

MATH 223, Linear Algebra
Fall, 2007
Solutions to Assignment 3

1. Suppose that A and B are invertible matrices of the same size.

(a) Show that $B(A^{-1} + B^{-1}) = (A + B)A^{-1}$.

Solution: Easy. By distributivity, the left-hand side is $BA^{-1} + BB^{-1} = BA^{-1} + I$ and the right-hand side is $AA^{-1} + BA^{-1} = I + BA^{-1}$. By commutativity of $+$, the sides are equal.

(b) Suppose that $A + B$ is also invertible. Show that $A^{-1} + B^{-1}$ is invertible, and its inverse is $A(A + B)^{-1}B$.

Solution: Being a product of invertible matrices, $A(A + B)^{-1}B$ is invertible, and its inverse is $B^{-1}((A + B)^{-1})^{-1}A^{-1}$. This equals

$$B^{-1}(A+B)A^{-1} = (B^{-1}A+I)A^{-1} = B^{-1}AA^{-1}+IA^{-1} = B^{-1}+A^{-1} = A^{-1}+B^{-1}.$$

So $A^{-1} + B^{-1}$ is invertible and its inverse is $A(A + B)^{-1}B$.

2. (a) Let $i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, a matrix over the reals. Show that

$i^2 = -I$, that the set $C = \{aI + bi : a, b \in \mathcal{R}\}$ is a subspace of $M_4(\mathcal{R})$ (regarded as a vector space over \mathcal{R}), C is closed under matrix multiplication, and that every nonzero matrix in C has an inverse, which is also in C . (We can identify C with the field of complex numbers in a pretty obvious manner. Reminder: $M_4(\mathcal{R})$ is the collection of all 4×4 matrices with real entries.)

Solution: I'll skip the direct and easy calculation that $i^2 = -I$. The zero matrix is in C since we can take $a = b = 0$. Now if we have two elements of C , say they are $a_1I + b_1i$ and $a_2I + b_2i$ (with a_1, b_1, a_2 and b_2 real); their sum is $(a_1 + a_2)I + (b_1 + b_2)i$ by basic properties of matrix algebra, and this is in C . So C is closed under addition. If $\alpha \in \mathcal{R}$ and $aI + bi \in C$, then $\alpha(aI + bi) = (\alpha a)I + (\alpha b)i$, also an element of C , so C is closed under scalar multiplication. So it's a subspace of $M_4(\mathcal{R})$.

Next, suppose that $a_1I + b_1i$ and $a_2I + b_2i$ are elements of C . Using the basic rules, we get

$$\begin{aligned} (a_1I + b_1i)(a_2I + b_2i) &= (a_1I + b_1i)(a_2I) + (a_1I + b_1i)(b_2i) \\ &= (a_1I)(a_2I) + (b_1i)(a_2I) + (a_1I)(b_2i) + (b_1i)(b_2i) = a_1a_2I + b_1a_2i + a_1b_2i + b_1b_2i^2 \\ &= (a_1a_2 - b_1b_2)I + (b_1a_2 + a_1b_2)i, \end{aligned}$$

just as in the complex numbers; we use $i^2 = -I$ here. Finally, if $aI + bi \in C$ is not zero, then of course at least one of a and b is not zero. A special case of the calculation we just did tells us that $(aI + bi)(aI - bi) = (a^2 + b^2)I + (ba - ab)i = (a^2 + b^2)I$. $a^2 + b^2 \neq 0$, so the inverse of $aI + bi$ is the matrix $\frac{a}{a^2+b^2}I + \frac{-b}{a^2+b^2}i$ and that's in C .

[You could have written out $aI + bi = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{pmatrix}$ and so

on and done the calculations more directly, but why bother? It's messier. This part of the problem is really just a warm-up for part (b). I hope you see what I mean that C is "essentially the same" as \mathcal{C} — technically, the two structures are *isomorphic*, but you don't need to know that for the course.]

(b) Now let i be as above and $j = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $k = ij$

(the matrix product); calculate k and verify that $j^2 = k^2 = -I$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$ and $ik = -j$. Show that $H = \{aI + bi + cj + dk : a, b, c, d \in \mathcal{R}\}$ is a subspace of $M_4(\mathcal{R})$, that H is closed under matrix multiplication, and that every nonzero matrix in H has an inverse, which is also in H . [Hint for that bit: try multiplying $aI + bi + cj + dk$ by $aI - bi - cj - dk$ and see what happens.]

Solution: Direct calculation shows that $k = ij = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

and $ji = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -k$. It is routine to show that

$j^2 = k^2 = -I$. Also $jk = i$, $kj = jij = j(-k) = -i$ and so on are also tedious but easy.

