Math 593 Assignment # 8 Solutions

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2. (a) Any additive (finite) abelian group G may be realized as a $\mathbb{Z}$ module. The structure theorem tells us that we may decompose G in primary form, that is, we have

$$G \simeq \bigoplus \mathbb{Z}/(p_i^{a_i})$$

, where the (not necessarily distinct) p_j satisfy

$$\prod_i p_i^{a_i} = \#G.$$

Since $500 = 2^2 \cdot 5^3$, we have the following 6 possibilities for the primary decomposition of G, and hence for G up to isomorphism:

$$\begin{aligned}
&\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(5)\\
&\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(5^2)\\
&\mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(5^3)\\
&\mathbb{Z}/(2^2) \oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(5)\\
&\mathbb{Z}/(2^2) \oplus \mathbb{Z}/(5) \oplus \mathbb{Z}/(5^2)\\
&\mathbb{Z}/(2^2) \oplus \mathbb{Z}/(5^3).
\end{aligned}$$

(b) Let T be a matrix over a field k with characteristic polynomial $(x^2 - x)^5$. Let us determine all the possible rational canonical forms for T. Let M_T be the $k[x]$ module determined by the action of T on k^{10}. Then the structure theorem enables us to write

$$M_T \simeq k[x]/(f_1(x)) \oplus \ldots \oplus k[X]/(f_r(x)),$$

where f_r is the minimal polynomial of T, $f_i | f_{i+1}$, and $x^5(x-1)^5 = \prod_i f_i$. It follows that we have

$$f_i = x^{a_i}(x-1)^{b_i}$$

with $\sum_i a_i = \sum_i b_i = 5$. It follows that for any two partitions of 5 we have a (unique) list of invariant factors f_i. The partitions of 5 are precisely

$$\begin{aligned}
&5\\
&4+1\\
&3+2\\
&3+1+1\\
&2+2+1\\
&2+1+1+1\\
&1+1+1+1+1.
\end{aligned}$$

Therefore, there are $7 \cdot 7 = 49$ similarity classes of matricies T, enumerated by any two partitions p_1, p_2 of 5. If r_i denotes the largest part of the partition p_i, then the minimal polynomial of a class given by p_1, p_2 is

$$x^{r_1}(x-1)^{r_2}.$$

(c) Certainly the minimal polynomial of T must divide $x(x-1)^2$. Moreover, since the characteristic polynomial of T has the same roots as the minimal polynomial and has degree 4 (since T is a 4×4 matrix), we can easily list the possible invariant factors for the decomposition of the $k[x]$ module specified by T into cyclic factors. The possibilities for the minimal polynomial are:

$$x,\ x-1,\ (x-1)^2,\ x(x-1),\ x(x-1)^2.$$

It follows that the possible invariant factors are:

$$\begin{aligned}
&x, x, x, x\\
&x-1, x-1, x-1, x-1\\
&(x-1), (x-1), (x-1)^2\\
&(x-1)^2, (x-1)^2\\
&x, x, x(x-1)\\
&x-1, x-1, x(x-1)\\
&x(x-1), x(x-1)\\
&x, x(x-1)^2\\
&x-1, x(x-1)^2
\end{aligned}$$

for a total of 9 similarity classes. Each list of invariant factors above gives the matrix T in rational canonical form. For example, the list $x(x-1), x(x-1)$ tells us that we have two blocks, each with companion matrix $x^2 - x$, corresponding to

$$\begin{pmatrix} 0 & 0 & & \\ 1 & 1 & & \\ & & 0 & 0 \\ & & 1 & 1 \end{pmatrix}.$$

3. Let $A = (a_{ij})$ be an $n \times n$ matrix with integer entries presenting the $\mathbb{Z}$ module M. Recall that we can bring A to diagonal form via elementary row and column operations. That is, we have

$$B = QAP^{-1}$$

where Q, P are products of elementary matrices, so that $\det B = \pm \det A$ and

$$B = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix},$$

where

$$d_i | d_{i+1}$$

and $d_i > 0$ for $1 \le i \le l \le n$. Moreover, we have

$$M \simeq \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_l) \oplus \mathbb{Z}^{n-l}$$

and

$$\det B = |\det A| = |d_1 \cdots d_n|.$$

Clearly, $|\det A| \neq 0$ if and only if $d_i > 0$ for all i, that is, if and only if $n = l$, if and only if M is finite. If M is finite, we have

$$\#M = \#\mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_n) = \prod_{i=1}^{n} \#\mathbb{Z}/(d_i) = \prod_{i=1}^{n} d_i = \det B = |\det A|.$$

