MATH 223, Linear Algebra
Fall, 2007
Assignment 4 Solutions

1. Consider the vector space $V = P_5(\mathcal{R})$ of polynomials with real coefficients (in one variable t) of degree at most 5 (including the zero polynomial). Show that if $c \in \mathcal{R}$ is any real number, then the set
$$\{1, t-c, (t-c)^2, (t-c)^3, (t-c)^4, (t-c)^5\}$$
is a basis of V.

Solution: Since the set $\{1, t, t^2, t^3, t^4, t^5\}$ is visibly a basis of V, we know that this vector space has dimension 6. To show that the 6 vectors $\{1, t-c, (t-c)^2, (t-c)^3, (t-c)^4, (t-c)^5\}$ are a basis of V, it therefore suffices to show that they are linearly independent (as spanning is then automatic). So suppose that for some scalars $a_0, a_1, a_2, a_3, a_4, a_5$ we have
$$a_0 + a_1(t-c) + a_2(t-c)^2 + a_3(t-c)^3 + a_4(t-c)^4 + a_5(t-c)^5 = 0;$$
we wish to show that each a_i must be zero. Comparing coefficients of t^5 on both sides, we see that we must have $a_5 = 0$. Comparing coefficients of t^4 on both sides of the resulting equation then forces $a_4 = 0$ as well. Continuing in this manner, we find that $a_3 = a_2 = a_1 = a_0 = 0$, as desired.

2. Let V be as in problem (1). Let W be the subset of V consisting of those polynomials $p(t)$ such that $p(0) = 0$, and let U be the subset of polynomials $p(t)$ that are *even*, i.e. such that $p(t) = p(-t)$.
   (a) Show that U and W are subspaces of V.
   (b) Compute $\dim V$, $\dim W$, $\dim U$, $\dim U \cap W$.

Solution:

(a) It suffices to show that each set U, W contains the zero vector and is closed under addition and scalar multiplication. Since the value of the zero polynomial at zero is indeed zero, W contains zero; similarly, as $0 = -0$, we see that U contains zero as well. Now suppose that p, q are polynomials with $p(0) = q(0) = 0$. Then
$$(p+q)(0) = p(0) + q(0) = 0 + 0 = 0 \quad \text{and} \quad kp(0) = k(p(0)) = k \cdot 0 = 0,$$
so W is closed under addition and scalar multiplication, and is thus a subspace. As for V, if p, q are even, then we have
$$(p+q)(-t) = p(-t) + q(-t) = p(t) + q(t) = (p+q)(t) \quad \text{and} \quad kp(-t) = k(p(-t)) = kp(t),$$
so U is closed under addition and scalar multiplication and is therefore a subspace of V.

(b) Since $\{1, t, t^2, t^3, t^4, t^5\}$ is a basis of V we know that $\dim V = 6$. Moreover, for any $p(t) \in V$ we may write
$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$
for some real numbers a_i. Evaluating at $t = 0$, we see that $p(0) = a_0$. Thus, $p(t) \in W$ if and only if $a_0 = 0$. It follows at once that $\{t, t^2, t^3, t^4, t^5\}$ is a basis of W, so $\dim W = 5$. Changing t to $-t$ above, we see that
$$p(-t) = a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4 - a_5 t^5,$$

and hence $p(t) = p(-t)$ implies (by comparing coefficients) that

$$2a_1 = 2a_3 = 2a_5 = 0.$$

Since these equations are over $\mathcal{R}$, the only solution is $a_1 = a_3 = a_5 = 0$ (in contrast to $\mathcal{Z}_2$). It follows that $p(t) \in U$ if and only if

$$p(t) = a_0 + a_2t^2 + a_4t^4$$

for some scalars a_0, a_2, a_4, and hence that $\{1, t^2, t^4\}$ is a basis of U. Thus, $\dim U = 3$. Combining these analyses shows that $p(t) \in U \cap W$ if and only if we can write

$$p(t) = a_2t^2 + a_4t^4,$$

and hence that $\{t^2, t^4\}$ is a basis of $U \cap W$, so $\dim U \cap W = 2$.

3. Do the vectors $v_1 = (1, 2, 0, 3, 5)$, $v_2 = (2, 0, 3, 1, 7)$, $v_3 = (3, 5, 1, 0, 2)$, $v_4 = (1, 1, 0, 9, 4)$, $v_5 = (8, 6, 7, 5, 3)$ form a basis of $\mathcal{R}^5$?

