with the usual component-wise structure. For we have

$$\phi((h_1,k_1)(h_2,k_2)) = \phi(h_1h_2, k_1k_2) = h_1h_2k_1k_2 = (h_1k_1)(h_2k_1) = \phi(h_1,k_1)\phi(h_2,k_2)$$

by the above, and trivially, $\phi(1,1) = 1$. It is obvious that ϕ is surjective. Moreover, if $\phi(h,k) = hk = 1$ then $h = k^{-1} = 1$ since $H \cap K = \{1\}$. Thus, ϕ is an isomoprphism and $G = HK \simeq H \times K$.

2. Let G be a finite abelian group, $g_0 \in G$ an element with *maximal* order, say n. Thus, $H = \langle g_0 \rangle$ is a maximal cyclic group of G. The group $\mu_n(\mathbf{C})$ of nth roots of unity in $\mathbf{C}$ is also cyclic of order n (it has generator $e^{\pm 2\pi i/n}$, for example). Pick an isomorphism between these groups, so we get an injective group homomorphism $\chi_0 : H \hookrightarrow \mathbf{C}^\times$. By HW2, this can be extended to a group homomorphism $\chi : G \to \mathbf{C}^\times$ since G is finite abelian.

(*i*) Let $K = \ker \chi$. Show that $\chi(G) = \mu_n(\mathbf{C}) = \chi_0(H)$, and deduce that the natural multiplication map $H \times K \to G$ is an isomorphism.

Solution: First, we claim that every element of G has order dividing n. Indeed, suppose that there exists $h \in G$ of order $m \nmid n$. Then by HW 1, Problem 7 (*ii*), since G is abelian, there would exist an element of order $\mathrm{lcm}(m,n) > n$, contradicting the maximality of n. It follows that $g^n = 1$ for all $g \in G$ and hence $\chi(g^n) = \chi(g)^n = 1$ so that $\chi(G) \subset \mu_n(\mathbf{C})$. On the other hand, we have already seen that $\chi_0(H) = \mu_n(\mathbf{C})$ (since g_0 has order n and χ_0 gives an isomorphism of $\langle g_0 \rangle$ with $\mu_n(\mathbf{C})$), so that $\chi(G) = \mu_n(\mathbf{C})$. Now we claim that $H \cap K = 1$. Indeed, if $h \in H \cap K$, then $\chi(h) = 1$ since $h \in K$, but χ induces an isomorphism $\chi_0 : H \to \mu_n(\mathbf{C})$, so $\chi(h) = \chi_0(h) = 1$ forces $h = 1$ since, in particular, χ_0 is injective. We conclude that the multiplication map $H \times K \to G$ (which is a group homomorphism since G is *abelian*) has trivial kernel (since if $hk = 1$ then $h = k^{-1} \in H \cap K$ must be trivial) and hence is injective. To prove surjectivity it suffices to count. More specifically, since $|G| = |K||G/K|$ and $|H \times K| = |H||K|$, it suffices to show $|G/K| = |H|$. But $|H| = n$, and χ_0 induces an isomorphism $G/K \to \mu_n(C)$, so $|G/K| = n = |H|$, as desired.

(*ii*) Using induction on $|G|$, deduce that every finite abelian group is a product of cyclic groups, and conclude that if a prime p divides $|G|$ then G contains an element of order p.

Solution: If $|G| = 1$ then G is trivially a direct product of cyclic groups. Now suppose that for all abelian K groups of order $|K| < n$ we have that K is a direct product of cyclic groups. Let $|G| = n > 1$. Then by part (*i*), we have $G \simeq H \times K$ where H is cyclic of order m, the maximal order of elements of G, and $K, H \leqslant G$. Clearly $m \neq 1$, since if this were the case then every element of G would have order 1, i.e. G would be trivial. It follows that $|K| < n$ so that by induction, K is a direct product of cyclic groups (since it is abelian). Thus, G is a direct product of cyclic groups.

