Math 593 Assignment # 9 Solutions

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1. Let $R = \mathbb{C}[x,y]$ and $I = (x,y) \subset R$. Define $\phi : I \otimes_R I \to \mathbb{C}$ by

$$\phi(f \otimes g) = f_x(0,0)g_y(0,0) - f_y(0,0)g_x(0,0).$$

We claim that ψ is an R module homomorphism, where $\mathbb{C}$ is considered as an R module by

$$f \cdot z := f(0,0)z$$

for any $f \in R$ and $z \in \mathbb{C}$. We first show that the map $\psi : I \times I \to \mathbb{C}$ given by

$$\psi(f,g) = f_x(0,0)g_y(0,0) - f_y(0,0)g_x(0,0)$$

is R bilinear. This is essentially a sequence of straightforward calculations. For example, for any $r \in R$ and $f, g, h \in I$ we have

$$\begin{aligned}
&\psi(rf+g,h)\\
&\quad = (rf+g)_x(0,0)h_y(0,0) - h_x(0,0)(rf+g)_y(0,0)\\
&\quad = (r_x(0,0)f(0,0) + f_x(0,0)r(0,0) + g_x(0,0))h_y(0,0) - h_x(0,0)(r_y(0,0)f(0,0) + f_y(0,0)r(0,0) + g_y(0,0))\\
&\quad = (f_x(0,0)r(0,0) + g_x(0,0))h_y(0,0) - h_x(0,0)(f_y(0,0)r(0,0) + g_y(0,0))
\end{aligned}$$

since $f \in I$ and hence $f(0,0) = 0$,

$$\begin{aligned}
&\quad = r(0,0)(f_x(0,0)h_y(0,0) - h_x(0,0)f_y(0,0)) + g_x(0,0)h_y(0,0) - h_x(0,0)g_y(0,0)\\
&\quad = r \cdot \psi(f,h) + \psi(g,h).
\end{aligned}$$

The calculations to show R linearity in the second component are similar, so we omit them. By the universal property of the tensor product, we have that ϕ is R linear, as required.

Now if $x \otimes y - y \otimes x = 0$ then since ϕ is an R module homomorphism we have

$$\begin{aligned}
0 &= \phi(0)\\
&= \phi(x \otimes y - y \otimes x)\\
&= \phi(x \otimes y) - \phi(y \otimes x)\\
&= (1 \cdot 1 - 0 \cdot 0) - (0 \cdot 0 - 1 \cdot 1)\\
&= 2,
\end{aligned}$$

a contradiction.

2. For $p, q \in \mathbb{Q}$ write $p = a/b,\ q = c/d$ for integers a, b, c, d. Then for $p \otimes q \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ we have

$$\begin{aligned}
p \otimes q &= \frac{a}{b} \otimes \frac{c}{d} \\
&= \left(\frac{acd}{bd} \otimes \frac{1}{d}\right) \\
&= d\left(\frac{ac}{bd} \otimes \frac{1}{d}\right) \\
&= \left(\frac{ac}{bd} \otimes \frac{d}{d}\right) \\
&= \left(\frac{ac}{bd} \otimes 1\right) \\
&= (pq \otimes 1),
\end{aligned}$$

where we have only used $\mathbb{Z}$-bilinearity of the tensor. This suggests that we consider the map $\phi : Q \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ given by $p \otimes q \mapsto pq$. Observe that from our calculations above, it is clear that this is a $\mathbb{Q}$ module homomorphism. Indeed, we have

$$\begin{aligned}
\phi(p \otimes q + r \otimes s) &= \phi((pq \otimes 1) + (rs \otimes 1)) \\
&= \phi(((pq + rs) \otimes 1)) \\
&= pq + rs \\
&= \phi(p \otimes q) + \phi(r \otimes s),
\end{aligned}$$

and

$$\begin{aligned}
\phi(\frac{x}{y}(p \otimes q)) &= \phi(\frac{x}{y}(\frac{y}{y}pq \otimes 1)) \\
&= \phi((\frac{x}{y}pq \otimes 1)) \\
&= \frac{x}{y}pq \\
&= \frac{x}{y}\phi(p \otimes q),
\end{aligned}$$

for any $x/y \in \mathbb{Q}$. Surjectivity of ϕ is clear since $\phi(p \otimes 1) = p$ for any $p \in \mathbb{Q}$. Injectivity is evident from the above calculation since if $\phi(p \otimes q) = \phi(r \otimes s)$ then $pq = rs$ so that

$$p \otimes q = pq \otimes 1 = rs \otimes 1 = r \otimes s.$$

It follows that ϕ is an isomorphism of $\mathbb{Q}$ modules, as required.

