# Math 594. Homework 8 (due March 19)

1. Let $L_1$ and $L_2$ be two finite extensions of $k$ inside of an extension $L/k$.

(*i*) Prove $[L_1L_2 : k] \leq [L_1 : k][L_2 : k]$.

(*ii*) Assume that $[L_1 : k]$ and $[L_2 : k]$ are relatively prime. Show that $[L_1L_2 : k] = [L_1 : k][L_2 : k]$. Is the converse true? If we instead assume $L_1 \cap L_2 = k$, then does the equality necessarily hold?

(*iii*) As an application of (*ii*), if $\alpha \in L$ is algebraic over $k$ with $[k(\alpha) : k]$ relatively prime to $[L_1 : k]$, prove that $[L_1(\alpha) : L_1] = [k(\alpha) : k]$, and conclude that minimal polynomial for $\alpha$ over $k$ is irreducible over $L_1$ and hence serves as the minimal polynomial for $\alpha$ over $L_1$.

* 2. (*i*) Let $L_1$ and $L_2$ be two quadratic extensions of $k$. Assume $k$ does not have characteristic 2, so $L_i = k(a_i)$ with $a_i^2 = b_i$ a non-square in $k^\times$. Show that $L_1 \simeq L_2$ as extensions of $k$ if and only if $b_1/b_2$ is a square in $k^\times$. Use this to give a complete list (without repetitions) of all quadratic extensions of $\mathbf{Q}$, up to isomorphism.

(*ii*) Consider the identification $\iota$ between isomorphism classes of quadratic extensions of $k$ and elements of the *group* $k^\times/{k^\times}^2$, as explained in (*i*). If $L_1$ and $L_2$ are two quadratic distinct extensions of $k$ inside of an extension $L/k$, show that the composite $L_1L_2$ is a degree 4 extension of $k$ and the non-trivial subextensions over $k$ are $L_1$, $L_2$, and the field corresponding to the 'product' of $L_1$ and $L_2$ under $\iota$ (i.e., $k(\sqrt{b_1b_2})$, if $L_i = k(\sqrt{b_i})$).

3. Let $f \in k[T, X]$, with $f$ not divisible by any non-constants in $k[T]$ or $k[X]$. Show that $f$ is irreducible when viewed in $k(X)[T]$ if and only if it is irreducible when viewed in $k(T)[X]$.

4. For each of the following extensions $L/k$, determine $[L : k]$ and find a basis for $L$ as a $k$-vector space:

$k = \mathbf{Q}$, $L = \mathbf{Q}(a, b)$ with $a^2 = 6$, $b^3 = 2$

$k = \mathbf{C}(T)$, $L$ is the splitting field of $X^n - T$ over $k$

$k = \mathbf{F}_p(T)$, $L$ is the splitting field of $X^p - T$ over $k$, with $p$ a prime (same $p$ in both places!).

5. Let $L/K$ be a field extension, and $\alpha \in K$ be algebraic over $L$. Consider the multiplication map $m_\alpha : K(\alpha) \to K(\alpha)$ on the finite-dimensional $K$-vector space $K(\alpha) = K[\alpha]$. Using a matrix for this relative to a suitable basis, prove that the characteristic polynomial of this linear map is the minimal polynomial of $\alpha$ over $K$. In terms of this minimal polynomial, what are the trace and determinant of this map?

6. Prove that $f(X) = X^3 + 3X + 1$ is irreducible over $\mathbf{Q}$. If we let $\alpha$ denote a root of $f$ in some extension, use the fact that $f$ vanishes at $\alpha$ and $f(X - 1)$ vanishes at $\alpha + 1$ to express $1/\alpha$ and $1/(\alpha + 1)$ as quadratic polynomials in $\alpha$ with $\mathbf{Q}$ coefficients.

7. Prove that $X^4 - 5X^2 + 6$ and $X^4 + 5X^2 + 6$ are reducible over $\mathbf{Q}$ with splitting fields of degree 4 which you should describe concretely (give a basis and express in the form $\mathbf{Q}(\alpha)$ for suitable $\alpha$).

Prove also that $X^4 - 5$ is irreducible over $\mathbf{Q}$ but with splitting field of degree 8 over $\mathbf{Q}$ which you should describe in terms of some field generators and a basis (don't try to verify a candidate for primitive generator; this is painful without Galois theory).