MATH 223, Linear Algebra
Fall, 2007
Solutions to Assignment 5

1. $W_1 = span\left\{\begin{pmatrix}1\\0\\5\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\15\\4\end{pmatrix}, \begin{pmatrix}1\\-1\\5\\-12\end{pmatrix}\right\}$ and $W_2 = span\left\{\begin{pmatrix}5\\0\\25\\-9\end{pmatrix}, \begin{pmatrix}13\\2\\65\\-5\end{pmatrix}, \begin{pmatrix}-11\\-4\\-55\\-17\end{pmatrix}\right\}$ are subspaces of $\mathcal{R}^4$. Find a basis for each of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

   Solution: We start by row-reducing the big matrix $\begin{pmatrix}1 & 3 & 1 & | & 5 & 13 & -11\\ 0 & 1 & -1 & | & 0 & 2 & -4\\ 5 & 15 & 5 & | & 25 & 65 & -55\\ -2 & 4 & -12 & | & -9 & -5 & -17\end{pmatrix}$, getting $\begin{pmatrix}1 & 0 & 4 & | & 0 & 2 & -4\\ 0 & 1 & -1 & | & 0 & 2 & -4\\ 0 & 0 & 0 & | & 1 & 1 & 1\\ 0 & 0 & 0 & | & 0 & 0 & 0\end{pmatrix}$. It is evident from the left half of this that the first two columns are independent, but not all three. A basis for W_1 is then $\left\{\begin{pmatrix}1\\0\\5\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\15\\4\end{pmatrix}\right\}$. Only slightly less obvious is that two of the three last columns are independent, but not all three; a basis for W_2 is $\left\{\begin{pmatrix}5\\0\\25\\-9\end{pmatrix}, \begin{pmatrix}13\\2\\65\\-5\end{pmatrix}\right\}$. Of the columns of the big row-reduced matrix, we can choose only three independent ones, most obviously the first, second and fourth. A basis for $W_1 + W_2$ is $\left\{\begin{pmatrix}1\\0\\5\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\15\\4\end{pmatrix} \begin{pmatrix}5\\0\\25\\-9\end{pmatrix}\right\}$. Finally, if we label the columns C_1,. . . ,C_6, then from the row-reduced matrix, it's clear that (say) $C_5 = 2C_1 + 2C_2 + C_4$; i.e $C_5 - C_4 = 2C_1 + 2C_2$. This is true for the original big matrix and that vector $C_5 - C_4$ is in the intersection. Since $2 + 2 = 3 + 1$, the intersection has dimension 1, and a basis for $W_1 \cap W_2$ is $\left\{\begin{pmatrix}8\\2\\40\\4\end{pmatrix}\right\}$.

2. Let $V = M_3(\mathcal{R})$ be the real vector space of 3×3 matrices with real

entries. Let $A = \begin{pmatrix} 3 & 5 & 2 \\ 1 & 0 & -1 \\ 7 & 5 & -2 \end{pmatrix}$. Now let $T : V \longrightarrow V$ be defined by $T(X) = AXA^T$ for any $X \in V$.

(a) Show that T is a linear operator on V.

Solution: The particular matrix A is irrelevant here. For any $X_1, X_2 \in V$, $T(X_1 + X_2) = A(X_1 + X_2)A^T = (AX_1 + AX_2)A^T$ $= AX_1A^T + AX_2A^T = T(X_1) + T(X_2)$ using the distributive laws. Also, for any $X \in V$ and scalar α, $T(\alpha X) = A(\alpha X)A^T = \alpha AXA^T = \alpha T(X)$. That does it.

(b) Suppose that

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

is the standard ordered basis for V. Find $[T]_B$.

Solution: Here the particular matrix A matters. Routine matrix multiplication shows us that $T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 21 \\ 3 & 1 & 7 \\ 21 & 7 & 9 \end{pmatrix}$. Letting the basis matrices be $B_1, \ldots, B_9$, we have $T(B_1) = 9B_1 + 3B_2 + 21B_3 + 3B_4 + 1B_5 + 7B_6 + 21B_7 + 7B_8 + 49B_9$. Those 9 coefficients go into the first column of $[T]_B$. $T(B_2) = AB_2A^T = 15B_1 + 0B_2 + 15B_3 + 5B_4 + 0B_5 + 5B_6 + 35B_7 + 0B_8 + 35B_9$, telling us the second column of $[T]_B$. We skip the rest of the details. $[T]_B =$

$$\begin{pmatrix} 9 & 15 & 6 & 15 & 25 & 10 & 6 & 10 & 4 \\ 3 & 0 & -3 & 3 & 0 & -5 & 2 & 0 & -2 \\ 21 & 15 & -6 & 35 & 25 & -10 & 14 & 10 & -4 \\ 3 & 5 & 2 & 0 & 0 & 0 & -3 & -5 & -2 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 7 & 5 & -2 & 0 & 0 & 0 & -7 & -5 & 2 \\ 21 & 35 & 14 & 15 & 25 & 10 & -6 & -10 & -4 \\ 7 & 0 & -7 & 3 & 0 & -5 & -2 & 0 & 2 \\ 9 & 35 & 14 & 35 & 25 & -10 & -14 & -10 & 4 \end{pmatrix}.$$

(c) Find a basis for each of $ker(T)$ and $im(T)$.