   Solution: Since $\dim \mathcal{R}^5 = 5$, the above 5 vectors form a basis if and only if they are linearly independent; equivalently they form a basis if and only if the 5×5 matrix A with rows $v_1, \ldots, v_5$ has rank 5. By elementary row operations, we readily compute that the reduced row echelon form of A is the 5×5 identity matrix, so A has rank 5 and the v_i form a basis of $\mathcal{R}^5$.

4. Let A and B be matrices, and suppose that the product AB is defined. If the column space of B is contained in the nullspace of A, show that $AB = 0$ (i.e. the product AB is equal to the zero matrix).

   Solution: A matrix is the zero matrix if and only if the corresponding linear transformation is the zero transformation. Thus, to show that $AB = 0$, it suffices to show that ABv is the zero vector for every vector v. By the definition of matrix multiplication, $w = Bv$ is a vector in the column space of B, and hence $Aw = 0$ since the column space of B is contained in the null space of A by hypothesis. Since we have

   $$AB(v) = A(Bv)$$

   (since matrix multiplication is associative), we conclude that $ABv = 0$. As this holds for every choice of v, we deduce that AB is the zero matrix.

5. Let $V = \mathcal{C}$ be the complex numbers, considered as a vector space over the real numbers $\mathcal{R}$. It has dimension 2, and the vectors $e_1 = 1$, $e_2 = i$ form a basis for $\mathcal{C}$ over $\mathcal{R}$. Consider the mapping

   $$m_i : \mathcal{C} \to \mathcal{C}$$

   defined by

   $$m_i(z) = i \cdot z,$$

   (where the multiplication $i \cdot z$ is as complex numbers). Show that m_i is a linear map and determine its matrix relative to the ordered basis $\{e_1, e_2\}$ of $\mathcal{C}$ given above.

   Solution: Let $z, w \in \mathcal{C}$ be any complex numbers. If $k \in \mathcal{R}$ we have

   $$m_i(z + kw) = i \cdot (z + kw) = iz + ikw = iz + k(iw) = m_i(z) + km_i(w),$$

and it follows that m_i is linear. To determine its matrix relative to the ordered basis e_1, e_2, we compute

$$m_i(e_1) = m_i(1) = i \cdot 1 = i = e_2$$

and

$$m_i(e_2) = m_i(i) = i \cdot i = -1 = -e_1.$$

Thus, the matrix of m_i in this ordered basis is:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

6. Let $V = P_3(\mathcal{R})$ be the real vector space of polynomials with real coefficients (in one variable t) with degree at most 3 . Define a mapping

$$f : V \to V$$

by

$$f(p(t)) = t^3 p\left(\frac{1}{t}\right).$$

(a) Show that f is a linear map.

(b) Find the matrix of f relative to the ordered basis $\{e_1, e_2, e_3, e_4\}$ of V given by $e_1 = 1$, $e_2 = t$, $e_3 = t^2$, $e_4 = t^3$.

(c) Find a basis for the kernel of the linear mapping

$$g : V \to V$$

given by

$$g(p(t)) = f(p(t)) - p(t).$$

Solution:

(a) First observe that since any $p(t) \in V$ has degree at most 3, the (a priori rational) function $f(p(t))$ is in fact a polynomial, so f is well-defined. If $p(t)$ and $q(t)$ are polynomials and k is any scalar, we compute

$$f((p + kq)(t)) = t^3 \left(p\left(\frac{1}{t}\right) + kq\left(\frac{1}{t}\right)\right) = f(p(t)) + kf(q(t)),$$

so f is indeed linear.

(b) Since we readily compute that

$$f(e_i) = e_{5-i},$$

the matrix of f relative to the ordered basis e_i is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(c) Writing $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, we find that

$$f(p(t)) = a_3 + a_2 t + a_1 t^2 + a_0 t^3$$

and hence that

$$g(p(t)) = f(p(t)) - p(t) = (a_3 - a_0) + (a_2 - a_1)t + (a_1 - a_2)t^2 + (a_0 - a_3)t^3.$$

It follows that $p(t)$ is in the kernel of g if and only if $a_0 = a_3$ and $a_1 = a_2$, i.e. if and only if

$$p(t) = a_0(1 + t^3) + a_1(t + t^2).$$

Since $v_1 = 1 + t^3$ and $v_2 = t + t^2$ are clearly linearly independent, it follows that they form a basis of the kernel of g.