Taking $a = b = c = d = 0$, we again see that the zero matrix is in H . Now if we have two elements of H , say they are $a_1I + b_1i + c_1j + d_1k$ and $a_2I + b_2i + c_2j + d_2k$ (with $a_1, b_1, c_1, d_1, a_2, b_2, c_2$ and d_2 real); their sum is $(a_1 + a_2)I + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k$ by basic properties of matrix algebra, and this is in H . So H is closed under addition. If $\alpha \in \mathcal{R}$ and $aI + bi + cj + dk \in H$, then $\alpha(aI + bi + cj + dk) = (\alpha a)I + (\alpha b)i + (\alpha c)j + (\alpha d)k$, also an element

of H , so H is closed under scalar multiplication. So it's a subspace of $M_4(\mathcal{R})$.

If $a_1I + b_1i + c_1j + d_1k$ and $a_2I + b_2i + c_2j + d_2k$ are two elements of H , their product is

$$\begin{aligned} &(a_1I)(a_2I) + (b_1i)(a_2I) + (c_1j)(a_2I) + (d_1k)(a_2I) + \\ &(a_1I)(b_2i) + (b_1i)(b_2i) + (c_1j)(b_2i) + (d_1k)(b_2i) + \\ &(a_1I)(c_2j) + (b_1i)(c_2j) + (c_1j)(c_2j) + (d_1k)(c_2j) + \\ &(a_1I)(d_2k) + (b_1i)(d_2k) + (c_1j)(d_2k) + (d_1k)(d_2k) \end{aligned}$$

by repeated use of the distributive laws. Using the facts mentioned above, and collecting terms, we see that this is just

$$\begin{aligned} &(a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)I + (b_1a_2 + a_1b_2 - d_1c_2 + c_1d_2)i + \\ &(c_1a_2 + d_1b_2 + a_1c_2 - b_1d_2)j + (d_1a_2 - c_1b_2 + b_1c_2 + a_1d_2)k. \end{aligned}$$

So it's an element of H , which is then closed under multiplication.

In case $a_1 = a_2 = a$, $b_1 = b$, $b_2 = -b$, $c_1 = c$, $c_2 = -c$, $d_1 = d$ and $d_2 = -d$, the entire mess above reduces to $(a^2 + b^2 + c^2 + d^2)I$ since everything else cancels. Unless $a = b = c = d = 0$, $a^2 + b^2 + c^2 + d^2$ is a positive real number, so the inverse of $aI + bi + cj + dk$ is $\frac{a}{a^2+b^2+c^2+d^2} + \frac{-b}{a^2+b^2+c^2+d^2}i + \frac{-c}{a^2+b^2+c^2+d^2}j + \frac{-d}{a^2+b^2+c^2+d^2}k$, which is in H .

The collection H then satisfies all the field axioms (with matrix addition and multiplication as operations) except that multiplication is *not* commutative. H is thus what's called a *division ring* — it's (isomorphic to) the so-called *quaternions* (or *Hamiltonians*). This last paragraph is just gratuitous information-spreading; it's not asking you to do anything.

3. Let $W = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where $\vec{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 5 \\ -7 \end{pmatrix}$, $\vec{w}_3 = \begin{pmatrix} 0 \\ 5 \\ 13 \\ -2 \end{pmatrix}$, be a subspace of \mathcal{R}^4 . For each of the vectors \vec{v}_1 , \vec{v}_2 and \vec{v}_3 below, determine whether it is in W . If it is, express it as a linear combination of \vec{w}_1 , \vec{w}_2 and \vec{w}_3 ; if not, explain why not. $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ -11 \\ -15 \\ 7 \end{pmatrix}$,

$$\vec{v}_3 = \begin{pmatrix} 2 \\ 7 \\ 20 \\ -5 \end{pmatrix}.$$

Solution: Clearly $\vec{v}_1 = \vec{0} = 0\vec{w}_1 + 0\vec{w}_2 + 0\vec{w}_3$ is in W . We check \vec{v}_2 and \vec{v}_3 simultaneously by considering the “doubly augmented” matrix

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 2 \\ 2 & 0 & 5 & -11 & 7 \\ 3 & 5 & 13 & -15 & 20 \\ 4 & -7 & -2 & 7 & -5 \end{array} \right). \text{ This row-reduces to } \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 13 & -1 \end{array} \right).$$

From this we see that we cannot solve either $x_1\vec{w}_1 + x_2\vec{w}_2 + x_3\vec{w}_3 = \vec{v}_2$ or $x_1\vec{w}_1 + x_2\vec{w}_2 + x_3\vec{w}_3 = \vec{v}_3$. Neither \vec{v}_2 nor \vec{v}_3 is in W . (That 20 should have been a 21, but what do you do?)

4. (a) Show that, if $p(X)$ is any polynomial over F , A is any square matrix over F , P is any invertible matrix over F , and $B = P^{-1}AP$, then $p(B) = P^{-1}p(A)P$. [Hint: First do this in the case $p(X) = X^m$.]

