MATH 593
Assignment # 2
(Due Monday, September 23)

HAND IN: #'s 2, 3,4.

(#1). (a). Starting with the field $\mathbf{Z}/3\mathbf{Z}[x]$ of three elements, consider the polynomial $x^2 + 1 \in (\mathbf{Z}/3\mathbf{Z})[x]$. Prove that $F = (\mathbf{Z}/3\mathbf{Z})[x]/(x^2 + 1)$ is a field, and write out the multiplication table in.

(b). By contrast, show that $(\mathbf{Z}/5\mathbf{Z})[x]/(x^2 + 1)$ is not a field.

(#2). Let A be a commutative ring (always with 1), and let $I, J \subset R$ be ideals of A such that $I + J = A$.

(a). Show that given any $a, b \in A$, there exists an element $x \in A$ such that $x \equiv a \pmod I$ and $x \equiv b \pmod J$.

(b). Prove that if in addition $I \cap J = \emptyset$, then

$$A \;\cong\; A/I \,\oplus\, A/J.$$

[Note concerning (b): Given rings R, S, the direct sum $R \oplus S$ (sometimes written as a product $R \times S$) is defined in the expected way: elements of $R \oplus S$ are ordered pairs (r, s) with $r \in R$, $s \in S$, and addition and multiplication is defined componentwise.]

(#3).Let R be a commutative ring (always with 1). An ideal $P \subseteq R$ is *prime* if given $x, y \in R$,

$$xy \in P \implies x \in P \text{ or } y \in P.$$

(a). Show that an ideal $I \subseteq \mathbf{Z}$ is prime iff either $I = (0)$ or $I = (p)$ for a prime number p.

(b). Show that a maximal ideal is prime.

(c). Show that the ideal $(x) \subseteq \mathbf{Q}[x, y]$ is prime but not maximal.

(d). Prove that R is an integral domain iff the zero ideal (0) is prime.

(e). Prove that an ideal $P \subseteq R$ is prime if and only if R/P is an integral domain.

(#4). Let R be a commutative ring (always with 1). An element $f \in R$ is called *nilpotent* if $f^n = 0$ for some $n > 0$. The *nilradical* $N = N(R) \subset R$ of R is the subset consisting of all nilpotent elements. R is *reduced* if $N(R) = 0$.

(a). Prove that $N(R)$ is an ideal.

(b). Show that $R/N(R)$ is reduced.

(c). More generally, let $I \subset R$ be any ideal. The *radical* $\sqrt{I}$ of I is defined as:

$$\sqrt{I} = \{r \in R \mid f^n \in I \text{ for some } n > 0. \}.$$

Prove that $\sqrt{I}$ is an ideal, and that $R/\sqrt{I}$ is reduced.