Math 594. Homework 3 (due January 29)

Book problems: §5.1: 14. §5.4: 4, 5, 15.

1. If H, K are normal subgroups of a group G with $H \cap K = \{1\}$ and $H.K = G$, show $hk = kh$ for all $k \in K$ and $h \in H$. Conclude that $H \times K \simeq G$.

2. Let G be a finite abelian group, $g_0 \in G$ an element with *maximal* order, say n. Thus, $H = \langle g_0 \rangle$ is a maximal cyclic group of G. The group $\mu_n(\mathbf{C})$ of nth roots of unity in $\mathbf{C}$ is also cyclic of order n (it has generator $e^{\pm 2\pi i/n}$, for example). Pick an isomorphism between these groups, so we get an injective group homomorphism $\chi_0 : H \hookrightarrow \mathbf{C}^\times$. By HW2, this can be extended to a group homomorphism $\chi : G \to \mathbf{C}^\times$ since G is finite abelian.

(i) Let $K = \ker \chi$. Show that $\chi(G) = \mu_n(\mathbf{C}) = \chi_0(H)$, and deduce that the natural multiplication map $H \times K \to G$ is an isomorphism.

(ii) Using induction on $|G|$, deduce that every finite abelian group is a product of cyclic groups, and conclude that if a prime p divides $|G|$ then G contains an element of order p.

(iii) For relatively prime nonzero integers a and b, show that the natural map $\mathbf{Z}/ab \to \mathbf{Z}/a \times \mathbf{Z}/b$ (defined by $r \bmod ab \mapsto (r \bmod a, r \bmod b)$ is a well-defined group homomorphism which is injective, and by counting deduce it is an isomorphism. Can you describe an inverse easily?

(iv) Using (iii), show that every finite abelian group G can be written in the form $\phi : G \simeq C_{n_1} \times \cdots \times C_{n_r}$ with cyclic groups C_{n_j} of order n_j where $n_1 | n_2 | \ldots | n_r$. Prove that r and the parameters $n_1, \ldots, n_r$ are uniquely determined by G. These are called the *invariant* factors of G. Use this to make a list of *all* finite abelian groups (up to isomorphism) of order $p^3 q^2$ for distinct primes p and q. Make sure your list has *no* repetitions.

3. Let F be a field and $G = \mathrm{GL}_n(F)$ for a positive integer n. Let B_n denote the subgroup of upper triangular matrices, U_n the subgroup of "strictly" upper triangular matrices (i.e., all 1's down the diagonal), and T_n the subgroup of diagonal matrices.

(i) Construct a natural group homomorphism from B_n onto T_n with kernel U_n, and use this to show that U_n is normal in B_n with $B_n = T_n.U_n$.

(ii) We must have $B_n = T_n \ltimes U_n$ using the conjugation action $u \mapsto tut^{-1}$ of $t \in T_n$ on U_n. Describe this action explicitly in the cases $n = 2, 3$, and then establish a general formula.

(iii) Show that elements of U_n have the form $1 + M$ where M is an upper triangular matrix having 0's on the diagonal. By looking at the nonzero entry of M nearest to the diagonal and lower right corner, show that if $a \neq 0$ in F for all nonzero integers a (e.g., $F = \mathbf{R}$ but not $F = \mathbf{F}_p$), then U contains no non-trivial elements of finite order. Try $n = 2, 3$ first to see what is happening.

(iv) In constrast, when $F = \mathbf{F}_p$, show that U_n is a finite p-group (hint: $u \in U_n$ can be written $u = 1 + M$ where 1 denotes the $n \times n$ identity matrix and M has 0's on and below the main diagonal. Deduce $M^n = 0$, but $(1 + M)^p = 1 + M^p$ when working over $\mathbf{F}_p$ because ...).

Remark: In 3(iv), you are implicitly going to use that $(1 + M)^r$ can be expanded as in the binomial formula. This is valid because 1 and M commute with each other!