Now suppose that $p||G|$ for some prime p. Then writing $G \simeq C_{n_1} \times \cdots \times C_{n_s}$ for cyclic groups C_{n_i}, we see that $p|n_{i_0}$ for some i_0 since $|G| = n_1n_2\cdots n_s$ and p is prime. But we saw on HW 1, Problem 5 (*i*) that if $p|n_{i_0}$ then $C_{n_{i_0}}$ has an element of order p, say g. Then

$$(1,\ldots,1,g,1,\ldots,1) \in C_{n_1} \times C_{n_1} \times \cdots \times C_{n_s} \simeq G$$

has order p, where g occurs in the i_0 component.

(*iii*) For relatively prime nonzero integers a and b, show that the natural map $\phi : \mathbf{Z}/ab \to \mathbf{Z}/a \times \mathbf{Z}/b$ (defined by $r \bmod ab \mapsto (r \bmod a, r \bmod b)$ is a well-defined group homomorphism which is injective, and by counting deduce it is an isomorphism. Can you describe an inverse easily?

Solution: Suppose that $r \equiv s \bmod ab$. Then $ab|(r-s)$ so certainly $a|(r-s)$ and $b|(r-s)$ so that $r \equiv s \bmod a$ and $r \equiv s \bmod b$, i.e. $\phi(r) = \phi(s)$, whence ϕ is well defined. Observe that $\phi(xy \bmod ab) = (xy \bmod a, xy \bmod b)$. But $(x \bmod a)(y \bmod a) \equiv xy \bmod a$ as is easily deduced from $(x + an_1)(y + an_2) = xy + a(yn_1 + xn_2 + an_1n_2)$, so that

$$\phi(xy) = ((x \bmod a)(y \bmod a), (x \bmod b)(y \bmod b)) = (x \bmod a, x \bmod b)(y \bmod a, y \bmod b) = \phi(x)\phi(y)$$

and ϕ is a group homomorphism. Now suppose that $x \equiv 0 \bmod a$ and $x \equiv 0 \bmod b$. Then $a|x$ and $b|x$, but since $(a,b) = 1$ we have $ab|x$ so that $x \equiv 0 \bmod ab$ and ϕ is injective. Again, since $(a,b) = 1$, there exist integers r, s with $ar + bs = 1$ so that with $x = arv + bsu$ we have $\phi(x) = (u,v)$ whence ϕ is surjective. It follows that ϕ is an isomorphism. An inverse map is $(u, v) \mapsto arv + bsu \bmod ab$, as we have just seen (where r, s are integers chosen so that $ar + bs = 1$).

(iv) Using (iii), show that every finite abelian group G can be written in the form $\phi : G \simeq C_{n_1} \times \cdots \times C_{n_r}$ with cyclic groups C_{n_j} of order n_j where $n_1|n_2|\dots|n_r$. Prove that r and the parameters $n_1, \dots, n_r$ are uniquely determined by G. These are called the *invariant* factors of G. Use this to make a list of *all* finite abelian groups (up to isomorphism) of order p^3q^2 for distinct primes p and q. Make sure your list has *no* repetitions.

Solution: We first observe that if C is a cyclic group of order n, then for any $d|n$ there are exactly d elements in C of order dividing d (we have an isomorphism $C \simeq \mathbf{Z}/n\mathbf{Z}$, and the elements of $\mathbf{Z}/n\mathbf{Z}$ of order dividing d are precisely $0, n/d, 2n/d, 3n/d, \dots, (d-1)n/d$). Since the group $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r}$ with $1 < n_1|n_2|\dots|n_r$ has n_1^r elements of order dividing n_1 the only way such a group could be cyclic is if $n_1^r = n_1$, which is to say $r = 1$ (since $n_1 > 1$). Thus, when G is cyclic of order n then its only decomposition as a product of C_{n_j}'s satisfying the given hypotheses is the case $r = 1$ and $n_1 = |G|$. This settles existence and uniqueness for the cyclic case.

