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11/7/11 Problems p. 305 #2, p. 313 # 11, 13.

Recall: A congruence transformation is a 1-1 function $T: \mathbb{R}^2 \to \mathbb{R}^2$ which preserves distance (isometry): $\forall$ $P, Q \in \mathbb{R}^2$, $d(T(P), T(Q)) = d(P, Q)$

where d = usual Euclidean distance

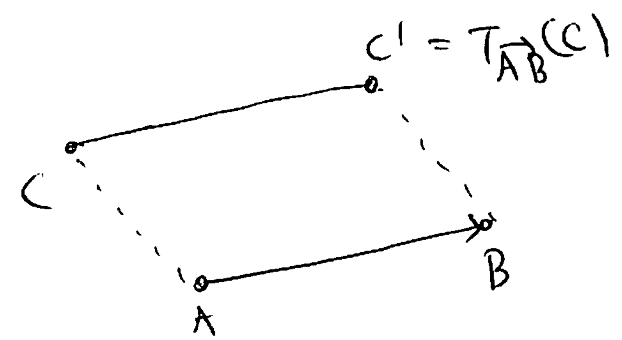
$$d((x,y),(x',y')) = \sqrt{(x-x')^2+(y-y')^2}$$

Today: Study some special types of congruence transformations, using 3) perspectives
a) Synthetic b) Analytic ($\mathbb{R}^2$) c) complex analytic ($\mathbb{C}$).

* Translations

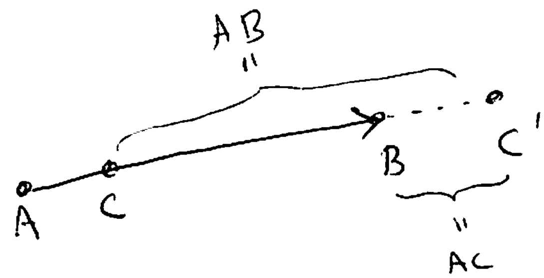
a) Let $\vec{AB}$ be a directed line segment and define $T_{\vec{AB}}$ = translation along $\vec{AB}$ as follows: $T_{\vec{AB}}(C) = C'$, where

i) If C is not on $\vec{AB}$, then C' = unique point s.t. $ABC'C$ is a parallelogram.



ii) If C is on $\overrightarrow{AB}$, then

C' = unique point such that $AB = CC'$ and $AC = BC'$

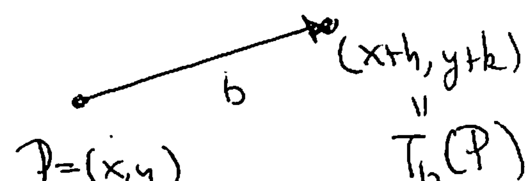


Ex: Prove C', as defined above, really is unique

b) Let b = the vector in $\mathbb{R}^2$ represented by $\overrightarrow{AB}$
$= (h, k)$

Translation along $\overrightarrow{AB}$ $\longleftrightarrow$ addition of b

$$T_b(x,y) := (x+h,\, y+k)$$



Rems: • $T_0(x,y) = (x,y)$, so

$T_0 = id$ is the identity transformation

• We can compose transformations

$$T_{b_2} \circ T_{b_1}(x,y) = T_{b_2}(x+h_1, y+k_1)$$
$$= (x + h_1 + h_2,\, y + k_1 + h_2)$$
$$= T_{b_1+b_2}(x,y) \qquad \forall\, x, y$$

So $\quad T_{b_2} \circ T_{b_1} = T_{b_1+b_2} = T_{b_2+b_1} = T_{b_1} \circ T_{b_2}$

Since $\quad T_{b} \circ T_{-b} = T_{b+(-b)} = T_0 = \mathrm{id}$,

the set $\quad \mathbb{T} := \{ \text{Translations of the plane} \}$

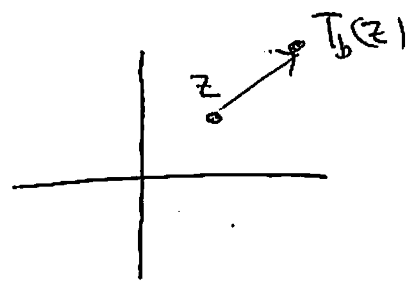
is a <u>commutative</u> group under composition.

Moreover, the map

$$\begin{aligned} (\mathbb{R}^2, +) &\longrightarrow (\mathbb{T}, \circ) \\ b &\longmapsto T_b \end{aligned}$$

is an isomorphism of groups.

c) $\quad b = h + ik \in \mathbb{C}$.



Then $T_b(z) = z + b$

for all $z \in \mathbb{C}$, so

$$\begin{aligned} (\mathbb{C}, +) &\longrightarrow (\mathbb{T}, \circ) \\ b &\longmapsto T_b \end{aligned}$$

is an isom. of gps.

<u>Thm</u>: For all $b \in \mathbb{C}$, $\quad T_b$ is an isometry.

