MATH 223, Linear Algebra
Fall, 2007
Assignment 6 Solutions

1. Diagonalize the following matrices over $\mathcal{R}$:

   (a) $\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

   Solution: For each matrix A above, we compute the characteristic polynomial to find the eigenvalues, then for each eigenvalue, we find a basis of the corresponding eigenspace. Finally we determine the matrix P such that $P^{-a}AP$ is diagonal.

   (a) We have
   $$\Delta_A(t) = \det\left(\begin{bmatrix} 4-t & 2 \\ 3 & -1-t \end{bmatrix}\right) = t^2 - 3t - 10 = (t-5)(t+2),$$
   so the eigenvalues are $5, -2$. To determine bases of the corresponding eigenspaces, we find
   $$E_5 = \text{nullsp}\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - 5 \cdot I_2\right) = \text{nullsp}\left(\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$
   and
   $$E_{-2} = \text{nullsp}\left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} + 2 \cdot I_2\right) = \text{nullsp}\left(\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right).$$
   It follows that $B := \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$ is a basis for $\mathcal{R}^2$ consisting of eigenvectors for A. Therefore, if we set
   $$P = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$$
   we have
   $$P^{-1}AP = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}.$$

   (b) Here we find
   $$\Delta_A(t) = t^2 - 3 = (t - \sqrt{3})(t + \sqrt{3}),$$
   so the eigenvalues are $\sqrt{3}, -\sqrt{3}$. We compute
   $$E_{\sqrt{3}} = \text{nullsp}\left(\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} - \sqrt{3} \cdot I_2\right) = \text{nullsp}\left(\begin{bmatrix} 1-\sqrt{3} & 1 \\ 2 & -1-\sqrt{3} \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1+\sqrt{3} \\ 2 \end{bmatrix}\right).$$
   and
   $$E_{-\sqrt{3}} = \text{nullsp}\left(\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} + \sqrt{3} \cdot I_2\right) = \text{nullsp}\left(\begin{bmatrix} 1+\sqrt{3} & 1 \\ 2 & -1+\sqrt{3} \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1-\sqrt{3} \\ 2 \end{bmatrix}\right).$$

It follows that $B := \left\{ \begin{bmatrix} 1+\sqrt{3} \\ 2 \end{bmatrix}, \begin{bmatrix} 1-\sqrt{3} \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathcal{R}^2$ consisting of eigenvectors for A. Therefore, if we set

$$P = \begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 2 & 2 \end{bmatrix}$$

we have

$$P^{-1}AP = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}$.
   (a) Explain why A can not be diagonalized over $\mathcal{R}$.
   (b) Diagonalize A over $\mathcal{C}$.

   Solution: A short calculation shows that

$$\Delta_A(t) = t^2 + 9.$$

It follows that A has no real eigenvalues, and hence that A is not diagonalizable over $\mathcal{R}$. To diagonalize A over $\mathcal{C}$, we first observe that A has complex eigenvalues $3i$ and $-3i$. We find

$$E_{3i} = \text{nullsp}\left(\begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} - 3i \cdot I_2 \right) = \text{nullsp}\left(\begin{bmatrix} 4-3i & -5 \\ 5 & -4-3i \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 4+3i \\ 5 \end{bmatrix} \right)$$

and

$$E_{-3i} = \text{nullsp}\left(\begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} + 3i \cdot I_2 \right) = \text{nullsp}\left(\begin{bmatrix} 4+3i & -5 \\ 5 & -4+3i \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 4-3i \\ 5 \end{bmatrix} \right).$$

Thus, a basis for $\mathcal{C}^2$ consisting of eigenvectors for A is: $B := \left\{ \begin{bmatrix} 4+3i \\ 5 \end{bmatrix}, \begin{bmatrix} 4-3i \\ 5 \end{bmatrix} \right\}$. Therefore, if

$$P = \begin{bmatrix} 4+3i & 4-3i \\ 5 & 5 \end{bmatrix}$$

we have

$$P^{-1}AP = \begin{bmatrix} 3i & 0 \\ 0 & -3i \end{bmatrix}.$$

3. Let V be a finite dimensional vector space over a field K, and suppose that $T : V \to V$ is a linear operator on V with eigenvalues $\lambda_1, \ldots, \lambda_m \in K$. For any nonnegative integer j, show that the linear operator $T^j : V \to V$ has eigenvalues $\lambda_1^j, \ldots, \lambda_m^j$.

   Solution: Let V and T be as above, and let λ be any eigenvalue of T. If v is an eigenvector of T with eigenvalue λ, then by definition

$$Tv = \lambda v.$$

Applying the linear operator T to both sides of this equation yields

$$T^2 v = T(\lambda v) = \lambda(Tv) = \lambda(\lambda v) = \lambda^2 v.$$

Repeating this procedure (i.e. applying T; or by induction) we arrive at the equality

$$T^j v = \lambda^j v$$

for all positive integers j. Thus (as v is assumed nonzero) we see that v is an eigenvector of T^j with eigenvalue λ^j for any positive integer j. By convention, $T^0 v = Iv = v$ and $\lambda^0 = 1$, so this equality holds for $j = 0$ too.

4. Let $V = P_2(\mathcal{C})$ be the complex vector space of polynomials with coefficients in $\mathcal{C}$ of degree at most 2, and consider the linear operator

$$L : V \to V$$

defined by

$$L(a_0 + a_1 t + a_2 t^2) = (a_0 - a_1) + (a_1 - a_2)t + (a_2 - a_0)t^2.$$

(a) Find all the eigenvalues of L (note that V is a complex vector space!).

