MATH 593
Assignment # 7
(Due Monday, November 4)

HAND IN: #'s 2,3,4

NOTE: In fairness to the grader, due dates will henceforth be taken seriously!

(#1). (a). Write down three matrices of integers, and use row and column operations to diagonalize them.

(b). Use row and column operations to diagonalize the matrix:

$$\begin{pmatrix} x & 2 & -14 \\ 0 & x & 7 \\ 0 & 0 & x \end{pmatrix}$$

over $\mathbf{Q}[x]$.

(# 2). Let $R = \mathbf{Q}[x, y]$ be the polynomial ring in two variables, and let $M = (x, y) \subseteq R$ be the the ideal generated by x and y, considered as a module over R. Find a presentation matrix for M with respect to the natural two generators (or other generators if you prefer).

(#3). Let M be the abelian group generated by elements e, f, g subject to the relations

$$3e + 6f + 12g = 0$$
$$2e - 4f + 10g = 0.$$

Express M as a direct sum of cyclic groups.

(# 4)[1]. Let A be an integral domain, and let M be an A-module. Recall that an element $m \in M$ is a *torsion* element if $am = 0$ for some $0 \neq a \in A$.

(i). Prove that the subset $T(M) \subset M$ of all torsion elements is a submodule. It is called the *torsion submodule* of M.

Recall that M is *torsion-free* if $T(M) = 0$.

(ii). Prove that $M/T(M)$ is torsion free.

[1] This was (#5) in the previous assignment, which wasn't required to be handed in. Now I will collect it.

(iii). Let $A = \mathbf{C}[x, y]$. Give an example of a finitely generated torsion-free A-module which is not free. Give an example of an A-module M for which $T(M)$ is non-trivial proper submodule of M.

(#3). Let A be a Euclidean ring and let M be a finitely generated torsion-free A-module. In this exercise, we will sketch a proof of the theorem that M is free.[2] Let $m_1, \ldots, m_s \in M$ be a set of generators. After re-indexing, we may suppose that $\{m_1, \ldots, m_n\}$ are linearly independent, while no larger subset of the $\{m_i\}$ is linearly independent. Denoting by $F = Am_1 + \cdots + Am_n \subset M$ the subset of M generated by $\{m_1, \ldots, m_n\}$, the independence of $\{m_1, \ldots, m_n\}$ means that F is free of rank n.

(i). For each index $n + 1 \leq i \leq s$, show that there exists a non-zero element $0 \neq a_i \in A$ such that $a_i m_i \in F$.

(ii). Let $a = a_{n+1} \cdot \ldots \cdot a_s \in A$, and consider the map $\phi_a : M \longrightarrow M$ given by $\phi_a(m) = am$. Prove that ϕ_a is injective, and that $\mathrm{im}\phi_a \subset F$.

(iii). Using (ii), deduce that M is free.

[2]The same proof will work for torsion-free modules over a PID if you grant that a submodule of a finitely generated free module over a PID is free.