Solution: Sorry about this one. I promise, no 9×9 matrices on the exams. When I row-reduced $[T]_B$, I got the following. $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3} & 0 & -\frac{7}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{10}{3} & -1 & \frac{13}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{10}{3} & 0 & -\frac{10}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{5}{4} & 0 & \frac{15}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{49}{20} & 1 & \frac{21}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{7}{4} & 1 & -\frac{11}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

A basis for the column space of $[T]_B$ consists of its first six columns; the corresponding basis of $im(T)$ is $\left\{ \begin{pmatrix} 9 & 3 & 21 \\ 3 & 1 & 7 \\ 21 & 7 & 9 \end{pmatrix}, \begin{pmatrix} 15 & 0 & 15 \\ 5 & 0 & 5 \\ 35 & 0 & 35 \end{pmatrix}, \begin{pmatrix} 6 & -3 & -6 \\ 2 & -1 & -2 \\ 14 & -7 & 14 \end{pmatrix}, \begin{pmatrix} 15 & 3 & 35 \\ 0 & 0 & 0 \\ 15 & 3 & 35 \end{pmatrix}, \begin{pmatrix} 25 & 0 & 25 \\ 0 & 0 & 0 \\ 25 & 0 & 25 \end{pmatrix}, \begin{pmatrix} 10 & -5 & -10 \\ 0 & 0 & 0 \\ 10 & -5 & -10 \end{pmatrix} \right\}$. To find the null space of $[T]_B$ we solve the homogeneous system and get $\begin{array}{rcrcrcr} x_1 & = & -\frac{7}{3}r & + & 0s & + & \frac{7}{3}t \\ x_2 & = & \frac{10}{3}r & + & 1s & + & -\frac{13}{3}t \\ x_3 & = & -\frac{10}{3}r & + & 0s & + & \frac{10}{3}t \\ x_4 & = & \frac{5}{4}r & + & 0s & + & -\frac{15}{4}t \\ x_5 & = & -\frac{49}{20}r & + & -1s & + & -\frac{21}{4}t \\ x_6 & = & \frac{7}{4}r & + & 0s & + & \frac{11}{4}t \\ x_7 & = & 1r & + & 0s & + & 0t \\ x_8 & = & 0r & + & 1s & + & 0t \\ x_1 & = & 0r & + & 0s & + & 1t \end{array}$.

We get 3 matrices in our basis for $ker(T)$; the basis is $\left\{ \begin{pmatrix} -\frac{7}{3} & \frac{10}{3} & -\frac{10}{3} \\ \frac{5}{4} & -\frac{49}{20} & \frac{7}{4} \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{7}{3} & -\frac{13}{3} & \frac{10}{3} \\ -\frac{15}{4} & -\frac{21}{4} & \frac{11}{4} \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

[Note to the markers: I hope you don't get too many pages of hairy calculations. The main things I'm looking for are: the matrix $[T]_B$ should be 9×9 and they should know in principle how to calculate it — that's not so hard; the row-reduction is a bit much, but they should at the very least knwo that the dimensions of the kernel and image add up to 9; also their bases should come out as sets of 3×3 matrices. If they get these right, don't worry too much about the numbers being off.]

3. Show the *exchange property* for linear span. That is, suppose that V is a vector space over the field F and $S \cup \{\vec{v}, \vec{w}\}$ is a subset of F; also suppose that $\vec{w} \in span(S \cup \{\vec{v}\})$ but $\vec{w} \notin span(S)$. Show that (in this case) $\vec{v} \in span(S \cup \{\vec{w}\})$.

Solution: Since $\vec{w} \in span(S \cup \{\vec{v}\}$, there must be some vectors $\vec{u}_1, \ldots, \vec{u}_n \in S$ and some scalars $a_1, \ldots, a_n$ and b such that $\vec{w} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n + b\vec{v}$. Since $\vec{w} \notin span(S)$, $b \neq 0$. So we can solve for $\vec{v}$. To wit, $\vec{v} = -\frac{a_1}{b}\vec{u}_1 - \cdots - \frac{a_n}{b}\vec{u}_n + \frac{1}{b}\vec{w}$. So indeed $\vec{v} \in span(S \cup \vec{w})$.