More generally, we induct on $|G|$ for proving existence and uniqueness (the case $|G| = 1$ is trivial). First we prove existence of the desired decomposition by induction. Suppose that N is the maximal order of an element of G. We can assume G is non-trivial, so $N > 1$. By (i), we have an isomorphism $G \simeq C_N \times H$ and (as is shown in the proof of (i)) all elements of G have order dividing N. This latter property is inherited by H. If H is trivial then G is cyclic, a case for which existence and uniqueness have been settled. Thus, we can assume H is nontrivial. Since $|H| = |G|/N < |G|$, by induction we conclude that $H = C_{n_1} \times \cdots \times C_{n_s}$ with $1 < n_1|n_2|\dots|n_s$. But H contains an element of order n_s (coming from the final factor group of H), so $n_s|N$. This shows the existence aspect for G in general.

For uniqueness, we must prove that if

$$G \simeq C_{n_1} \times \ldots \times C_{n_r} \simeq C_{m_1} \times \ldots \times C_{m_s}$$

with $1 < n_1|n_2|\dots|n_r$ and $1 < m_1|\dots|m_s$ then $r = s$ and $n_j = m_j$ for all j. We argue by induction on $|G|$. If a prime p divides n_{i_0} but not n_{i_0-1}, we have seen above that $C_{n_1} \times \ldots \times C_{n_r}$ has p^{r-i_0+1} elements of order dividing p (there are p elements of order dividing p for each factor C_{n_i} with $i \geq i_0$ in the direct product). Without loss of generality, suppose that $r \geq s$ and let p be any prime dividing m_1. Then since $G \simeq C_{m_1} \times \ldots \times C_{m_s}$, we see that G has exactly p^s elements of order dividing p. Now clearly, $p \nmid n_i$ for any i is absurd, since $p||G|$ and $|G| = n_1 n_2 \cdots n_r$, so suppose that $p|n_{i_0}$. Then by the above, we have $p^{r-i_0+1} = p^s$, from which it follows (since $p > 1$) that $r - i_0 + 1 = s$, and since we have assumed that $r \leq s$, we have $r = s$ and $i_0 = 1$, i.e. $p|n_1$. Now since $n_1|n_i$ for all i, we see that $p|n_i$ for all i, and we already know that $p|m_j$ for all j.

It remains to show $n_j = m_j$ for $j \geq 1$, or equivalently that $n_j/p = m_j/p$ for $j \geq 1$. Let

$$H = \{g \in G | g^p = 1\}.$$

Then by inspection

$$G/H \simeq C_{n_1/p} \times \cdots \times C_{n_r/p}$$

and

$$G/H \simeq C_{m_1/p} \times \cdots \times C_{m_r/p}$$

by using the fact that $C_{pt}/\{g \in C_{pt} | g^p = 1\}$ is isomorphic to C_t (just use the map $x \mapsto x^p$; it is not hard to see this is surjective and has kernel $\{g \in C_{pt} | g^p = 1\}$). Using the inductive hypothesis for the group G/H of order $|G|/|H| < |G|$, we get $n_j/p = m_j/p$ for all $j \geq 1$, as desired.

If G is abelian of order p^3q^2 then the possible distinct lists of invariant factors of G are

$$\{p, p, pq^2\};\ \{p, pq, pq\};\ \{pq, p^2q\};\ \{p, p^2q^2\};\ \{q, p^3q\};\ \{p^3q^2\}$$

as is easily checked. Thus, the possibilities for G are:

Invariant factors	G
p, p, pq^2	$C_p \times C_p \times C_{pq^2}$
p, pq, pq	$C_p \times C_{pq} \times C_{pq}$
pq, p^2q	$C_{pq} \times C_{p^2q}$
p, p^2q^2	$C_p \times C_{p^2q^2}$
q, p^3q	$C_q \times C_{p^3q}$
p^3q^2	$C_{p^3q^2}$

3. Let F be a field and $G = \mathrm{GL}_n(F)$ for a positive integer n. Let B_n denote the subgroup of upper triangular matrices, U_n the subgroup of "strictly" upper triangular matrices (i.e., all 1's down the diagonal), and T_n the subgroup of diagonal matrices.