(b) For each eigenvalue found in part a), determine a basis of eigenvectors for the corresponding eigenspace.

(c) Fnd a basis B such that $[L]_B$ is a diagonal matrix and determine $[L]_B$.

Solution: Let $E = \{1, t, t^2\}$ be the standard basis of $P_2(\mathcal{C})$. To determine $[L]_E$, we compute $L(1) = 1 - t^2$, $L(t) = -1 + t$, $L(t^2) = -t + t^2$. Thus,

$$[L]_E = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

and hence

$$\Delta_{[L]_E}(t) = \det\left(\begin{bmatrix} 1-t & -1 & 0 \\ 0 & 1-t & -1 \\ -1 & 0 & 1-t \end{bmatrix}\right) = (1-t)(1-t)^2 - 1(-1)(-1) = -(t^3 - 3t^2 + 3t).$$

The eigenvalues of L are therefore the complex roots of $t^3 - 3t^2 + 3t = t(t^2 - 3t + 3)$, so by the quadratic formula (for example) are:

$$0,\ \alpha := \frac{3 + \sqrt{-3}}{2},\ \overline{\alpha} := \frac{3 - \sqrt{-3}}{2}.$$

To determine a basis of each corresponding eigenspace of L, we first determine bases for the eigenspaces of $[L]_E$ (and then translate to bases of the eigenspaces of L using coordinate vectors). We find:

$$E_0 = \text{nullsp}\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\right).$$

Row reducing, we have

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$E_0 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).$$

Similarly,

$$E_\alpha = \operatorname{nullsp}\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} - \alpha \cdot I_3\right) = \operatorname{nullsp}\left(\begin{bmatrix} 1-\alpha & -1 & 0 \\ 0 & 1-\alpha & -1 \\ -1 & 0 & 1-\alpha \end{bmatrix}\right).$$

We then row reduce to determine a basis, using the formula $(1-\alpha)^3 - 1 = 0$

$$\begin{bmatrix} 1-\alpha & -1 & 0 \\ 0 & 1-\alpha & -1 \\ -1 & 0 & 1-\alpha \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & (1-\alpha)^2 \\ 0 & 1-\alpha & -1 \\ -1 & 0 & 1-\alpha \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & (1-\alpha)^2 \\ 0 & 0 & -1+(1-\alpha)^3 \\ -1 & 0 & 1-\alpha \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1+\alpha \\ 0 & 1 & -(1-\alpha)^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We conclude that

$$E_\alpha = \operatorname{span}\left(\begin{bmatrix} (1-\alpha) \\ (1-\alpha)^2 \\ 1 \end{bmatrix}\right).$$

A similar calculation shows that

$$E_{\overline{\alpha}} = \operatorname{span}\left(\begin{bmatrix} (1-\overline{\alpha}) \\ (1-\overline{\alpha})^2 \\ 1 \end{bmatrix}\right).$$

It follows that

$$B := \left\{1 + t + t^2,\ \frac{-1-\sqrt{-3}}{2} + \frac{-1+\sqrt{-3}}{2}t + t^2,\ \frac{-1+\sqrt{-3}}{2} + \frac{-1-\sqrt{-3}}{2}t + t^2\right\}$$

is a basis of $P_2(\mathcal{C})$ consisting of eigenvectors for L, and we have

$$[L]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \overline{\alpha} \end{bmatrix}.$$

5. Consider the *Fibonacci sequence* defined recursively by $F_0 = 0, F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

(a) Compute F_n for $n \leq 10$.

(b) Show that the F_n satisfy the matrix equation

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

(c) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(d) Show that for all n, we have the formula

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Solution:

(a) We readily compute the first 11 Fibonacci numbers to be:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.$$

(b) The matrix equation is equivalent to the system of equations

$$F_{n+2} = F_{n+1} + F_n \qquad F_{n+1} = F_{n+1};$$

the second equation is tautological, while the first is the defining relation of the Fibonacci sequence.

(c) The characteristic polynomial is

$$t^2 - t - 1,$$

so the eigenvalues are

$$\rho := \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\rho} := \frac{1-\sqrt{5}}{2} = 1 - \rho.$$

We easily determine that the corresponding eigenspaces are

$$E_\rho = \operatorname{nullsp}\left(\begin{bmatrix} 1-\rho & 1 \\ 1 & -\rho \end{bmatrix} \right) = \operatorname{span}\left(\begin{bmatrix} \rho \\ 1 \end{bmatrix} \right)$$

and

$$E_{\overline{\rho}} = \operatorname{nullsp}\left(\begin{bmatrix} 1-\overline{\rho} & 1 \\ 1 & -\overline{\rho} \end{bmatrix} \right) = \operatorname{span}\left(\begin{bmatrix} \overline{\rho} \\ 1 \end{bmatrix} \right).$$

Thus, if we set

$$P := \begin{bmatrix} \rho & \overline{\rho} \\ 1 & 1 \end{bmatrix}$$

we have

$$P^{-1}AP = \begin{bmatrix} \rho & 0 \\ 0 & \overline{\rho} \end{bmatrix} = D,$$

or what is the same thing, $A = PDP^{-1}$.

(d) From part b), we easily deduce the equality

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for every nonnegative integer n. Using part c), we get

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = (PDP^{-1})^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

However, from a previous assignment, we have

$$(PDP^{-1})^n = PD^nP^{-1},$$

so we get

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} \rho & \overline{\rho} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \rho^n & 0 \\ 0 & \overline{\rho}^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\overline{\rho} \\ -1 & \rho \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \rho^{n+1} - \overline{\rho}^{n+1} \\ \rho^n - \overline{\rho}^n \end{bmatrix},$$

which is the desired equality.