Honors Algebra 4, MATH 371 Winter 2010

Assignment 2

Due Monday, January 25 at 08:35

1. Let R be a ring.

   (a) Let I be an ideal of R and denote by $\pi : R \to R/I$ the natural ring homomorphism defined by $\pi(x) := x \bmod I$ ($= x + I$ using coset notation). Show that an arbitrary ring homomorphism $\phi : R \to S$ can be factored as $\phi = \psi \circ \pi$ for some ring homomorphism $\psi : R/I \to S$ if and only if $I \subseteq \ker(\phi)$, in which case ψ is unique.

   (b) Suppose that R is commutative with 1. An *R-algebra* is a ring S with identity equipped with a ring homomorphism $\phi : R \to S$ mapping 1_R to 1_S such that $\mathrm{im}(\phi)$ is contained in the center of S (*i.e.* the set

   $$c(S) := \{z \in S \mid zs = sz \text{ for all } s \in S\}$$

   of all elements of S that commute with every other element). If (S, ϕ) and (S', ϕ') are two R-algebras then a ring homomorphism $f : S \to S'$ is called a *homomorphism of R-algebras* if $f(1_S) = 1_{S'}$ and $f \circ \phi = \phi'$. For an R-algebra (S, ϕ) we will frequently simply write rx for $\phi(r)x$ whenever $r \in R$ and $x \in S$.

   Prove that the polynomial ring $R[X]$ in one variable is naturally an R-algebra, and that if S is an R-algebra then for any $s \in S$ there exists a unique R-algebra homomorphism $f : R[X] \to S$ such that $f(X) = s$. In other words, mapping $R[X]$ to S is the "same" as choosing an element s of S.

2. Let R be a ring with 1.

   (a) Prove that there is a unique map of rings $f_R : \mathbf{Z} \to R$. Conclude that every ring with 1 is a $\mathbf{Z}$-algebra in a unique way.

   (b) For a ring R with 1, the kernel of the ring homomorphism f_R as in (2a) is an ideal of $\mathbf{Z}$ so it has the form $c(R)\mathbf{Z}$ for a unique $c(R) \in \mathbf{Z}$ satisfying $c(R) \geq 0$. By definition, the *characteristic of* R is this integer $c(R)$. Convince yourself that when $c(R) > 0$, this number is the least number of times we have to add $1 \in R$ to itself to get $0 \in R$. Now prove that if R is a ring with 1 that is an integral domain, then the characteristic of R is either 0 or a prime number.

   (c) Prove that for $g : R \to S$ a homomorphism of rings with 1 taking 1_R to 1_S the characteristic of S divides the characteristic of R.

   (d) Let $g : R \to S$ be a homomorphism of rings with 1 taking 1_R to 1_S. If g is injective, prove that $c(R) = c(S)$. Give an example with g not injective where $c(R) \neq c(S)$.

3. Let I and J be ideals of a ring R. We define

(a) $I + J := \{a + b \mid a \in I,\ b \in J\}$

(b) $IJ := \{a_1 b_1 + \cdots + a_s b_s \mid a \in I,\ b \in J\}$

Prove that $I+J$ is the smallest ideal of R containing I and J and that IJ is an ideal contained in the intersection $I \cap J$. Convince yourself that $I \cap J$ is an ideal of R, and show that if R is commutative and $I + J = R$ then $IJ = I \cap J$. Show by giving examples that $IJ \neq I \cap J$ in general, and that $I \cup J$ (set-theoretic union) need not be an ideal.

4. Let R be a commutative ring and I, J ideals of R. If P is a prime ideal of R containing IJ, prove that P contains I or P contains J.

5. Let R be a commutative ring.

   (a) Show that the set of all nilpotent elements of R (called the *nilradical of* R) is an ideal. Hint: this is basically 1(b) from assignment 1, but be careful about showing that this set is really an abelian group under addition.

   (b) Prove that the nilradical of R is contained in the intersection of all prime ideals of R.

   (c) Let $G := \mathbf{Z}/p\mathbf{Z}$ as a group under addition (it is cyclic of order p). Let $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}$ as a ring, and note that this is a field with p elements. Let R be the group ring $R := \mathbf{F}_p G$. What is the nilradical of R?

6. Let R be a commutative ring. Prove that the set of prime ideals in R has minimal elements with respect to inclusion. Such minimal elements are called *minimal primes*.

7. Let R be a finite (as a set) commutative ring with 1. Prove that every prime ideal of R is maximal.

8. Let $\varphi : R \to S$ be a homomorphism of commutative rings and I an ideal of S. Prove that $\varphi^{-1}(I)$ (set-theoretic inverse image) is an ideal of R that is prime whenever I is a prime ideal of S. Show that this holds with "prime" replaced by "maximal" provided we assume that φ is surjective. Give a counterexample to this if we drop the surjectivity requirement.