Math 593 Assignment # 6 Solutions

Bryden R. Cais

November 8, 2002

1. Let k be a field and A_i a finite dimensional vector space over k. Consider the exact sequence:

$$0 \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} 0$$

The familiar rank nullity theorem tells us that

$$\dim \operatorname{im} f_i + \dim \ker f_i = \dim A_i.$$

However, since the sequence is exact we have $\operatorname{im} f_i = \ker f_{i+1}$ so that

$$\dim A_i = \dim \operatorname{im} f_i + \dim \operatorname{im} f_{i-1}$$

for $0 \leq i \leq n$. Hence

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i \dim A_i &= \sum_{i=0}^{n}(-1)^i \left(\dim \operatorname{im} f_i + \dim \operatorname{im} f_{i-1}\right) \\
&= \sum_{i=0}^{n}(-1)^i \dim \operatorname{im} f_i + \sum_{i=0}^{n}(-1)^i \dim \operatorname{im} f_{i-1} \\
&= \sum_{i=0}^{n}(-1)^i \dim \operatorname{im} f_i - \sum_{i=0}^{n}(-1)^i \dim \operatorname{im} f_i + (\dim \operatorname{im} f_{-1} - \dim \operatorname{im} f_n) \\
&= (\dim \operatorname{im} f_{-1} - \dim \operatorname{im} f_n) \\
&= 0,
\end{aligned}$$

since the image of 0 under f_{-1} is 0 and the image of A_n under f_n is 0.

2. (a) Suppose that the sequence

$$0 \longrightarrow N \xrightarrow{u} M \xrightarrow{v} L \longrightarrow 0 \tag{1}$$

splits, that is, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "u"] \arrow[d, "\phi_N"'] & M \arrow[r, "v"] \arrow[d, "\varphi"'] & L \arrow[r] \arrow[d, "\phi_L"] & 0 \\
0 \arrow[r] & N \arrow[r, "\iota_N"'] & N \oplus L \arrow[r, "\pi_L"'] & L \arrow[r] & 0
\end{tikzcd} \tag{2}$$

where $\iota_N(n) = (n, 0)$ is the natural inclusion map, $\pi_L(n, l) = l$ the natural projection map, and ϕ_N, ϕ_L, φ are isomorphisms. Define

$$j : M \longrightarrow N$$

by

$$j := \phi_N^{-1} \circ \pi_N \circ \varphi,$$

where $\pi_N(n, l) = n$ is the natural projection map (and ϕ_N^{-1} is well defined since ϕ_N is an isomorphism). It is obvious that j is a homomorphism since it is the composition of homomorphisms. Then we have

$$j \circ u = \phi_N^{-1} \circ \pi_N \circ \varphi \circ u = \mathrm{id}_N$$

since the diagram above is commutative. Observe that we need not assume that $\phi_N = \mathrm{id}_N$.
Similarly, define

$$s : L \longrightarrow M$$

by

$$s := \varphi^{-1} \circ \iota_L \circ \phi_L,$$

where $\iota_L(l) = (0, l)$ is the natural inclusion map. Again, it is obvious that s is a homomorphism and we have

$$v \circ s = v \circ \varphi^{-1} \circ \iota_L \circ \phi_L = \mathrm{id}_L$$

since the diagram commutes. This shows that (a) implies (b) and (c).

(b) Suppose that there is a homomorphism $j : M \longrightarrow N$ such that $j \circ u = \mathrm{id}_N$. Define

$$\varphi : M \longrightarrow N \oplus L$$

by

$$\varphi(m) = (j(m), v(m)).$$

Since j, v are homomorphisms, it is clear that φ is a homomorphism. Now suppose that $\varphi(m) = 0$. Then $j(m) = (0, 0)$ so that $m \in \ker j = \operatorname{im} u$ by exactness. It follows that $m = u(n)$ for some $n \in N$, and hence that $j(m) = j \circ u(n) = n = 0$. Therefore, $m = u(0) = 0$ so that φ is injective. Let $(n, l) \in N \oplus L$ be arbitrary. Since v is surjective, there exists $m' \in M$ such that $v(m') = l$. Set

$$m = u(n) + m' - u \circ j(m').$$

Then

$$v(m) = v(u(n) + m' - u \circ j(m')) = v(m') = l$$

since $\operatorname{im} u = \ker j$, and

$$j(m) = j(u(n) + m' - u \circ j(m')) = j \circ u(n) + j(m') - j \circ u \circ j(m') = n$$

since $j \circ u = \mathrm{id}_N$. Surjectivity of φ follows. Thus, φ is an isomorphism. Taking the obvious maps $\phi_N = \mathrm{id}_N$ and $\phi_L = \mathrm{id}_L$, it is easy to see that the diagram (2) is commutative and hence that the sequence (1) splits.
Similarly, suppose that we have a homomorphism $s : L \longrightarrow M$ such that $v \circ s = \mathrm{id}_L$. Define

$$\psi : N \oplus L \longrightarrow M$$

by

$$\psi(n,l) = u(n) + s(l).$$

Since u, s are homomorphisms, it is clear that ψ is a homomorphism. Now suppose that $\psi(n,l) = 0$. Then certainly $v(u(n) + s(l)) = l = 0$ since $\ker v = \operatorname{im} u$ and $v \circ s = \mathrm{id}_L$ so that $u(n) = 0$. Since u is injective, it follows that $(n,l) = (0,0)$. Now let $m \in M$ be arbitrary. Observe that

$$m = (m - s \circ v(m)) + s \circ v(m)$$

and that

$$v(m - s \circ v(m)) = v(m) - v \circ s \circ v(m) = v(m) - v(m) = 0$$

since $v \circ s = \mathrm{id}_L$. It follows that $(m - s \circ v(m)) \in \ker v = \operatorname{im} u$ so there exists $n \in N$ such that $u(n) = (m - s \circ v(m))$. Then set $l = v(m)$ whence $m = \psi(n,l)$ and ψ is surjective. Thus, ψ is an isomorphism and we let $\varphi = \psi^{-1}$, $\phi_N = \mathrm{id}_N$ and $\phi_L = \mathrm{id}_L$. As before, it is easy to see that the diagram (2) is commutative and hence that the sequence (1) splits.

