Honors Algebra 4, MATH 371 Winter 2010

Assignment 1

Due Friday, January 15 at 08:35

1. Let R be a ring. An element x of R is called *nilpotent* if there exists an integer $m \geq 0$ such that $x^m = 0$.
   (a) Show that every nilpotent element of R is either zero or a zero-divisor.
   (b) Suppose that R is commutative and let $x, y \in R$ be nilpotent and $r \in R$ arbitrary. Prove that $x + y$ and rx are nilpotent.
   (c) Now suppose that R is commutative with an identity and that $x \in R$ is nilpotent. Show that $1 + x$ is a unit and deduce that the sum of a unit and a nilpotent element is a unit.
2. Let R be a commutative ring with 1 and let $f := a_0 + a_1 x + \cdots + a_n x^n$ be an element of the ring $R[x]$ (*i.e.* a polynomial in one variable over R).
   (a) Prove that f is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent.
   (b) Prove that f is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent.
   (c) Prove that f is a zero-divisor in $R[x]$ if and only if f is nonzero and there exists $r \in R$ with $r \neq 0$ satisfying $rf = 0$.
3. Let n be a positive integer.
   (a) Determine the zero-divisors of the ring $\mathbf{Z}/n\mathbf{Z}$. Prove your answer.
   (b) For a prime p, let $G := \mathbf{Z}/p\mathbf{Z}$ as an abelian group (under addition of residue classes). Determine the zero divisors of the group-ring $\mathbf{Z}G$. Hint: it may help to write G multiplicatively.
4. List all subrings of $\mathbf{Z}/60\mathbf{Z}$. Which of these have an identity?
5. Prove that $x \in M_n(\mathbf{C})$ is nilpotent if and only if its only eigenvalue is zero. Show in particular that every strictly upper-triangular matrix (i.e. zeroes along and below the main diagonal) is nilpotent.