Math 593 Assignment # 2 Solutions

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2. Let A be a commutative ring with unit and let $I, J \subset A$ be ideals with $I + J = A$.

   (a) Since $I + J = A$ and $1 \in A$, there exist $i \in I$ and $j \in J$ such that $i + j = 1$. Now let $a, b \in A$ and consider
   $$x = aj + bi \in A.$$
   We have $x \equiv bi \equiv b \mod J$ since $i \equiv 1 \mod J$. Similarly, $x \equiv a \mod I$.

   (b) By part (a), we have a surjective map
   $$A \longrightarrow A/I \oplus A/J$$
   given by
   $$a \mapsto (a \mod I, a \mod J).$$
   This is obviously a homomorphism. An element a of A is in the kernel of this map if and only if $a \equiv 0 \mod I$ and $a \equiv 0 \mod J$, that is, $a \in I$ and $a \in J$. In other words, we have an exact sequence
   $$0 \longrightarrow I \cap J \longrightarrow A \longrightarrow A/I \oplus A/J \longrightarrow 0.$$
   Therefore,
   $$A/(I \cap J) \simeq A/I \oplus A/J,$$
   and if $I \cap J = 0$, then $A \simeq A/I \oplus A/J$.

   Remark 1. This is just a special case of the **Chinese Remainder Theorem**.

3. (a) From class, we know that $\mathbb{Z}$ is a PID, so that every ideal is of the form $n\mathbb{Z}$. If $0 \neq n = xy$ is composite (with $1 < x, y < n$) then $x, y \notin (n)$ since $n \not| x, y$ but $xy = n \in (n)$. Therefore (n) is not prime. Conversely, if $p \in \mathbb{Z}$ is a prime and $xy \in (p)$ then $p|xy$ so that $p|x$ or $p|y$; that is, $x \in (p)$ or $y \in (p)$. In short, (p) is prime. Since $\mathbb{Z}$ is an integral domain, by part d, (0) is prime.

   (b) Let $M \subset R$ be a maximal ideal and suppose that $ab \in M$ and $a \notin M$. Then $M + (a)$ is an ideal of R properly containing M. It follows that we can write
   $$1 = m + ax$$
   for some $m \in M$ and $x \in R$. Therefore, $b = mb + abx \in M$ since $ab \in M$. It follows that M is prime.

Remark 2. Of course, we could just observe that R/M is a field and hence an integral domain, and then apply part (e).

(c) The ideal (x) is prime since x is irreducible. Moreover, $(x) \subsetneq (x, y) \subsetneq \mathbb{Q}[x, y]$ so that (x) is not maximal.

Remark 3. If you want to use overkill, observe that (x) is the kernel of the homomorphism

$$\varphi : \mathbb{Q}[x, y] \longrightarrow \mathbb{Q}[y]$$

given by

$$p(x, y) \mapsto p(0, y).$$

We therefore have an exact sequence

$$0 \longrightarrow (x) \longrightarrow \mathbb{Q}[x, y] \longrightarrow \mathbb{Q}[y] \longrightarrow 0,$$

and since $\mathbb{Q}[y]$ is an integral domain, it follows that $\mathbb{Q}[x, y]/(x) \simeq \mathbb{Q}[y]$ is an integral domain and hence, by part (e), that (x) is prime.

(d) This follows from part (e) since $R/(0) \simeq R$.

(e) We have that R/P is an integral domain if and only if for any $x, y \in R/P$, $xy = 0$ implies $x = 0$ or $y = 0$. Let $\tilde{x}, \tilde{y}$ be any lifts of x, y to R. Then R/P is an integral domain if and only if $\tilde{x}\tilde{y} \in P$ implies $\tilde{x} \in P$ or $\tilde{y} \in P$, i.e. if and only if P is prime.

4. Since parts (a) and (b) follow from (c), we prove only (c). Let $I \subset R$ be an ideal and let $\sqrt{I}$ be the radical of I. Let $x, y \in \sqrt{I}$. Then there exist m, n such that $x^m, y^n \in I$. Set $N = m + n$ and consider

$$(x + y)^N = \sum_{i=0}^{N} \binom{N}{i} x^{N-i} y^i,$$

(by the binomial theorem, which holds in any commutative ring). Observe that for any $0 \leq i \leq N$, either $N - i \geq m$ or $i \geq n$. Therefore, every summand is in I (since I is a two sided ideal of R) and hence $(x + y)^N \in I$. It follows that $x + y \in \sqrt{I}$.

Remark 4. Notice that we don't need R to be commutative for this to hold. All we need is that $xy = cyx$ for come c in the center of R when considered as a group under multiplication.

Now let $r \in R, x \in \sqrt{I}$. Then for some n, we have $x^n \in I$ and hence, since R is commutative,

$$(rx)^n = r^n x^n \in I.$$

It follows that $rx \in \sqrt{I}$ and hence that $\sqrt{I}$ is an ideal.

Let $f \in R/\sqrt{I}$ and suppose that $f^n = 0$. Let $\tilde{f}$ be any lift of f to R. Then $\tilde{f}^n \in \sqrt{I}$, and hence there exists some m such that $\tilde{f}^{nm} \in I$. It follows that $\tilde{f} \in \sqrt{I}$ and hence that $f = 0$. Therefore $R/\sqrt{I}$ is reduced.