Solution: Contrary to what a lot of people think, it is usually *false* that $(P^{-1}AP)^m = P^{-m}A^mP^m$. Instead, $(P^{-1}AP)^m = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$. Using associativity, we see that all those PP^{-1} 's in the middle cancel out, leaving $P^{-1}A^mP$. So if $p(X) = X^m$, $p(B) = B^m = (P^{-1}AP)^m = P^{-1}A^mP = P^{-1}p(A)P$, as advertized.

For the general case, let $p(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$, so that $p(B) = a_n(P^{-1}AP)^n + a_{n-1}(P^{-1}AP)^{n-1} + \cdots + a_1(P^{-1}AP) + a_0I$. Note that $I = P^{-1}IP$ and using the special case in the previous paragraph, we see that $p(B) = a_nP^{-1}A^nP + a_{n-1}P^{-1}A^{n-1}P + \cdots + a_1P^{-1}AP + a_0P^{-1}IP$; factoring out P^{-1} from the left gives $p(B) = P^{-1}(a_nA^nP + a_{n-1}A^{n-1}P + \cdots + a_1AP + a_0IP)$, which equals (please, not “equals to”) $P^{-1}(a_nA^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)P = P^{-1}p(A)P$, as per our request.

- (b) Suppose that $p(X)$ is a polynomial over F , that $p(0) \neq 0$, and that $p(A) = 0$ (the $n \times n$ zero matrix) for some matrix A . Show that A is invertible, and in fact there is a polynomial $q(X)$ (of degree one less than that of $p(X)$) such that $A^{-1} = q(A)$.

Solution: Let $p(X) = a_nX^n + \cdots + a_1X + a_0$; we are assuming that $a_0 \neq 0$ but that $p(A) = a_nA^n + \cdots + a_1A + a_0I = 0$. Thus $I = -a_0^{-1}(a_nA^n + \cdots + a_1A) = (-a_0^{-1}a_nA^{n-1} - \cdots - a_0^{-1}a_1I)A = A(-a_0^{-1}a_nA^{n-1} - \cdots - a_0^{-1}a_1I)$. Let $q(X) = -a_0^{-1}a_nX^{n-1} - \cdots - a_0^{-1}a_1$ and we have $Aq(A) = q(A)A = I$, which is what the doctor ordered.

- (c) Let $A = \begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix}$. Find a quadratic polynomial $p(X)$ over \mathcal{R} such that $p(0) \neq 0$ and $p(A) = 0$; and thus a degree one polynomial $q(X)$

such that $A^{-1} = q(A)$. [To find p , first find a nontrivial dependence relation involving I , A and A^2 .]

Solution: $A^2 = \begin{pmatrix} 25 & 4 \\ 0 & 9 \end{pmatrix}$, so we see directly that $A^2 - 2A - 15I = 0$.

Thus $I = \frac{1}{15}A^2 - \frac{2}{15}A$ and $A^{-1} = \frac{1}{15}A - \frac{2}{15}I$. $p(X) = X^2 - 2X - 15$ and $q(X) = \frac{1}{15}X - \frac{2}{15}$ will do it.

5. Find a basis for each of the null space, row space and column space of the

following matrix over \mathcal{C} . $\begin{pmatrix} 1 & 2-3i & i & 0 \\ 1+i & 6-i & 3-2i & 4i \\ 2-2i & 1-11i & 5-5i & 4+4i \end{pmatrix}$

Solution: Aargh! My arithmetic went haywire again. After a certain

amount of fuss, the given matrix row-reduces to $\begin{pmatrix} 1 & 0 & 0 & \frac{4}{3} + 4i \\ 0 & 1 & 0 & \frac{2}{3} - \frac{2}{3}i \\ 0 & 0 & 1 & -\frac{2}{3} + \frac{2}{3}i \end{pmatrix}$.

Anyway, a basis for the row space is

$\left\{ \left(1 \ 0 \ 0 \ \frac{4}{3} + 4i \right), \left(0 \ 1 \ 0 \ \frac{2}{3} - \frac{2}{3}i \right), \left(0 \ 0 \ 1 \ -\frac{2}{3} + \frac{2}{3}i \right) \right\}$. We

take the first three columns of the original matrix as a basis for the col-

umn space; that is $\left\{ \begin{pmatrix} 1 \\ 1+i \\ 2-2i \end{pmatrix}, \begin{pmatrix} 2-3i \\ 6-i \\ 1-11i \end{pmatrix}, \begin{pmatrix} i \\ 3-2i \\ 5-5i \end{pmatrix} \right\}$. Finally, if

we solve the system represented by the RREF form we found, we see that

a basis for the null space is $\left\{ \begin{pmatrix} -\frac{4}{3} - 4i \\ -\frac{2}{3} + \frac{2}{3}i \\ \frac{2}{3} - \frac{2}{3}i \\ 1 \end{pmatrix} \right\}$.