4. i Let T be an $n \times n$ invertible matrix over a field k, and let $P_T(x)$ denote the characteristic polynomial of T. We have

$$\begin{aligned} P_T(x) &= \det(xI - T) \\ &= \det(T)\det(xT^{-1} - I) \\ &= (-x)^n \det(T)\det(\frac{1}{x}I - T^{-1}) \\ &= (-x)^n \det(T) P_{T^{-1}}\left(\frac{1}{x}\right), \end{aligned}$$

so that

$$P_{T^{-1}}(x) = \frac{(-x)^n}{\det(T)} P_T\left(\frac{1}{x}\right).$$

ii Suppose that $V_T = M_1 \oplus M_2 \oplus \cdots \oplus M_r$ where $M_i \subset V$ is a T submodule of V_T, isomorphic (with the structure of multiplication by x given as multiplication by T) to $k[x]/(d_i(x))$ with $d_i|d_{i+1}$. Clearly, d_i is the characteristic polynomial of multiplication by T on the T-submodule M_i of V_T. It is not difficult to see that M_i also has the structure of a T^{-1} module. Indeed, $TM_i \subset M_i$ since M_i is a

T-submodule; equality follows from the fact that M_i is finite dimensional, so that $M_i = T^{-1}M_i$. Moreover, by part (i), the characteristic polynomial of multiplication by T^{-1} on M_i is

$$f_i(x) = \frac{(-x)^{\deg d_i}}{\det(T|_{M_i})} d_i\left(\frac{1}{x}\right) = x^{r_i} + \frac{a_{1i}}{a_{0i}} x^{r_i-1} + \cdots + \frac{a_{(r_i-1)i}}{a_{0i}},$$

where

$$d_i(x) = x^{r_i} + a_{(r_i-1)i}x^{r_i-1} + \cdots + a_{0i}.$$

It follows that M_i with the structure of multiplication by x as multiplication by T^{-1} is isomorphic to $k[x]/(f_i(x))$. It follows that

$$V_{T^{-1}} \simeq \bigoplus_{i=1}^{r} k[x]/(f_i(x)).$$

Moreover, we have $f_i | f_{i+1}$ since we can write

$$d_{i+1}(x) = e_i(x)d_i(x)$$

so that

$$\begin{aligned}
f_{i+1}(x) &= \frac{(-x)^{\deg d_{i+1}}}{\det(T|_{M_{i+1}})} d_{i+1}\left(\frac{1}{x}\right) \\
&= \frac{(-x)^{\deg e_i}(-x)^{\deg d_i}}{\det(T|_{M_{i+1}})} d_i\left(\frac{1}{x}\right) e_i\left(\frac{1}{x}\right) \\
&= \frac{\det(T_{M_i})}{\det(T|_{M_{i+1}})} f_i(x)(-x)^{\deg e_i} e_i\left(\frac{1}{x}\right).
\end{aligned}$$

There is another way to see this. The decomposition

$$V_T \simeq \bigoplus k[x]/d_i$$

is giving us the rational canonical form of T. Let d_i be as above. Then the companion matrix to d_i is

$$C_i = \begin{pmatrix}
0 & & & & & -a_{0i} \\
1 & 0 & & & & -a_{1i} \\
 & 1 & 0 & & & \vdots \\
 & & \ddots & \ddots & & \vdots \\
 & & & \ddots & 0 & \vdots \\
 & & & & 1 & -a_{(r_i-1)i}
\end{pmatrix}.$$

By a direct calculation, it is easy to verify that

$$C_i^{-1} = \begin{pmatrix}
-\frac{a_{1i}}{a_{0i}} & 1 & 0 & & & 0 \\
-\frac{a_{2i}}{a_{0i}} & & 1 & 0 & & 0 \\
\vdots & & & \ddots & \ddots & \vdots \\
-\frac{1}{a_{0i}} & & & & 0 & 1
\end{pmatrix}.$$

By appropriately flipping columns and rows, it can be seen that the rational canonical form for C_i^{-1} is the companion matrix D_i to $f_i(x)$, where f_i is above. Since the rational canonical form of T is

$$\begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_r \end{pmatrix}$$

it follows that the rational canonical form of T^{-1} is

$$\begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_r \end{pmatrix}.$$

This then gives us the decomposition

$$V_{T^{-1}} \simeq \bigoplus_{i=1}^{r} k[x]/f_i$$