(i) Construct a natural group homomorphism from B_n onto T_n with kernel U_n, and use this to show that U_n is normal in B_n with $B_n = T_n.U_n$.

Solution: First observe that the determinant of any upper triangular matrix is the product of the diagonal entries. Thus, any invertible upper triangular matrix has all nonzero entries along the diagonal. Therefore, we define $\phi : B_n \to T_n$ by sending an upper triangular matrix B to the diagonal matrix T whose diagonal is exactly the diagonal of B. It is not difficult to see that ϕ is a surjective homomorphism, since $T_n \leqslant B_n$ and $\phi|_{T_n} = \mathrm{id}_{T_n}$, and

$$\begin{pmatrix} a_1 & \cdots & & * \\ & a_2 & & \vdots \\ & & \ddots & \\ & & & a_n \end{pmatrix} \begin{pmatrix} b_1 & \cdots & & * \\ & b_2 & & \vdots \\ & & \ddots & \\ & & & b_n \end{pmatrix} = \begin{pmatrix} a_1b_1 & \cdots & & * \\ & a_2b_2 & & \vdots \\ & & \ddots & \\ & & & a_nb_n \end{pmatrix}.$$

Moreover, $\ker \phi = U_n$ since a matrix $B \in B_n$ maps to the identity if and only if it has 1's along the diagonal, e.g. if and only if $B \in U_n$. It follows that $U_n \lhd B_n$ as it is the kernel of a homomorphism. Observe that $T_n \cap U_n = \{1\}$ since the only diagonal matrix with 1's along the diagonal is the identity. Since ϕ is a surjective homomorphism with kernel U_n, we have $B_n/U_n \simeq T_n$. On the other hand, $T_nU_n/U_n \simeq T_n$ by the Second Isomorphism Theorem, and certainly $T_nU_n \leqslant B_n$. This tells us that the natural map $T_nU_n/U_n \to B_n/U_n$ is an isomorphism, from which it follows that $T_nU_n = B_n$.

(ii) We must have $B_n = T_n \ltimes U_n$ using the conjugation action $u \mapsto tut^{-1}$ of $t \in T_n$ on U_n. Describe this action explicitly in the cases $n = 2, 3$, and then establish a general formula.

Solution: Let

$$t = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, \; u = \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ & 1 & x_{23} & \cdots & x_{2n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{(n-1)n} \\ & & & & 1 \end{pmatrix}.$$

Then by straightforward matrix multiplication, we have

$$tut^{-1} = \begin{pmatrix} 1 & a_1x_{12}a_2^{-1} & a_1x_{13}a_3^{-1} & \cdots & a_1x_{1n}a_n^{-1} \\ & 1 & a_2x_{23}a_3^{-1} & \cdots & a_2x_{2n}a_n^{-1} \\ & & \ddots & & \vdots \\ & & & 1 & a_{n-1}x_{(n-1)n}a_n^{-1} \\ & & & & 1 \end{pmatrix},$$

that is, the entry in the (i, j) position of tut^{-1} is

$$\begin{cases} 1 & \text{if } i = j \\ a_ix_{ij}a_j^{-1} & \text{if } j > i \\ 0 & \text{otherwise} \end{cases}.$$

(*iii*) Show that elements of U_n have the form $1 + M$ where M is an upper triangular matrix having 0's on the diagonal. By looking at the nonzero entry of M nearest to the diagonal and lower right corner, show that if $a \neq 0$ in F for all nonzero integers a (e.g., $F = \mathbf{R}$ but not $F = \mathbf{F}_p$), then U contains no non-trivial elements of finite order. Try $n = 2, 3$ first to see what is happening.

