Math 594. Finite nilpotent groups

Here is a proof of a result whose proof in the text depends on an exercise which was not assigned.

Theorem 0.1. *Let G be a finite group. Then G is nilpotent if and only if G is a product of its Sylow subgroups* (*i.e., G is a product of p-groups for various primes p*).

Proof. We argue (surprise, surprise) by induction on $|G|$. The case $|G| = 1$ is trivial, so assume $|G| > 1$.

By virtue of nilpotence, the upper central series eventually reaches the (non-trivial) top, G, so certainly the first step $Z = Z(G)$ is non-trivial. Hence, the quotient G/Z is nilpotent (by theorem from class) and of smaller size (!). Thus, all of its Sylow subgroups are normal. By the lemma below (a basic fact which seems to not be proven in the book), the image of a p-Sylow of G is a p-Sylow in G/Z. Hence, for every p-Sylow P of G, the image ZP/Z is a p-Sylow of G/Z. But this is normal from the inductive hypothesis, so ZP is normal in G. Aha, but p-all Sylows of G are conjugate, so by normality of ZP, they're all in ZP. But ZP contains P, so a p-Sylow of ZP has same size as one of G, so the p-Sylows of G are the same thing as the p-Sylows of ZP. But all p-Sylows of ZP are conjugate *within* ZP, and P is one of them. Since P is trivially normal in ZP (as Z commutes with everything), we conclude that P is the unique p-Sylow. In other words, Sylows in G are unique (for each prime).

If $\{P_1, ..., P_n\}$ is the list of nontrivial Sylows of G, then we've just shown these are normal subgroups. Hence, $P_i.P_j = P_j.P_i$ for all i and j. Since P_i meets P_j trivially for $i \neq j$ (as these have relatively prime orders), by an old homework problem these groups pairwise commute with each other, so there is a well-defined *group homomorphism*

$$P_1 \times \cdots \times P_n \to G$$

which is surjective (any finite group is generated by one of each type of Sylow; why?), hence an isomorphism by counting. ∎

It remains to prove the lemma alluded to above:

Lemma 0.2. *If G is a finite group and H is a normal subgroup, then for any prime p and p-Sylow P of G, the image HP/H of P in G/H is a p-Sylow of G/H.*

Proof. The image HP/H is a p-group in G/H, so lies in a p-Sylow Q of G/H. Now we have $Q = G'/H$ for a subgroup G' of G containing HP, so we may replace G with G' without affecting P or Q (and hence not affecting power of p in the order of the ambient group and its quotient by H). Hence, we can assume $G/H = Q$ (a p-group), and we want P to surject onto this. That is, we claim in general that if we have a quotient situation $G \twoheadrightarrow Q$ with Q a p-group, then every p-Sylow of G surjects onto Q. We shall prove this by induction on $|Q|$, and by conjugacy of Sylows it is enough to prove *some* Sylow of G surjects onto Q (as then all of its conjugates certainly do).

When $|Q| = 1$ this is trivial. In general, from structure of p-groups, there exists an index p normal subgroup Q' in Q, and this has the form G'/H for a normal subgroup G' in G containing H. Since $G/G' = Q/Q'$ is order p, G' has index p in G. By induction, since $|Q'| < |Q|$, there exists a p-Sylow P' of G' surjecting onto Q'. Since G' has index p in G, P' is not a p-Sylow of G, so lies strictly inside of a p-Sylow P, and P' must be normal in P (since by the homework, a proper subgroup of a p-group has strictly bigger normalizer, and the only subgroup of P strictly containing P' is P). Consequently, P/P' makes sense and $P/P' \to G/G'$ is a nontrivial map between groups of size p (recall P can't be in G', since P' is already a p-Sylow of G'), so this is an isomorphism. But $G/G' \simeq Q/Q'$ via the natural map, so the upshot is that the composite map $P \to G \twoheadrightarrow Q$ induces a map $P/P' \to Q/Q'$ which is an isomorphism. Since P' surjects onto Q', it follows that P surjects onto Q, as desired. ∎