4. Find the inverse of the following matrix, and express it as a product of elementary matrices. (It is, of course, over $\mathcal{C}$, the complex numbers.) $$\begin{pmatrix} i & 0 & 2-i \\ 0 & 1 & 0 \\ 1+3i & 0 & 5-3i \end{pmatrix}.$$

Solution: We work with the big matrix $\begin{pmatrix} i & 0 & 2-i & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 1+3i & 0 & 5-3i & | & 0 & 0 & 1 \end{pmatrix}$ and row-reduce it step-by-step. Our first row operation is $R_1 \mapsto -iR_1$, corresponding to the elementary matrix $E_1 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and yields

$$\begin{pmatrix} 1 & 0 & -1-2i & | & -i & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 1+3i & 0 & 5-3i & | & 0 & 0 & 1 \end{pmatrix}.$$

Our second row operation is $R_3 \mapsto R_3 + (-1-3i)R_1$, corresponding to the elementary matrix $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1-3i & 0 & 1 \end{pmatrix}$, and yields $\begin{pmatrix} 1 & 0 & -1-2i & | & -i & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 2i & | & -3+i & 0 & 1 \end{pmatrix}$.

Our third row operation is $R_3 \mapsto -\frac{1}{2}iR_3$, corresponding to the elementary matrix $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2}i \end{pmatrix}$, and yields $\begin{pmatrix} 1 & 0 & -1-2i & | & -i & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & \frac{1}{2}+\frac{3}{2}i & 0 & -\frac{1}{2}i \end{pmatrix}$.

Our fourth and final row operation is $R_1 \mapsto R_1 + (1+2i)R_3$, corresponding to the elementary matrix $E_4 = \begin{pmatrix} 1 & 0 & 1+2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and yields

$$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{5}{2}+\frac{3}{2}i & 0 & 1-\frac{1}{2}i \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & \frac{1}{2}+\frac{3}{2}i & 0 & -\frac{1}{2}i \end{pmatrix}.$$

The inverse is then $\begin{pmatrix} -\frac{5}{2}+\frac{3}{2}i & 0 & 1-\frac{1}{2}i \\ 0 & 1 & 0 \\ \frac{1}{2}+\frac{3}{2}i & 0 & -\frac{1}{2}i \end{pmatrix}$. It equals $E_4E_3E_2E_1$, so the given matrix is then $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} =$

$$\begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1+3i & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2i \end{pmatrix} \begin{pmatrix} 1 & 0 & -1-2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. Let $V = P_5(t)$ be the real vector space of polynomials of degree at most 5. Which of the following subsets of V are subspaces of it? Justify your answers.

   (a) $S_1 = \{p \in V : \frac{1}{4}(p'(t))^2 = p(t)\}$.

   (b) $S_2 = \{p \in V : p(2) = p(-2) = 0\}$.

   (c) $S_3 = \{p \in V : tp'(t) = 5p(t)\}$.

   Solution: S_1 is not a subspace, because for instance $t^2 \in S_1$, but $2t^2 \notin S_1$.

   S_2 is a subspace, because clearly the zero polynomial satisfies the condition. Also, if p_1 and p_2 are in S_2, then $p_1(2) = p_1(-2) = p_2(2) = p_2(-2) = 0$, so that $(p_1+p_2)(2) = p_1(2)+p_2(2) = 0+0 = 0$ and also $(p_1+p_2)(-2) = p_1(-2) + p_2(-2) = 0 + 0 = 0$. Hence $p_1 + p_2 \in S_2$ and it is closed under addition. Finally, if $p \in S_2$ and r is a real number $p(2) = p(-2) = 0$ so $(rp)(2) = rp(2) = r \cdot 0 = 0$ and also $(rp)(-2) = rp(-2) = r \cdot 0 = 0$; thus $rp \in S_2$ and S_2 is also closed under scalar multiplication.

   S_3 is also a subspace. Again, the zero polynomial satisfies the condition. If p_1 and p_2 are in S_3, then $tp_1'(t) = 5p_1(t)$ and $tp_2'(t) = 5p_2(t)$. Hence

   $$t(p_1+p_2)'(t) = t(p_1'(t)+p_2'(t)) = tp_1'(t)+tp_2'(t) = 5p_1(t)+5p_2(t) = 5(p_1+p_2)(t).$$

   So $p_1+p_2 \in S_3$. Finally, if $p \in S_3$ and r is real, then $t(rp)'(t) = tr(p'(t)) = rtp'(t) = r(5p(t)) = 5(rp)(t)$ and $rp \in S_3$.

6. Find a basis for each of the row space, column space, and null space of the following matrix. It is over $\mathcal{Z}_2$. $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$.

   What is the dimension of each of these spaces, and how many elements does each have?

   Solution: The given matrix row-reduces quickly to $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

   A basis for the row space is then $\{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \end{pmatrix}\}$.

   A basis for the null space is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$. Finally, for the null space, we see that we can choose parameters for x_4 and x_5; we have $x_3 = x_5$, $x_2 = x_4 + x_5$ and $x_1 = 0$. A basis for the null space is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The row and column spaces each have dimension 3 and 8 elements; the null space has dimension 2 and 4 elements.