Math 594. Homework 5 (due February 12)

Book problems. §6.1: 7, 12, 13.

1. Compute the upper and lower central series for $G = U_3(\mathbf{R})$, the group of strictly upper triangular invertible matrices in $\mathrm{GL}_3(\mathbf{R})$.

The remaining exercises in this assignment develop some basic notions in representation theory. We fix throughout a group G and a field F. An (linear) *representation* of G on a vector space V over F is a homomorphism $\rho : G \to \mathrm{GL}(V)$. Concretely, we're "representing" elements of G by endomorphisms of a vector space, but it must be stressed that ρ might *not* be injective. The case of infinite G (e.g., Lie groups) and infinite-dimensinal V (e.g., function spaces) are extremely fundamental, but intuition is developed by first understanding finite G and $\dim V < \infty$. We'll often refer to the data of (V, ρ) as a *representation space of* G. The most classical example is $G = \mathfrak{S}_n$ with its representation on F^n through permutations of the standard basis. Also, when $\dim V = 1$ then a representation of G on V is just a group homomorphism $\chi : G \to F^\times$.

2. We say that a representation space (V, ρ) is *irreducible* if $V \neq 0$ and there do not exist any G-stable subspaces aside from 0 and V.

(i) Prove that the natural representation of $\mathrm{GL}(V)$ on V is irreducible when $0 < \dim V < \infty$.

(ii) Consider the permutation representation of $G = \mathfrak{S}_3$ on F^3. If $3! = 6 \neq 0$ in F, show that the hyperplane $H : x_1 + x_2 + x_3 = 0$ and the line spanned by $(1, 1, 1)$ are complementary G-stable subspaces on which G acts irreducibly. If $2 = 0$ in F, show the same claims. If $3 = 0$ in F (e.g., $F = \mathbf{F}_3$), show that there is *no* line in F^3 complementary to H which is G-stable, and that the action of G on H is also *not* irreducible (find a G-stable line in H).

(iii) Changing the ground field can make a big difference. For example, consider the representation ρ of $\mathbf{Z}/3$ on $\mathbf{R}^2$ in which $1 \in \mathbf{Z}/3$ acts by the counterclockwise rotation through an angle of $2\pi/3$. Write down the matrix for this and prove this representation is irreducible. Over $\mathbf{C}^2$, the same matrix defines a representation of $\mathbf{Z}/3$ but use eigenvalues to explicitly determine two G-stable lines and prove these are *the only two* such lines in the representation space.

(iv) If (V, ρ) and (V', ρ') are irreducible, prove that any G-compatible linear map $T : V \to V'$ is either zero or an isomorphism (note that $\ker T$ and $\mathrm{im} T$ are G-stable!).

(v) If (V, ρ) is irreducible, $\dim V$ is finite, and F is *algebraically closed* (so characteristic polynomials make sense and have roots in F), show that the only G-compatible linear endomorphisms $T : V \to V$ (i.e., $T \circ \rho(g) = \rho(g) \circ T$ for all $g \in G$) are scalar multiplications. Hint: T has some nonzero eigenspace.

3. Suppose (V, ρ) is a finite-dimensional representation space and G is finite with $|G| \neq 0$ in F. If $W \subseteq V$ is a subspace which is stable under the G-action, prove as follows that there exists a complementary subspace W' which is also stable under the G-action (so (V, ρ) is build up as direct sum of W and W' *together with their G-actions*).

(i) Let $\pi : V \to W$ be an arbitrary linear map which restricts to the identity on W (i.e., $\pi(w) = w$ for all $w \in W \subseteq V$). Show that if π is G-equivariant in the sense that $\pi \circ \rho(g) = \rho(g) \circ \pi$ for all $g \in G$, then $\ker \pi$ is G-stable and provides a G-stable complement to W.

(ii) In general, we won't be so lucky, so just pick *any* linear $\pi : V \to W$ which lifts the identity on W (make this by using bases to make a complementary subspace away from which we're projecting). Now we average! Using that W is G-stabe and $|G| \neq 0$ in F, it *makes* sense to form the F-linear map

$$\pi' = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ \pi \circ \rho(g^{-1}).$$

Using the fraction out front and that π lifts the identity on the G-stable W, prove that π' restricts to the identity on W and *is G-equivariant*. Deduce that $\ker \pi'$ provides the desired *G-stable* complement. This averaging trick is due to Maschke.

(*iii*) Consider the permutation representation of $\mathfrak{S}_n$ on F^n, with W the hyperplane $\sum x_j = 0$ and $n! \neq 0$ in F. Find an $\mathfrak{S}_n$-stable complement to W.

4. A very important representation of a finite group G is its *left regular representation*. Let $V = F[G] \stackrel{\text{def}}{=} \oplus F e_g$ be a vector space whose basis is indexed by the elements of G, and define $\rho_{\text{reg}} : G \to \text{GL}(V)$ by $\rho_{\text{reg}}(g) : e_{g'} \mapsto e_{gg'}$. This "$F$-linearizes" the left multiplication action of G on itself as a set. It's a big space (think of $G = \mathfrak{S}_n$)!

(*i*) If (V, ρ) is a nonzero finite-dimensional F-linear representation of G and $|G| \neq 0$ in F, use induction on dimension and Exercise 3 to show that V is a direct sum of G-stable subspaces V_i on which G acts *irreducibly*. Thus, to describe all finite-dimensional representations of G up to isomorphism it is "enough" in such cases to describe the irreducible ones (of course, the real art is to actually locate the V_i's explicitly in interesting situations).

(*ii*) Show that if (V, ρ) is a nonzero representation of G, then by choosing a nonzero $v_0 \in V$ the natural map $\pi_{v_0} : F[G] \to V$ defined by $e_g \mapsto \rho(g)(v_0)$ is a map of representation spaces (i.e., π_{v_0} is linear commutes with the G-actions: $\pi_{v_0} \circ \rho_{\text{reg}}(g) = \rho(g) \circ \pi_{v_0}$ for all $g \in G$). Prove that the image of π_{v_0} is nonzero and G-stable, and conclude that if (V, ρ) is *irreducible* then π_{v_0} is *surjective* (in particular, $\dim V \leq |G|$ is finite!). Consequently, all irreducible G-representations are quotients of the left regular representation (over F).

(*iii*) Prove that any finite-dimensional representation space (V, ρ) of G admits a rising chain of G-stable subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

such that each V_i a G-stable subspace and V_i/V_{i-1} (for $1 \leq i \leq n$) with its natural G-action is *irreducible*. We call this a *length n filtration* of V by G-stable subspaces. By a Jordan-Hölder style of argument, prove that any two such filtrations with irreducible successive quotients have the same length and give rise to the same collection of irreducible successive quotient representations (up to isomorphism), perhaps in different orderings.

(*iv*) Using (*iii*), deduce that *up to isomorphism* there are only *finitely many* irreducible representations of G on F-vector spaces of finite dimension. The first real theorems in representation theory provide systematic ways to "explicitly" determine these irreducibles, and in real life one certainly wants to realize them in geometric ways (rather than just as "abstract" creatures) if at all possible. The effective version of this finiteness result makes it possible to determine the structure of various molecules and crystals, given enough knowledge about the symmetry. Ask your friends in physical chemistry about this.