Math 594. Homework 8 (due March 19)

1. Let L_1 and L_2 be two finite extensions of k inside of an extension L/k.

(*i*) Prove $[L_1L_2 : k] \leq [L_1 : k][L_2 : k]$.

(*ii*) Assume that $[L_1 : k]$ and $[L_2 : k]$ are relatively prime. Show that $[L_1L_2 : k] = [L_1 : k][L_2 : k]$. Is the converse true? If we instead assume $L_1 \cap L_2 = k$, then does the equality necessarily hold?

(*iii*) As an application of (*ii*), if $\alpha \in L$ is algebraic over k with $[k(\alpha) : k]$ relatively prime to $[L_1 : k]$, prove that $[L_1(\alpha) : L_1] = [k(\alpha) : k]$, and conclude that minimal polynomial for α over k is irreducible over L_1 and hence serves as the minimal polynomial for α over L_1.

$*$ 2. (*i*) Let L_1 and L_2 be two quadratic extensions of k. Assume k does not have characteristic 2, so $L_i = k(a_i)$ with $a_i^2 = b_i$ a non-square in $k^\times$. Show that $L_1 \simeq L_2$ as extensions of k if and only if b_1/b_2 is a square in $k^\times$. Use this to give a complete list (without repetitions) of all quadratic extensions of $\mathbf{Q}$, up to isomorphism.

(*ii*) Consider the identification ι between isomorphism classes of quadratic extensions of k and elements of the *group* $k^\times/{k^\times}^2$, as explained in (*i*). If L_1 and L_2 are two quadratic distinct extensions of k inside of an extension L/k, show that the composite L_1L_2 is a degree 4 extension of k and the non-trivial subextensions over k are L_1, L_2, and the field corresponding to the 'product' of L_1 and L_2 under ι (i.e., $k(\sqrt{b_1b_2})$, if $L_i = k(\sqrt{b_i})$).

3. Let $f \in k[T, X]$, with f not divisible by any non-constants in $k[T]$ or $k[X]$. Show that f is irreducible when viewed in $k(X)[T]$ if and only if it is irreducible when viewed in $k(T)[X]$.

4. For each of the following extensions L/k, determine $[L : k]$ and find a basis for L as a k-vector space:

$k = \mathbf{Q}$, $L = \mathbf{Q}(a, b)$ with $a^2 = 6$, $b^3 = 2$

$k = \mathbf{C}(T)$, L is the splitting field of $X^n - T$ over k

$k = \mathbf{F}_p(T)$, L is the splitting field of $X^p - T$ over k, with p a prime (same p in both places!).

5. Let L/K be a field extension, and $\alpha \in K$ be algebraic over L. Consider the multiplication map $m_\alpha : K(\alpha) \to K(\alpha)$ on the finite-dimensional K-vector space $K(\alpha) = K[\alpha]$. Using a matrix for this relative to a suitable basis, prove that the characteristic polynomial of this linear map is the minimal polynomial of α over K. In terms of this minimal polynomial, what are the trace and determinant of this map?

6. Prove that $f(X) = X^3 + 3X + 1$ is irreducible over $\mathbf{Q}$. If we let α denote a root of f in some extension, use the fact that f vanishes at α and $f(X - 1)$ vanishes at $\alpha + 1$ to express $1/\alpha$ and $1/(\alpha + 1)$ as quadratic polynomials in α with $\mathbf{Q}$ coefficients.

7. Prove that $X^4 - 5X^2 + 6$ and $X^4 + 5X^2 + 6$ are reducible over $\mathbf{Q}$ with splitting fields of degree 4 which you should describe concretely (give a basis and express in the form $\mathbf{Q}(\alpha)$ for suitable α).

Prove also that $X^4 - 5$ is irreducible over $\mathbf{Q}$ but with splitting field of degree 8 over $\mathbf{Q}$ which you should describe in terms of some field generators and a basis (don't try to verify a candidate for primitive generator; this is painful without Galois theory).