<u>pf</u>: Let $z, w \in \mathbb{C}$. Then $\quad |T_b(z) - T_b(w)| = |(z+b) - (w+b)|$

$$= |z - w|. \qquad \blacksquare$$

* Rotations:

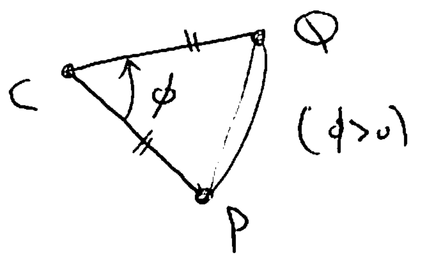
a) C a point in the plane, ϕ a real number

$-\pi < \phi \leq \pi$ (radians)

$(-180° < \phi \leq 180°)$

$R_{C,\phi} :=$ ϕ-rotation about C, defined by

- $R_{C,\phi}(C) = C$
- $R_{C,\phi}(P) = Q$ for Q the unique point s.t.
  - i) $PC = QC$
  - ii) $m\angle PCQ = |\phi|$, where
    - $\triangle PCQ$ is oriented $\underline{CCW}$ if $\phi > 0$
    - $\triangle PCQ$ is oriented $\underline{CW}$ if $\phi < 0$



Can generalize to allow arbitrary ϕ:

$R_{C,\phi}(P) =$ rotate P around C an angle of ϕ in $\begin{Bmatrix} CCW \\ CW \end{Bmatrix}$ direction if $\begin{Bmatrix} \phi > 0 \\ \phi < 0 \end{Bmatrix}$

Then if $\underbrace{\phi \equiv \phi' \bmod 2\pi}_{\text{i.e. } \phi - \phi' = 2\pi n,\ n \in \mathbb{Z}.}$, $R_{C,\phi} = R_{C,\phi'}$

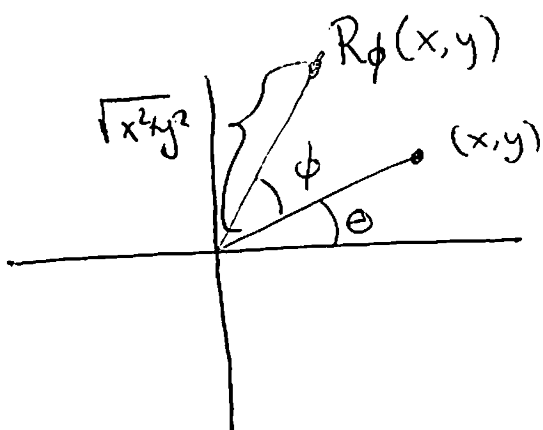
We can compose two rotations w/ same center: $R_{C,\phi_2} \circ R_{C,\phi_1} = R_{C,\phi_1+\phi_2}$

and $R_{C,0} = \mathrm{id}$, so we deduce

$$\mathbb{R}/2\pi\mathbb{Z} = \left\{\begin{array}{l}\text{the group of} \\ \text{reals mod } 2\pi\mathbb{Z}, \\ \text{under } +\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{l}\text{Rotations about} \\ C, \text{ under } \circ\end{array}\right\}$$

$$\phi \longmapsto R_{C,\phi}$$

b) If C = origin, we set $R_\phi := R_{C,\phi}$.



$$(x,y) = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$
$$= \sqrt{x^2+y^2}\,(\cos\theta, \sin\theta)$$

So
$$R_\phi(x,y) = \sqrt{x^2+y^2}\,(\cos(\theta+\phi), \sin(\theta+\phi))$$
$$= \sqrt{x^2+y^2}\,(\cos\theta\cos\phi - \sin\theta\sin\phi,\ \sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$= (x\cos\phi - y\sin\phi,\ y\cos\phi + x\sin\phi)$$

In matrix form,

$$[R_\phi] = \begin{bmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{bmatrix}, \text{ as } R_\phi(x,y) = [R_\phi]\begin{bmatrix}x \\ y\end{bmatrix}$$

More generally, it's easy to see that

$$R_{c,\phi} = T_c \circ R_\phi \circ T_{-c}$$

[pf: Use that $T_c, R_\phi, R_{c,\phi}$ are <u>linear</u> transformations of $\mathbb{R}^2$, so equality may be checked on a basis]

c) $$R_\phi(x+iy) = (x\cos\phi - y\sin\phi) + i(x\sin\phi + y\cos\phi)$$

$$= (x+iy)(\cos\phi + i\sin\phi)$$

$$= z_\phi \cdot (x+iy) \quad \text{for } z_\phi = \cos\phi + i\sin\phi = e^{i\phi}$$

Hence, for $c \in \mathbb{C}$,

$$R_{c,\phi}(z) = z_\phi(z-c) + c$$

In particular,

$$\underbrace{\{z_\phi \in \mathbb{C} \mid |z_\phi| = 1\}}_{\text{under mult}} \xrightarrow[\cong]{\text{isom of gpr.}} \left\{\begin{array}{c}\text{Rotations about } c, \\ \text{under composition}\end{array}\right\}$$

<u>Thm</u> For any c, ϕ, $R_{c,\phi}$ is a congruence transformation.

pf. $$|R_{c,\phi}(z) - R_{c,\phi}(w)| = |(z_\phi(z-c)+c) - (z_\phi(w-c)+c)|$$

$$= |z_\phi(z-w)| \underset{|z_\phi|=1.}{=} |z-w|. \quad \blacksquare$$