MATH 223, Linear Algebra
Fall, 2007
Solutions to Assignment 7

1. A linear operator T on a vector space V is called a *projection* if $T^2 = T$. (We will be looking at *orthogonal projections* later.)

   (a) If $T = T_A$ is represented by the matrix $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$, show that T is a projection.

   Solution: Since T^2 will be represented by A^2, it's enough to show that $A^2 = A$. I skip this very easy calculation.

   (b) Show that, if T is a projection on V, then so is $I - T$; here I is the identity operator on V.

   Solution: For any $\vec{v} \in V$, $(I-T)^2\vec{v} = (I-T)(I-T)\vec{v} =$ $(I-T)(\vec{v} - T\vec{v}) = \vec{v} - T\vec{v} - T(\vec{v} - T\vec{v}) = \vec{v} - T\vec{v} - T\vec{v} + T^2\vec{v}$. Using $T^2 = T$, this reduces to $\vec{v} - T\vec{v} = (I-T)\vec{v}$. So $(I-T)^2 = I - T$ and $I - T$ is a projection, too (if T is).

   (c) Show that, if T is a projection on V, then $V = ker(T) \oplus im(T)$. (Recall that $\oplus$ is the direct sum, so that not only do we have $V = ker(T) + im(T)$ but also, $ker(T) \cap im(T) = \vec{0}\}$.)

   Solution: For any $\vec{v} \in V$, of course $T\vec{v} \in im(T)$. Now $T(\vec{v} - T\vec{v}) = T\vec{v} - T^2\vec{v} = \vec{0}$, so $\vec{v} - T\vec{v} \in ker(T)$. As $\vec{v} = (\vec{v} - T\vec{v}) + T\vec{v}$, we have $\vec{v} \in ker(T) + im(T)$ and so $V = ker(T) + im(T)$.

   To show the sum is direct, suppose that $\vec{v} \in ker(T) \cap im(T)$. Then $T\vec{v} = \vec{0}$ and also $\vec{v} = T\vec{w}$ for some $\vec{w} \in V$. Then $\vec{0} = T\vec{v} = T(T\vec{w}) = T^2\vec{w} = T\vec{w} = \vec{v}$. So $ker(T) \cap im(T) = \{\vec{0}\}$, and thus $V = ker(T) \oplus im(T)$.

   (d) Show that, if T is a projection on V, its only possible eigenvalues are 0 and 1.

   Solution: One can do this directly, using the definition of eigenvalues. Supposing that λ is an eigenvalue of the projection T, we have $T\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$. Then $T^2\vec{v} = T(T\vec{v}) = T(\lambda\vec{v}) = \lambda T\vec{v} = \lambda^2\vec{v}$. But as $T^2 = T$, this is also $T\vec{v} = \lambda\vec{v}$; since $\vec{v} \neq \vec{0}$, $\lambda^2 = \lambda$ and so λ is either 0 or 1.

   Quicker, but maybe less instructive, is to note that since $T^2 - T = 0$, the minimal polynomial must divide $\lambda^2 - \lambda$ and the roots of this are just 0 and 1. The eigenvalues must then be either 0 or 1. [They are all zero just if T is the zero operator, and all 1 just if $T = I$.]

   (e) Show that, if T is a projection on V and V is finite-dimensional, then T is diagonalizable.

Solution: Short answer — the minimal polynomial divides $\lambda^2 - \lambda = \lambda(\lambda - 1)$ and so has no repeated roots; this implies the operator is diagonalizable.

The long answer is that since $V = ker(T) \oplus im(T)$, if we choose a basis for $ker(T)$ and a basis for $im(T)$, together they form a basis for V. Clearly each element of the basis for $ker(T)$ is an eigenvector corresponding to 0. Also, any vector in the basis for $im(T)$ is $\vec{v} = T\vec{w}$ for some $\vec{w} \in V$; as such $T\vec{v} = T^2\vec{w} = T\vec{w} = 1\vec{v}$ and so an eigenvector corresponding to 1. We have a basis of V consisting of eigenvectors, so T is diagonalizable.

Either way of looking at it is acceptable — the second is definitely more instructive, though longer. In any case, with an appropriately chosen basis $[T]$ comes out diagonal with 1's and 0's on the diagonal.

2. For each of the following three matrices over the reals, find its characteristic polynomial and its minimal polynomial. Decide which ones are diagonalizable. $A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 5 & -1 & 2 \\ 1 & 4 & 1 \\ 0 & 3 & 3 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 1 & -2 \\ -1 & 2 & -1 \end{pmatrix}$.

Solution: $det(\lambda I - A_1) = det \begin{pmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{pmatrix} =$ $(\lambda - 1)[(\lambda - 1)^2 - (-2)^2] + 2[-2(\lambda - 1) - (-2)^2] - 2[(-2)^2 + 2(\lambda - 1)] = \lambda^3 - 3\lambda^2 - 9\lambda - 5$. This is $\chi_{A_1}(\lambda)$ (or $\Delta_A(\lambda)$), the characteristic polynomial. It factors as $(\lambda + 1)^2(\lambda - 5)$ so the possibilities for the minimal polynomial are either χ_{A_1} itself, or $(\lambda + 1)(\lambda - 5) = \lambda^2 - 4\lambda - 5$. A simple calculation shows that $A_1^2 - 4A_1 - 5I = 0$, so $min_{A_1}(\lambda) = \lambda^2 - 4\lambda - 5$. The matrix is diagonalizable.

