Math 593 Assignment # 3 Solutions

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1. (a) $\mathbb{Z}(x)$, the field of rational functions in x.
   (b) $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$ is a field.
   (c) $\mathbb{C}(x, y) = \mathbb{C}(x)(y)$.

2. Let F denote the field of fractions of $R = k[[x]]$ and for any $f \in k((x))$ let $v(f)$ be the unique integer such that $x^{v(f)} f = c_0 + c_1 x + \cdots$. Define

$$\phi : k((x)) \longrightarrow F$$

by

$$\phi(f) = \begin{cases} \frac{f}{1} & \text{if } f \in k[[x]] \\ x^{v(f)} f / x^{v(f)} & \text{otherwise} \end{cases}.$$

It is clear that ϕ is an injective homomorphism. To show surjectivity, let $f/g \in F$. Then $x^{v(g)} g$ is a unit in $k[[x]]$ (from lecture) so there exists $h \in k[[x]]$ such that $gh = 1$. Thus,

$$\frac{f}{g} = x^{v(g)} fh.$$

Clearly $fh \in k[[x]]$ and since $v(g) \in \mathbb{Z}$, f/g is in the image of ϕ.

3. For any maximal ideal $\mathfrak{M}$, it is clear that $1 \notin \mathfrak{M}$ and hence $a = a/1 \in A_{\mathfrak{M}}$ for any $a \in A$ and any $\mathfrak{M}$. It follows that

$$A \subset \bigcap_{\substack{\mathfrak{M} \subseteq A \\ \mathfrak{M} \text{ maximal}}} A_{\mathfrak{M}}.$$

For the reverse inclusion, let $x \in A_{\mathfrak{M}}$ for all maximal $\mathfrak{M} \subset A$. Then we can write $x = a_{\mathfrak{M}} / b_{\mathfrak{M}}$ for each $\mathfrak{M}$ with $a_{\mathfrak{M}} \in A, b_{\mathfrak{M}} \in A \setminus \mathfrak{M}$. Observe that the ideal B generated by the $b_{\mathfrak{M}}$ is all of A, for if not, then $B \subset \mathfrak{M}_0$ for some maximal ideal $\mathfrak{M}_0$, but $b_{\mathfrak{M}_0} \in B$ and $b_{\mathfrak{M}_0} \notin \mathfrak{M}_0 \supset B$, a contradiction. Hence we can write

$$1 = \sum_{\mathfrak{M} \in T} c_{\mathfrak{M}} b_{\mathfrak{M}}$$

for some finite set T where $c_{\mathfrak{M}} \in A$. It follows that

$$x = \sum_{\mathfrak{M} \in T} x c_{\mathfrak{M}} b_{\mathfrak{M}} = \sum_{\mathfrak{M} \in T} c_{\mathfrak{M}} a_{\mathfrak{M}} \in A,$$

and we are done.

4. (a) We use the result from class that a ring R is local with unique maximal ideal $\mathfrak{M}$ if any $f \in R \setminus \mathfrak{M}$ is a unit. So let $\mathfrak{M} \subset \mathcal{C}_0$ be the ideal of germs of functions vanishing at 0 (this is obviously an ideal). If $[(U, f)] \in \mathcal{C}_0 \setminus \mathfrak{M}$, then $f(0) \neq 0$ so that by continuity, there exists a neighborhood $V \subset U$ of 0 with f nonvanishing on V. Then $[(V, 1/f)] \in \mathcal{C}_0 \setminus \mathfrak{M}$ and

$$[V, 1/f][U, f] = [V, 1] = \mathrm{id} \in \mathcal{C}_0,$$

so f is invertable, as required.

(b) Now let $\mathcal{C}(\mathbb{R})$ denote the ring of all continuous functions on $\mathbb{R}$ and let $\mathfrak{M} \subset \mathcal{C}(\mathbb{R})$ be the maximal ideal of all functions vanishing at 0. Define

$$\varphi : \mathcal{C}(R)_{\mathfrak{M}} \longrightarrow \mathcal{C}_0$$

by

$$\varphi([f/g]) = [U_g, f/g],$$

where $[f/g]$ denotes the equivalence class of f/g and U_g is a neighborhood of 0 on which g does not vanish (such a neighborhood always exists since $g(0) \neq 0$ and g is continuous). We claim that φ is an isomorphism. There are several things to check:

i. **The map φ is well defined:** That is, we must show that it does not depend on the choice of representative f/g. So suppose that $f/g, h/k$ are any two representatives of the same equivalence class in $\mathcal{C}(\mathbb{R})$. Then there exists a continuous function $s \notin \mathfrak{M}$ such that $s(fk - gh) \equiv 0$ on $\mathbb{R}$. Since $s \notin \mathfrak{M}$, there exists a neighborhood U of 0 such that s does not vanish on U. It follows that $fk - gh$ must vanish on U and hence that $f/g = k/h$ on $U \cap U_g \cap U_h$ so that $[(U_g, f/g)] = [(U_h, k/h)]$, so φ is well defined.

ii. **Injectivity:** Suppose that $\varphi([f/g]) = \varphi([k/h])$. Then there exists some neighborhood U of 0 such that $f/g - k/h \equiv 0$ on U. Now there exists some $\epsilon > 0$ with $(-\epsilon, \epsilon) \subset U$. Let

$$\chi_\epsilon(t) = \begin{cases} 0 & \text{if } |t| > \epsilon \\ 1 & \text{if } |t| < \epsilon/2 \\ 2\left(1 - \frac{|t|}{\epsilon}\right) & \text{otherwise} \end{cases}.$$

Then clearly $\chi_\epsilon \in \mathcal{C}(\mathbb{R}) \setminus \mathfrak{M}$ and

$$\chi_\epsilon(fh - gk) \equiv 0$$

on all of $\mathbb{R}$, in other words, $[f/g] = [k/h]$ in $\mathcal{C}(\mathbb{R})_{\mathfrak{M}}$, as required.

iii. **Surjectivity:** Let $[(U, f)] \in \mathcal{C}_0$. Then for some $\epsilon > 0$ we have $(-\epsilon, \epsilon) \subsetneq U$. Now define

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \in (-\epsilon, \epsilon) \\ f(\epsilon) & \text{if } t \geq \epsilon \\ f(-\epsilon) & \text{if } t \leq -\epsilon \end{cases}.$$

Then it is not hard to see that $\tilde{f} \in \mathcal{C}(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})_{\mathfrak{M}}$ and that $\varphi([\tilde{f}]) = [(U, f)]$.

iv. **The map φ is a homomorphism:** This follows directly from the definition of addition and multiplication in $\mathcal{C}_0$.