Solution: Since every element $u \in U_n$ is an upper triangular matrix with 1's along the diagonal, it is clear that $u - 1$ is an upper triangular matrix with 0's along the diagonal. Now suppose that $u \neq 1$ and with $x = u - 1$ let x_{ij} be the (i, j) entry of x. Let (i_0, j_0) be such that $x_{ij} = 0$ if $i > i_0$ and $x_{i_0j} = 0$ if $j < j_0$ (i.e. $x_{i_0j_0}$ is the nonzero entry of x in the lower right corner, nearest the diagonal. Then we have

$$(x^2)_{i_0j_0} = \sum_l x_{i_0l}x_{lj_0} = \sum_{l\leq i_0} x_{i_0l}x_{lj_0} = \sum_{j_0\leq l\leq i_0} x_{i_0l}x_{lj_0}$$

since all other terms in the sum are zero. But $i_0 < j_0$ since $x_{i_0j} = 0$ for all $j \leq i_0$, so that the entire sum is 0. It follows from the Binomial Theorem (since 1 and u commute) that the i_0, j_0 entry of $(1 + u)^n = (1 + nu + \binom{n}{2}u^2 + \cdots + u^n)$ is $nx_{i_0j_0} \neq 0$ since $\mathbf{Z} \hookrightarrow F$ by assumption. Thus, every nontrivial $u \in U_n$ has infinite order.

(*iv*) In contrast, when $F = \mathbf{F}_p$, show that U_n is a finite p-group (hint: $u \in U_n$ can be written $u = 1 + M$ where 1 denotes the $n \times n$ identity matrix and M has 0's on and below the main diagonal. Deduce $M^n = 0$, but $(1 + M)^p = 1 + M^p$ when working over $\mathbf{F}_p$ because ...).

Remark: In 3(*iv*), you are implicitly going to use that $(1 + M)^r$ can be expanded as in the binomial formula. This is valid because 1 and M commute with each other!

Solution: Since M is upper triangular with zeroes along the diagonal, we have $M_{ij} = 0$ for $j < i + 1$. It is not difficult to see that $M^r_{ij} = 0$ for $j < i+r$. Explicitly, proceed by induction and suppose that $M^{r-1}_{ij} = 0$ for $j < i + r - 1$. Then we have $M^r_{ij} = \sum_l M^{r-1}_{il} M_{lj} = \sum_{i+r-1<l} M^{r-1}_{lj} M_{il}$, but $M_{il} = 0$ for $l \geq j$ so that $M^r_{ij} = \sum_{i+r-1<l<j} M^{r-1}_{lj} M_{il}$. If $j < i + r$ then this sum contains no terms and is therefore 0. It follows that $M^n_{ij} = 0$ for all i, j so $M^n = 0$ and obviously $M^r = 0$ for $r > n$.

Now since $p|\binom{p}{k}$ for $1 < k < p$, we see that $(1 + M)^p = 1 + M^p$ when F has characteristic p (e.g. $F = \mathbf{F}_p$) since $1, M$ commute. Then by induction, one has $(1 + M)^{p^r} = 1 + M^{p^r} = 1$ whenever $p^r \geq n$ so that every element of U_n has order p^r for some r when F has characteristic p. Clearly $\mathrm{GL}_n(\mathbf{F}_p)$ is finite so that U_n is a finite group. One way to see that U_n is a p-group is to observe that any $n \times n$ matrix with entries in $\mathbf{F}_p$ and 1's along the diagonal is invertible. Thus, the entries for the $n(n - 1)/2$ positions above the diagonal can be any elements of $\mathbf{F}_p$, so there are $p^{n(n-1)/2}$ elements in U_n.