We have thus shown that (b) and (c) imply (a). This completes the proof.

3. (a) Let F be a free R module with basis $\{x_i\}_{i \in I}$, and N an arbitrary R module. Let $n_i \in N$ for $i \in I$. Because F is free, every element $f \in F$ has a *unique* representation

$$f = \sum_{i \in I} a_i x_i \tag{3}$$

for $a_i \in R$. Define

$$\phi : F \longrightarrow N$$

by $\phi(x_i) = n_i$ and extend R-linearity to all of F. By definition, ϕ is a R module homomorphism. Moreover, the uniqueness of (3) guarantees that ϕ is well defined. Since $\{x_i\}$ is a basis for F, any homomorphism $F \to N$ agreeing with ϕ on $\{x_i\}$ agrees with ϕ on all of F. Hence ϕ is unique.

(b) By part (a), there is a unique homomorphism

$$\phi : F \to \bigoplus_{i \in I} R$$

given by $\phi(x_i) = (0, 0, \ldots, 0, 1, 0, \ldots)$ where the 1 is i^{th} position. (OK, we are supposing that I is countable.) If

$$\phi\left(\sum_{i \in I} a_i x_i\right) = (0, 0 \ldots),$$

then we clearly have $a_i = 0$ for all i. This shows that ϕ is injective. Surjectivity is also clear: if $a = (a_i)_{i \in I} \in \oplus_{i \in I} R$ then $\phi(\sum_{i \in I} a_i x_i) = a$. Thus ϕ is an isomorphism.

(c) Suppose that $M \supset N$ are R modules and $M/N \simeq F$ is free. Then we have an exact sequence

$$0 \longrightarrow N \xrightarrow{\ u\ } M \xrightarrow{\ v\ } F \longrightarrow 0. \tag{4}$$

Let $\{x_i\}$ be a basis for F. Since v is surjective, there exist $\{m_i\}_{i \in I}$ such that $v(m_i) = x_i$. Since F is free, by part (a) we have a unique homomorphism $s : F \to M$ given by $s(x_i) = m_i$. It is clear that $v \circ s = \mathrm{id}_L$, so that by problem 2, the sequence (4) splits and $M \simeq F \oplus N$, as desired.

4. (a) Let P be a free module, and suppose we have an exact sequence $M \to N \to 0$ together with a homomorphism $s : P \to N$. Let $\{x_i\}$ be a generating set for P. Since u is surjective, for each i there exists $m_i \in M$ such that $u(m_i) = x_i$. By part (a) of problem 3, there exists a unique homomorphism

$$\bar{s} : P \longrightarrow M$$

given by $\bar{s}(x_i) = m_i$. It follows that the diagram

$$\begin{tikzcd} & P \arrow[dl, "\bar{s}"'] \arrow[d, "s"] & \\ M \arrow[r, "u"] & N \arrow[r] & 0 \end{tikzcd}$$

is commutative, and hence that P is projective.

(b) Suppose that P is projective and we have an exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$$

Then since P is projective, there exists a homomorphism $\bar{s}$ such that the following diagram is commutative:

$$\begin{tikzcd} & & & P \arrow[dl, "\bar{s}"'] \arrow[d, "\mathrm{id}_P"] & \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r, "u"] & P \arrow[r] & 0 \end{tikzcd}$$

We now apply problem 2(c), from which it follows that any such exact sequence splits. Thus (a) implies (b).

Next, assume (b) and recall that every module is the quotient of some free module. We therefore have an exact sequence

$$0 \longrightarrow Q \longrightarrow F \longrightarrow P \longrightarrow 0,$$

where F is free. Since we are assuming that any such sequence splits, we have an isomorphism

$$F \longrightarrow P \oplus Q,$$

that is, P is a direct summand of a free module. Thus (b) implies (c).

Finally, suppose (c) and let $M \xrightarrow{u} N \longrightarrow 0$ be exact. Suppose that we have a homomorphism $s : P \to N$. Let $\pi_P : P \oplus Q \to P$ be the natural projection map. Since there exists a module Q such that $P \oplus Q$ is free, by part (a) of this problem, we have a homomorphism $\bar{s}'$ such that the diagram

$$\begin{tikzcd}
 & P \oplus Q \arrow[dl, "\bar{s}'"'] \arrow[d, "\pi_P"] & \\
 & P \arrow[d, "s"] & \\
M \arrow[r, "u"] & N \arrow[r] & 0
\end{tikzcd}$$

commutes. We therefore have a map $\bar{s} : P \longrightarrow M$ given by

$$\bar{s} = \bar{s}' \circ \iota_P$$

where $\iota_P : P \longrightarrow P \oplus Q$ is the natural inclusion map, such that

$$s = u \circ \bar{s}.$$

It follows that P is projective and hence (c) implies (a).

5. Until next time ...