Math 593 Assignment # 5 Solutions

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October 23, 2002

1. The $k[x]$ module
$$M = \frac{k[x]}{(x^3+x+1)} \oplus \frac{k[x]}{x^2}$$
is given to us in primary decomposition form. As such, we can read off the rational canonical form of the matrix T: it has two blocks corresponding to the two summands, and each block is the companion matrix of the polynomial in the "denominator" of the summand. That is, we have
$$T = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$
up to a change of basis of $V = k^5$.
What is going on is simple. Let $V = \kappa^5$. Then we know that V has standard basis e_i for $1 \leq i \leq 5$. Observe that M has basis
$$f_1 = (1,0),\ f_2 = (x,0),\ f_3 = (x^2,0),\ f_4 = (0,1),\ f_5 = (0,x).$$
We then have an isomorphism (because we are sending a basis to a basis)
$$M \longrightarrow V$$
given by
$$f_i \mapsto e_i.$$
To determine T we simply need to see what multiplication by x does to the basis f_i. We have
$$\begin{aligned} xf_1 &= (x,0) = f_2 \\ xf_2 &= (x^2,0) = f_3 \\ xf_3 &= (-x-1,0) = -f_2 - f_1 \\ xf_4 &= (0,x) = f_5 \\ xf_5 &= (0,x^2) = 0. \end{aligned}$$
It follows that multiplication by x on M has the matrix T above.

2. (a) Let ρ be a representation of G on V. We make V into a $k[G]$ module by defining

$$gv := \rho(g)v$$

for any $g \in G$ and $v \in V$ and extending by linearity to all of $k[G]$. Clearly V is an abelian group under $+$. It is easy to verify that

i. $1v = \rho(1)v = v$ since $\rho(1)$ is the identity matrix (since ρ is a homomorphism).
ii. $(gh)v = \rho(gh)v = \rho(g)\rho(h)v = \rho(g)(hv) = g(hv)$, since ρ is a homomorphism.
iii. $(g+h)v = (\rho(g)+\rho(h))v = \rho(g)v + \rho(h)v = gv + hv$
iv. $g(v+w) = \rho(g)(v+w) = \rho(g)v + \rho(g)w = gv + gw$, since $\rho(g)$ is a linear transformation.

Hence, we have made V into a $k[G]$ module.

(b) Now suppose that V has the structure of a $k[G]$ module where scalars $\lambda \in k[G]$ act by scalar (vector space) multiplication. Define $\rho : G \to \mathrm{GL}(V)$ by

$$\rho(g)v := gv$$

for any $g \in G, v \in V$. We must check that ρ defines a homomorphism $\rho : G \to \mathrm{GL}(V)$. Observe that

$$\rho(gh)v := (gh)v = g(hv) = g(\rho(h)v) = \rho(g)\rho(h)v$$

since V is a $k[G]$ module. Moreover, since

$$\rho(1g)v = \rho(1)\rho(g)v = \rho(g)\rho(1)v = \rho(g)v$$

for all $v \in V$, we have $\rho(1) = 1$. It follows that ρ is a homomorphism. The usual axioms for a $k[G]$ module imply that $\rho(g)$ is a linear transformation of V. Since

$$\rho(g)\rho(g^{-1}) = \rho(1) = 1$$

we see that $\rho(g)$ is an invertible linear transformation. It follows that ρ is a representation of G on V. The fact that $V \simeq V_\rho$ as $k[G]$ modules follows from our definition in part a).

3. Let k be a field and $f, g \in k[x]$ nonzero polynomials. For any $p, q \in k[x]$, we let $\bar{p}_q$ denote the image of p in $k[x]/(q)$ under the natural quotient map π_q. Let

$$R = \mathrm{Hom}_{k[x]}\left(k[x]/f, k[x]/g\right),$$

and suppose that $\phi \in R$. Let $\tilde{\phi}$ be any lift of ϕ to an element of $\mathrm{Hom}_{k[x]}\left(k[x], k[x]\right)$; that is, $\tilde{\phi}$ is any $k[x]$ module homomorphism making the following diagram commute:

$$\begin{array}{ccc} k[x] & \xrightarrow{\tilde{\phi}} & k[x] \\ {\scriptstyle \pi_f}\downarrow & & \downarrow{\scriptstyle \pi_g} \\ k[x]/(f) & \xrightarrow[\phi]{} & k[x]/(g) \end{array}$$

Then we have

$$\tilde{\phi}(f) = f\tilde{\phi}(1),$$

but since

$$\phi \circ \pi_f(f) = \phi(\bar{0}_f) = \bar{0}_g,$$

we must have

$$\pi_g(f\tilde{\phi}(1)) = \bar{0}_g,$$

or what is the same, $g|f\tilde{\phi}(1)$. Since k is a field, $k[x]$ is a PID (therefore also a UFD), so putting

$$(g, f) = (h)$$

we see that

$$g_1 = g/h$$

divides $\tilde{\phi}(1)$. Now define the map

$$\Pi : k[x] \longrightarrow R$$

by

$$p \mapsto \phi_p$$

where ϕ_p is defined by

$$\phi_p(\bar{1}_f) = \bar{p}_g \bar{g_1}_g.$$

(Realize that an element of R is completely determined by where it sends $\bar{1}_f$). It is not difficult to check that Π is a $k[x]$ module homomorphism. Indeed,

(a)

$$\begin{aligned}\Pi(p+q)(\bar{1}_f) &= \phi_{p+q}(\bar{1}_f)\\ &= (\bar{p}_g + \bar{q}_g)\bar{g_1}_g\\ &= \bar{p}_g\bar{g_1}_g + \bar{q}_g\bar{g_1}_g\\ &= (\Pi(p) + \Pi(q))(\bar{1}_f),\end{aligned}$$

so that $\Pi(p+q) = \Pi(p) + \Pi(q)$.

