Math 593 Assignment # 7 Solutions

Bryden R. Cais

November 8, 2002

1.

2. Put $R = \mathbb{Q}[x, y]$ and let $M = (x, y) \subset R$ (considered as an R module). We claim that we have an exact sequence

$$0 \longrightarrow R \xrightarrow{A} R^2 \xrightarrow{B} M \longrightarrow 0$$

where

$$Ar = r\begin{pmatrix} -y \\ x \end{pmatrix}$$

and

$$B\begin{pmatrix} f \\ g \end{pmatrix} = fx + gy.$$

To verify this assertion, it suffices to check that $\operatorname{Im} A = \ker B$ since it is obvious that B is surjective. Suppose that $fx + gy = 0$. Then $fx = -gy$ and $y|f$ since R is a UFD and y is irreducible. Similarly, $x|g$ and it follows that $f = -ry,\ g = rx$ for some $r \in R$, that is, $\ker B = \operatorname{Im} A$. It follows that the presentation matrix for M is the matrix of multiplication by A as a map $R \to R^2$. It is clear that this matrix is

$$\begin{pmatrix} -y \\ x \end{pmatrix}.$$

3. The presentation matrix is

$$\begin{pmatrix} 3 & 2 \\ 6 & -4 \\ 12 & 10 \end{pmatrix}.$$

We use row and column operations to simplify this matrix. We have

$$\begin{pmatrix} 3 & 2 \\ 6 & -4 \\ 12 & 10 \end{pmatrix} \xrightarrow{\text{col } 1 - \text{col } 2} \begin{pmatrix} 1 & 2 \\ 10 & -4 \\ 2 & 10 \end{pmatrix} \xrightarrow{\text{row } 2 - 10\cdot\text{row } 1} \begin{pmatrix} 1 & 2 \\ 0 & -24 \\ 2 & 10 \end{pmatrix} \xrightarrow{\text{row } 3 - 2\cdot\text{row } 1}$$
$$\begin{pmatrix} 1 & 2 \\ 0 & -24 \\ 0 & 6 \end{pmatrix} \xrightarrow{\text{col } 2 - 2\cdot\text{col } 1} \begin{pmatrix} 1 & 0 \\ 0 & -24 \\ 0 & 6 \end{pmatrix} \xrightarrow{\text{row } 2 + 4\cdot\text{row } 3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 6 \end{pmatrix},$$

from which it is evident that

$$M \simeq \mathbb{Z} \oplus \mathbb{Z}/(6).$$

4. Let A be an integral domain and M an A module.

   (a) Let $T(M) \subset M$ denote the set of all torsion elements and let $m_1, m_2 \in T(M)$. Then there exist $a_1, a_2 \in A \setminus \{0\}$ such that $a_i m_i = 0$ for $i = 1, 2$. Then for any $a \in A$ we have

   $$a_1 a_2 (m_1 + m_2) = a_2(a_1 m_1) + a_1(a_2 m_2) = 0$$

   and

   $$a_1(am_1) = a(a_1 m_1) = 0.$$

   Since A is an integral domain, $a_1 a_2 \neq 0$ and therefore $m_1 + m_2,\ am_1 \in T(M)$ which shows that $T(M)$ is a submodule of M.

   (b) Suppose that $m \in T(M/T(M))$. Let $\tilde{m}$ be any lift of m to M. Then there exists some $a \in A \setminus \{0\}$ such that $am = 0$, or equivalently, such that $a\tilde{m} \in T(M)$. Then there exists $b \in A \setminus \{0\}$ such that $ba\tilde{m} = 0$ in M. Since A is an integral domain, $ba \neq 0$ so that $\tilde{m} \in T(M)$. Therefore, $m = 0$ and we are done.

   (c) Let $A = \mathbb{C}[x, y]$. Considering $M = (x, y)$ as an A module, it is not hard to see that A is torsion free: this follows directly from the fact that A is an integral domain and $M \subset A$. However M is not free since we have a nontrivial relation between the generators x, y; namely $x(y) - y(x) = 0$ and since M is not generated by any single element of $\mathbb{C}[x, y]$. Now consider the A module

   $$M = A \oplus A/(x).$$

   It is clear that $N = A/(x)$ considered as an A submodule of M by the obvious inclusion is the torsion submodule of M and is a proper submodule.

5. Let A be a Euclidean ring and let M be a finitely generated torsion free A module. Let $S = \{m_i\}_{i=1}^{s}$ generate M and without loss of generality suppose that $m_1, \ldots, m_n$ is a maximal linearly independent subset of S. Let F be the free A module generated by $m_1, \ldots, m_n$.

   (a) By our choice of m_j for $1 \leq j \leq n$, for any $n + 1 \leq i \leq s$ the set $m_1, \ldots, m_n, m_i$ is linearly dependent. We therefore have a nontrivial relation

   $$\sum_{j=1}^{n} a_j m_j + a_i m_i = 0,$$

   where not all a_j are zero. In particular, $a_i \neq 0$ since $m_1, \ldots, m_n$ are linearly independent. It follows that

   $$a_i m_i \in F.$$

   (b) Set $a = a_{n+1} \cdots a_s \in A$. Since each $a_i \neq 0$ and A is an integral domain, we have $a \neq 0$. Define $\phi_a : M \to M$ by $\phi_a(m) = am$. Since $a \neq 0$ and M is torsion free, ϕ_a is injective. Moreover, any $m \in M$ has the form

   $$m = \sum_{j=1}^{n} b_j m_j + \sum_{i=n+1}^{s} b_i m_i$$

   for some $b_l \in A$. We have

   $$am = a \sum_{j=1}^{n} b_j m_j + \sum_{i=n+1}^{s} (b_i a_{n+1} \cdots \hat{a}_i \cdots a_s)(a_i m_i),$$

where $\hat{a}_i$ denotes the omission of a_i from the product. Since $a_i m_i \in F$ by construction, it follows that $am \in F$ and hence that $\operatorname{Im} \phi_a \subset F$.

(c) Since ϕ_a is injective is gives an isomorphism

$$\phi_a : M \to \operatorname{Im} \phi_a \subset F.$$

Since F is free and finitely generated, it follows that $\operatorname{Im} \phi_a$ and hence M is also free.