For A_2, the characteristic polynomial is $det(\lambda - A_2) = \lambda^3 - 12\lambda^2 + 45\lambda - 54$ by a calculation as above. This factors as $(\lambda - 3)^2(\lambda - 6)$. $(\lambda - 3)(\lambda - 6) = \lambda^2 - 9\lambda + 18$ is not the minimal polynomial, since $A_2^2 - 9A_2 + 18I = \begin{pmatrix} -3 & 6 & -3 \\ 0 & 0 & 0 \\ 3 & -6 & 3 \end{pmatrix} \neq 0$. It must be the same as χ_{A_2} (and it is — check it by multiplying this last matrix by $A_2 - 3I$).

For A_3, we get $det(\lambda I - A_3) = \lambda^3 + 9\lambda = \lambda(\lambda - 3i)(\lambda + 3i)$. This must also be the minimal polynomial, and the matrix is diagonalizable over $\mathcal{C}$, though not over $\mathcal{R}$.

3. (a) Suppose that the complex matrix A is diagonalizable and has only one

complex (possibly real) eigenvalue. Show that A is already diagonal.

Solution: We have, for any nonconstant polynomial $p(\lambda)$ over the complex numbers, that $p(\lambda)$ factors over $\mathcal{C}$ as $a_n(\lambda-\alpha_1)\cdots(\lambda-\alpha_n)$. Applying this to the characteristic polynomial of A, the α_j's are the eigenvalues of A. Assuming they are all the same α, but that A is diagonalizable, we must have that $P^{-1}AP$ is the diagonal matrix αI. But then $A = P(\alpha I)P^{-1} = \alpha I$ (since α_I commutes with every matrix of the same size).

(b) Give a 3×3 example of a real matrix B such that B has only one real eigenvalue, that B is diagonalizable (over $\mathcal{C}$), but B is not diagonal.

Solution: How about A_3 from the last problem?

4. Find the determinant of the following matrix over the reals: $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 8 & 18 & 9 \\ 3 & 17 & 53 & 4 \\ -4 & -24 & -75 & 10 \end{pmatrix}$.

Solution: We can do this by cofactor expansions, of course, but it's easy to start row-reducing it. Performing the three operations $R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 - 3R_1$ and $R_4 \mapsto R_4 + 4R_1$ does not change the determinant, and turns the given matrix into $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 8 & -5 \\ 0 & 8 & 38 & -17 \\ 0 & -12 & -55 & 38 \end{pmatrix}$. Next $R_3 \mapsto R_3 - 4R_2$ and $R_4 \mapsto R_4 + 6R_2$ still leaves the determinant unchanged, and yields $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 8 & -5 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & -7 & 8 \end{pmatrix}$. Finally $R_4 \mapsto R_4 + \frac{7}{6}R_3$ gives us the matrix $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & 8 & -5 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & \frac{23}{2} \end{pmatrix}$, which has the same determinant as our original matrix. It is $(1)(2)(6)(\frac{23}{2}) = 138$.

5. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ and $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g' & h' & j' \end{pmatrix}$ be matrices with complex entries, $det(A) = 2-i$ and $det(B) = 1+3i$. If $C = \begin{pmatrix} -3ie & b+(1+i)e & 3h+2ih' \\ -3id & a+(1+i)d & 3g+2ig' \\ -3if & c+(1+i)f & 3j+2ij' \end{pmatrix}$, what is $det(C)$? Justify.

Solution: Let's start by taking the transpose. This does not affect the determinant. $det(C) = det(C^T) = det \begin{pmatrix} -3ie & -3id & -3if \\ b+(1+i)e & a+(1+i)d & c+(1+i)f \\ 3ih+2ih' & 3g+2ig' & 3j+2ij' \end{pmatrix}$.

Factoring out $-3i$ from the first row, $det(C) = -3idet \begin{pmatrix} e & d & f \\ b+(1+i)e & a+(1+i)d & c+(1+i)f \\ 3h+2ih' & 3g+2ig' & 3j+2ij' \end{pmatrix}$.

Now we perform the row-operation $R_2 \mapsto R_2 - (1+i)R_1$, without altering the determinant, to find that $det(C) = -3idet \begin{pmatrix} e & d & f \\ b & a & c \\ 3h+2ih' & 3g+2ig' & 3j+2ij' \end{pmatrix}$.

Let's now flip the first two rows, changing the sign.

$det(C) = +3idet \begin{pmatrix} b & a & c \\ e & d & f \\ 3h+2ih' & 3g+2ig' & 3j+2ij' \end{pmatrix}$. Tossing the two first columns also switches the signs, so $det(C) = -3idet \begin{pmatrix} a & b & c \\ d & e & f \\ 3g+2ig' & 3h+2ih' & 3j+2ij' \end{pmatrix}$.

Now we're getting somewhere. By additivity in the third row, we have that

$$det(C) = -3i \left[det \begin{pmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3j \end{pmatrix} + det \begin{pmatrix} a & b & c \\ d & e & f \\ 2ig' & 2ih' & 2ij' \end{pmatrix} \right].$$

The first of these determinants is $3det(A) = 6 - 3i$ and the second one is $2idet(B) = -6 + 2i$; adding these and multiplying by $-3i$ tells us that $det(C) = -3$.