3. Define

$$\phi : V^* \times W \to \mathrm{Hom}(V, W)$$

by

$$(f, w) \mapsto (v \mapsto f(v) \cdot w)\,.$$

It is not difficult to see that ϕ is bilinear. For every $v \in V$ we have

$$\begin{aligned}\phi(af+g, bw+u)(v) &= (af+g)(v)(bw+u) \\ &= (af(v)+g(v))(bw+u) \\ &= abf(v)w + af(v)u + bg(v)w + bg(v)u \\ &= ab\phi(f,w)(v) + a\phi(f,u)(v) + b\phi(g,w)(v) + b\phi(g,u)(v).\end{aligned}$$

Thus we obtain a homomorphism $\tilde{\phi} : V^* \otimes W \to \mathrm{Hom}(V, W)$ given by $\phi(f, w)(v) = f(v)w$. To check that this is an isomorphism, pick any basis $\{v_i\}$ of V and $\{w_j\}$ of W. Let $\{v_i^*\}$ be the dual basis to $\{v_i\}$. Then we know that $\{v_i^* \otimes w_j\}$ is a basis for $V^* \otimes W$. Suppose that $f \in \mathrm{Hom}(V, W)$ is given by

$$f(v_i) = \sum_j a_{ij} w_j.$$

Let

$$x = \sum_{k,j} a_{kj}(v_k^* \otimes w_j) \in V^* \otimes W.$$

Then clearly we have

$$\phi(x)(v_i) = \sum_{k,j} a_{kj}(v_k^*(v_i)w_j) = \sum_j a_{ij} w_j = f(v_i),$$

so that ϕ is surjective. Since $\dim \mathrm{Hom}(V, W) = \dim V^* \otimes W$, we must have that ϕ is an isomorphism.

Observe that although we chose a basis to show that ϕ was an isomorphism, the definition of ϕ is independent of any choice of basis. This is what it means for an isomorphism of vector spaces to be "canonical."

4. Let M be an R module and $I \subset R$ an ideal. Recall that we have an exact sequence

$$0 \longrightarrow I \xrightarrow{u} R \xrightarrow{v} R/I \longrightarrow 0.$$

By right exactness of the tensor product, this gives an exact sequence

$$I \otimes_R M \xrightarrow{u\otimes\mathrm{id}} R \otimes_R M \xrightarrow{v\otimes\mathrm{id}} R/I \otimes_R M \longrightarrow 0.$$

We have

$$\begin{aligned}u \otimes \mathrm{id}\left(\sum_l i_l \otimes m_l\right) &= u \otimes \mathrm{id}\left(\sum_l 1 \otimes i_l m_l\right) \\ &= \sum_l u(1) \otimes i_l m_l \\ &= u(1) \otimes \left(\sum_l i_l m_l\right)\end{aligned}$$

so that if $u \otimes \mathrm{id}(x) = 0$ then since u is injective and $u(1) \neq 0$ we have $x = 0$ so that $u \otimes \mathrm{id}$ is injective. We claim that $I \otimes_R M \simeq IM$. For we may define

$$\begin{aligned} f &: I \otimes M \to IM, \\ g &: IM \to I \otimes M \end{aligned}$$

by

$$\begin{aligned} f\left(\sum_l i_l \otimes m_l\right) &= \sum_l i_l m_l, \\ g(x) &= 1 \otimes x. \end{aligned}$$

A priori, it is unclear that the image of g is contained in IM. However, we have

$$\begin{aligned} g \circ f\left(\sum_l i_l \otimes m_l\right) &= 1 \otimes \sum_l i_l m_l \\ &= \sum_l 1 \otimes i_l m_l \\ &= \sum_l i_l \otimes m_l \end{aligned}$$

since the tensor product is over R and $I \subset R$. Moreover, for $x = \sum_l i_l m_l \in IM$ we have

$$\begin{aligned} f \circ g(x) &= f\left(1 \otimes \sum_l i_l m_l\right) \\ &= \sum_l i_l m_l \\ &= x, \end{aligned}$$

from which it follows that f is an isomorphism, since it is obviously a homomorphism. Similarly, one shows that $R \otimes_R M \simeq M$ (the isomorphism is the same as above). It follows that

$$M \otimes_R R/I \simeq R/I \otimes_R M \simeq M/IM,$$

as required.

5. Let $\{e_1, e_2\}$ be a basis of $\mathbb{R}^2$. Let $v = ae_1 + be_2$ and $w = ce_1 + de_2$. Then

$$\begin{aligned} v \otimes w &= ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + bde_2 \otimes e_2 \\ &= \alpha e_1 \otimes e_1 + \beta e_1 \otimes e_2 + \gamma e_2 \otimes e_1 + \delta e_2 \otimes e_2, \end{aligned}$$

with

$$\alpha\delta - \beta\gamma = 0.$$

This shows that any simple tensor corresponds to a point of the quadric hypersurface given by

$$\alpha\delta - \beta\gamma = 0.$$

Conversely, suppose that we have a point $(\alpha, \beta, \gamma, \delta)$ of the above hypersurface. If $\alpha, \beta, \gamma, \delta$ are all 0, it is clear that this corresponds to the simple tensor $0 \otimes 0$. Otherwise, without loss of generality, suppose that $\alpha \neq 0$ and that $\alpha\delta - \beta\gamma = 0$. Then

$$\delta = \frac{\beta\gamma}{\alpha}$$

and

$$\alpha e_1 \otimes e_1 + \beta e_1 \otimes e_2 + \gamma e_2 \otimes e_1 + \delta e_2 \otimes e_2 = (\alpha e_1 + \gamma e_2) \otimes (e_1 + \frac{\beta}{\alpha} e_2)$$

is a simple tensor.