Honors Algebra 4, MATH 371 Winter 2010

Assignment 4

Due Wednesday, February 17 at 08:35

1. Let R be a commutative ring with $1 \neq 0$.

   (a) Prove that the nilradical of R is equal to the intersection of the prime ideals of R. Hint: it's easy to show using the definition of prime that the nilradical is contained in every prime ideal. Conversely, suppose that f is not nilpotent and consider the set S of ideals I of R with the property that "$n > 0 \implies f^n \notin I$." Show that S has maximal elements and that any such maximal element must be a prime ideal.

   (b) Suppose that R is *reduced*, *i.e.* that the nilradical of R is the zero ideal. If $\mathfrak{p}$ is a minimal prime ideal of R, show that the localization $R_{\mathfrak{p}}$ has a unique prime ideal and conclude that $R_{\mathfrak{p}}$ is a field.

   (c) Again supposing R to be reduced, prove that R is isomorphic to a subring of a direct product of fields.

2. Let R be a commutative ring with $1 \neq 0$ and let $\varphi : R \to R$ be a ring homomorphism. If R is noetherian and φ is surjective, show that φ must be injective too, and hence an isomorphism. (Hint: Consider the iterates of φ and their kernels.) Can you give a counter-example to this when R is not noetherian?

3. As usual, for a prime p we write $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ for the field with p elements.

   (a) Find all monic irreducible polynomials in $\mathbf{F}_p[X]$ of degree ≤ 3 for $p = 2, 3, 5$.

   (b) Prove that for $f \in \mathbf{F}_p[X]$ monic and irreducible, the ideal $(f(X))$ is maximal and hence that $\mathbf{F}_p[X]/(f(X))$ is a field. Show that $\mathbf{F}_p[X]/(f(X))$ has finite cardinality $p^{\deg f}$ and use part (3a) to explicitly construct finite fields of orders $8, 9, 25, 125$.

   (c) Prove that $\mathbf{F}_7[X]/(X^2+2)$ and $\mathbf{F}_7[X]/(X^2+X+3)$ are both finite fields of size 49. Show that these fields are isomorphic by exhibiting an explicit isomorphism between them.

4. Let R ba a ring with $1 \neq 0$ and M an R-module. Show that if $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of M then $\cup_{i \geq 1} N_i$ is a submodule of N. Show by way of counterexample that modules over a ring need not have maximal proper submodules (in contrast to the special case of ideals in a ring with 1).

5. Let R be any commutative ring with $1 \neq 0$ and M and R-module. Show that the canonical map
$$\mathrm{Hom}_R(R, M) \to M$$
sending φ to $\varphi(1)$ is an isomorphism of R-modules.

6. Let $F = \mathbf{R}$ and let $V = \mathbf{R}^3$. Consider the linear map $\varphi : V \to V$ given by rotation through an angle of $\pi/2$ about the z-axis. Consider V as an $F[X]$-module by defining
$$(a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0)v := (a_n \varphi^n + a_{n-1} \varphi^{n-1} + \cdots + a_1 \varphi + a_0)v,$$
where φ^i is the composition of φ with itself i-times.

(a) What are the $F[X]$-submodules of V?

(b) Show that V is naturally a module over the quotient ring $F[X]/(X^3 - X^2 + X - 1)$.

7. Let R be a ring with $1 \neq 0$.

   (a) For a left ideal I of R and an R-module M, define

   $$IM := \{r_1 m_1 + r_2 m_2 + \cdots + r_k m_k \ : \ r_i \in R,\ m_i \in M,\ k \in \mathbf{Z}_{\geq 0}\}.$$

   Show that IM is an R-submodule of M.

   (b) Prove that for any ideal I of R and any positive integer n, there is a canonical isomorphism of R-modules

   $$R^n/IR^n \simeq R/IR \times R/IR \times \cdots \times R/IR$$

   with n-factors in the product on the right.

   (c) Suppose now that R is commutative and that $R^n \simeq R^m$ as R-modules. Show that $m = n$. Hint: reduce to the case of finite dimensional vector spaces over a field by applying (7b) with I a maximal ideal of R.

   (d) If R is commutative and A is any finite set of cardinality n, show that $F(A) \simeq R^n$ as R-modules (Hint: Show that R^n satisfies the same universal mapping property as $F(A)$ and deduce from this that one has maps in both directions whose composition in either order must be the identity). Conclude that the rank of a free module over a commutative ring is well-definied if it is finite.

8. Let R be a ring with $1 \neq 0$ and M an R-module. We say that M is *irreducible* if $M \neq 0$ and the only submodules of M are 0 and M.

   (a) Show that M is irreducible if and only if M is a nonzero cyclic R-module.

   (b) If R is commutative, show that M is irreducible if and only if $M \simeq R/I$ as R-modules for some maximal ideal I of R.

   (c) Prove Schur's lemma: if M_1 and M_2 are irreducible R-modules then any nonzero R-module homomorphism $\varphi : M_1 \to M_2$ is an isomorphism.