(b)

$$\begin{aligned}\Pi(pq)(\bar{1}_f) &= \phi_{pq}(\bar{1}_f)\\ &= \bar{p}_g\bar{q}_g\bar{g_1}_g\\ &= \bar{p}_g\Pi(q)(\bar{1}_f)\end{aligned}$$

so that $\Pi(pq) = p\Pi(q)$ (modulo g, of course, but that is all we care about).

That Π is surjective follows from our opening remarks. Now $\Pi(p) = 0$ if and only if $\Pi(p)(\bar{1}_f) = \bar{0}_g$, that is, if and only if $g|pg_1$, or equivalently, if and only if $h|p$. Therefore, $\ker \Pi = (h)$ and we have

$$k[x]/(h) \simeq \mathrm{Hom}_{k[x]}\left(k[x]/(f), k[x]/(g)\right).$$

4. (a) Let $V = \mathbb{R}^2$ and $T_\theta : V \to V$ be rotation by θ. Then V_θ is irreducible if and only if $\theta \notin \pi\mathbb{Z}$. First suppose that that $\theta \notin \pi\mathbb{Z}$. Then $\sin(\theta) \neq 0$ so that for any nonzero $v \in V$, the magnitude of $v \times T_\theta v$ is not zero; that is, $v, T_\theta v$ form the sides of a parallelogram in $\mathbb{R}^2$ with nonzero area. As such, $v, T_\theta v$ span $\mathbb{R}^2$. (There are lots of ways of saying this; the idea is that as long as θ is not a multiple of π, the vectors $v, T_\theta v$ are nor multiples of each other). Let $W \subset V$ be a T_θ invariant nontrivial subspace and let $w \in W$ be nonzero. Then $T_\theta w \in W$ so that W contains a basis for V and hence $W = V$ so that V is irreducible. Conversely, if $\theta = n\pi$ for some $n \in \mathbb{Z}$ then $T_\theta v = \pm v$ for any $v \in V$; as such V_θ has the nontrivial proper submodules $M_v = \{\lambda v : \lambda \in \mathbb{R}\}$ for all $v \neq 0$ in V and is thus reducible.

   (b) A nontrivial submodule of V_ρ is a nontrivial subspace $W \subset V$ that is preserver by the action of G. Therefore, V is irreducible if and only if for every $W \subset V$, there exists some $g \in G$ such that $\rho(g)W \not\subseteq W$. Equivalently, V is irreducible if and only if the G orbit of any nonzero vector $v \in V$ spans V (if the orbit doesn't span V then it spans a proper, nontrivial subspace W, which is then preserved by the action of G).

   (c) V is reducible as a $k[S_3]$ module. To see this, let $W \subset V$ be the subspace given by

$$W = \{(a, a, a) : a \in k\}.$$

   Clearly this is a nontrivial proper subspace of V. Moreover, W is S_3 invariant since S_3 acts by permuting the components of any vector $v \in V$. It follows that W is a nontrivial, proper $k[S_3]$ submodule of V.

5. (a) Suppose that M is an irreducible R module. Let $\phi \in \mathrm{End}_R(M)$ and suppose that $\phi \neq 0$. Then since $\ker\phi \subset M$ is a submodule, $\ker\phi = 0$ or M; since $\phi \neq 0$, ϕ is then injective. Hence ϕ is an isomorphism and therefore invertible; that is, $\mathrm{End}_R(M)$ is a division ring. Notice that we really do need to show that ϕ is both injective and surjective: if M were finite, one implies the other, but in general injectivity of a map $\phi : M \to M$ does *not* imply surjectivity. For example, the map $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n) = 2n$ is certainly injective, but definitely not surjective.

   (b) There are many ways of seeing this. The intuition is that T acts just like $i = \sqrt{-1}$ and gives R^2 complex structure. Moreover, since x acts by T, $\mathbb{R}[x]$ is like $\mathbb{C}$. It is clear that the $\mathbb{C}$ linear endomorphisms of $\mathbb{C}$ is isomorphic to $\mathbb{C}$ since any such endomorphism is completely determined by where the identity is sent, and it can be sent anywhere. To make this intuition into a proof, we proceed as follows:

   We take the standard basis e_1, e_2 for V. In this basis, any endomorphism ov V is given as a 2×2 matrix with entries in $\mathbb{R}$. Suppose $\varphi \in \mathrm{End}_{\mathbb{R}[x]}(V_T)$ and let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   be the matrix of φ. Since φ must commute with T, and since the matrix of T is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

   a short calculation shows that we must have

$$\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix},$$

that is, $b = -c, a = d$. It follows that every $\varphi \in \mathrm{End}_{\mathbb{R}[x]}(V_T)$ has the form

$$a + bT$$

for some $a, b \in \mathbb{R}$. Therefore, we define

$$\phi : \mathbb{C} \longrightarrow \mathrm{End}_{\mathbb{R}[x]}(V_T)$$

by

$$\phi(a + bi) = a + bT.$$

By our remarks above, ϕ is surjective. Injectivity is also clear. That ϕ is a homomorphism follows from a short calculation:

$$\phi((a + bi)(c + di)) = \phi(ac - bd + (ad + bc)i) = (ac - bd) + (ad + bc)T,$$

on the other hand,

$$\phi(a + bi)\phi(c + di) = (a + bT)(c + dT) = ac + adT + bcT + bdT^2 = (ac - bd) + (ad + bc)T$$

since $T^2 = -1$. It is obvious that ϕ preserves addition and that $\phi(1) = 1$. Since ϕ is a bijective ring homomorphism, it is an isomorphism, and we are done.