Math 594. Algebra II.

Midterm Exam 1 (February 10–17, 2002)

No time limit

Name: ____________________

Read all questions carefully. *Do any* 5 *out of the* 6 *questions.* You will not be given partial credit on the basis of having misunderstood a question, and please show all work. *Unless otherwise indicated*, you may use without proof all results which were discussed in lecture, homework, or the course text. Be clear and precise in stating what you use.

If you are unable to solve part of a problem, you may still use the conclusion from that part to do subsequent parts of the problem. If your solution does not fit in the indicated space, please use the back of the same page. This is an open-book, open-notebook exam. It is due at the beginning of class on Monday, February 17. At that time solution sets will be handed out, so late submission will be unacceptable.

You must sign below (indicating your agreement with) the following honor pledge: *I pledge my honor that I have not used calculators, electronic computing machines of any sort, the Internet, contact with other human beings, or any published book other than the course text for mathematical assistance in connection with my work on this exam.*

Question	Possible	Actual
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total	100	

1. (20 pts) Let G be a group. We define its *automorphism group* $\mathrm{Aut}(G)$ to be the set of group isomorphisms $\phi : G \simeq G$.

(i) (5 pts) Prove that using composition of maps, $\mathrm{Aut}(G)$ is a group.

(ii) (5 pts) For $g \in G$, define $c_g : G \simeq G$ to be the left conjugation action: $c_g(g') = gg'g^{-1}$. Prove that $c_g \in \mathrm{Aut}(G)$ and that $g \mapsto c_g$ is a group homomorphism $G \to \mathrm{Aut}(G)$ with kernel $Z(G)$ (the center of G). The image of this map is denoted $\mathrm{Inn}(G)$ and its elements are called the *inner automorphisms* of G.

(*iii*) (10 pts) Prove $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. The quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is denoted $\mathrm{Out}(G)$, and is called the *outer automorphism group* of G (though its elements are not actually automorphisms of G, but are merely coset classes by the inner automorphism group).

It is an important problem to know when the outer automorphism group is trivial, or to understand its structure. By considering how an element of $\mathrm{Aut}(S_3)$ acts on the three transpositions in S_3, construct an injection of groups $\mathrm{Aut}(S_3) \to S_3$ and use the triviality of the center to conclude by pure thought (i.e., without grungy calculations) that $\mathrm{Inn}(S_3) = \mathrm{Aut}(S_3)$. That is, $\mathrm{Out}(S_3)$ is trivial.

2. (20 pts) Give examples of each of the following, briefly indicating why your examples satisfy the requirements, or explain (with brief argument) why no such example exists.
(*i*) (4 pts) A cyclic group which is a product of two non-trivial groups.

(*ii*) (4 pts) A group acting transitively on a set with trivial stabilizer at one point and non-trivial stabilizer at another point.

(*iii*) (4 pts) Two non-isomorphic non-abelian groups of order 20.

(*iv*) (4 pts) An infinite non-abelian solvable group.

(*v*) (4 pts) A non-normal subgroup H in a finite group G such that H is not equal to its normalizer in G.

3. (20 pts) Let G be a non-trivial finite p-group (i.e., $p|\#G$) and let V be a nonzero finite-dimensional vector space over $\mathbf{F}_p$. Suppose G acts linearly on V on the left (i.e., we're given a group homomorphism $G \to \mathrm{GL}(V)$).

(*i*) (7 pts) Prove that the orbits of G with more than 1 point have size a power of p, and conclude that $\{v \in V \,|\, \mathrm{Stab}_G(v) = G\}$ is divisible by p. Using that this set is non-empty (it contains 0!), show G fixes some nonzero $v \in V$.

(*ii*) (6 pts) Choose a nonzero $v_0 \in V$ fixed by G, say spanning a line L. By considering the induced action of G on V/L and using induction on $\dim V$, prove the existence of a basis of V with respect to which the image of G in $\mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbf{F}_p)$ lies in the subgroup of strictly upper triangular matrices.

(iii) (7 pts) Let $H = \mathrm{GL}_n(\mathbf{F}_p)$ and let G be a p-subgroup. Using *(ii)* for an appropriate V, but not using the Sylow theorems, deduce that some conjugate of G lies inside the subgroup of strictly upper triangular matrices.

4. (20 pts) Let C be the cube in $\mathbf{R}^3$ with side length 2 and the 8 vertices at the points $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ with $\varepsilon_j \in \{\pm 1\}$. Let Γ denote the group of "orientation-preserving symmetries of the cube", by which we mean permutations of the 8 vertices which preserve the relation "joined by a common edge" for pairs of vertices and "lie on a common face" for pairs of edges, and which respect the orientation of $\mathbf{R}^3$ formed by 3 edges emanating from a vertex.

(i) (10 pts) Observe that there are 4 "long diagonals" on the cube. Use this to define a group homomorphism $\Gamma \to S_4$, and explain why it is injective. Then by either using stabilizers of a long diagonal (watch the orientation!) to compute $\#\Gamma$, or by hunting for transpositions in the image (or using some other geometric method), prove this group map is an isomorphism.

(*ii*) (10 pts) Suppose a finite group G acts on a finite set X. For each $g \in G$, define $\mathrm{Fix}(g) = \#\{x \in \ | \, g.x = x\}$ to be the number of points fixed by g. Prove

$$\sum_{g \in G} \mathrm{Fix}(g) = \#\{(g,x) \in G \times X \mid g.x = x\} = \sum_{x \in X} \#\{g \in G \mid g.x = x\} = \sum_{x \in X} |G|/|G.x|,$$

and by breaking up the final sum over orbits deduce *Burnside's Lemma*: the number of orbits is the average number of fixed points (i.e., $|G|^{-1} \sum_{g \in G} \mathrm{Fix}(g)$).

(**Extra Credit**) (10 pts) Imagine we paint the 6 faces of the cube with 2 red faces, 2 blue faces, and 2 green faces. Let P be the set of such ways of painting the cube. There is an obvious action of G on P, and we regard two colorings as "equivalent" if they lie in the same G-orbit (why is this reasonable?). Thus, the number of "essentially different" colorings is the number of G-orbits. Use Burnside's Lemma to determine this number (hint: $\mathrm{Fix}(\gamma)$ only depends on the conjugacy class of γ, and in $\Gamma \simeq S_4$ we know the conjugacy classes!).

5. (20 pts) Classify all finite groups of order 40 with non-commutative 2-Sylow subgroup, and prove that your list does not contain any repetitions (up to isomorphism). You may use the classification of finite groups of order 8.

6. (20 pts) Let V be a vector space of finite dimension $n > 0$ over a field F. Define $\mathrm{Aff}(V)$ to be the group of "affine transformations" of V: this is the group generated by $\mathrm{GL}(V)$ together with translations $t_{v_0} : v \mapsto v + v_0$ for $v_0 \in V$ (note that t_{-v_0} is an inverse to t_{v_0}).

(*i*) (6 pts) In the special case $V = F$ (so $n = 1$), explain how to identify $\mathrm{Aff}(V)$ with the group of linear polynomials $ax + b$ under "composition of functions", and show that this is isomorphic to the group of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a, b \in F$ and $a \neq 0$. This is often called the "$ax + b$ group".

(*ii*) (8 pts) Viewing V as an additive group, consider the natural map of sets

$$\phi : \mathrm{GL}(V) \times V \to \mathrm{Aff}(V)$$

defined by $(T, v) \mapsto t_v \circ T$. Show that the composite of $g_1 = t_{v_1} \circ T_1$ and $g_2 = t_{v_2} \circ T_2$ can be written in "$t_v \circ T$" form for a unique $v \in V$ and $T \in \mathrm{GL}(V)$ (depending on the v_j's and T_j's), and likewise we can write $g_1^{-1} = t_{v_1'} \circ T_1'$ for some unique v_1', T_1' depending on v_1 and T_1. Conclude that ϕ is a bijection of sets, and explain how it describes $\mathrm{Aff}(V)$ as a semi-direct product of $\mathrm{GL}(V)$ and V (= translation group), with V normal in $\mathrm{Aff}(V)$.

(*iii*) (6 pts) Consider the natural (left) action of $G = \mathrm{Aff}(V)$ on V (via $g.v = g(v)$), and prove $\mathrm{GL}(V)$ is the stabilizer of the origin. Compute the stabilizer group at any $v_0 \in V$ as the conjugate of $\mathrm{GL}(V)$ by an explicit element of G (depending of course on v_0).