Mathematics 189-223B, Winter 2003
Linear Algebra

The purpose here is to explain and elaborate a little upon the exponential notation $z = re^{i\theta}$, where z is a complex number (of absolute value 1), θ and r are real and $r \geq 0$.

It is standard to both define and represent a complex number as $a+bi$ where a and b are real numbers, and to draw the point (a, b) in a plane so that a is the horizontal ("real") component, and b is the vertical ("imaginary") component.

If $z = a + bi$, we say that a is the *real part* of z and bi the *pure imaginary part.* a and b are know as the *rectangular coordinates* of z.

It is convenient sometimes to represent the complex number z in term of its distance r from the origin and the angle θ made by the vector from the origin to z; if we take θ counterclockwise starting from the positive real axis, we will have (if $z \neq 0 + 0i$),

$$r = |z| = \sqrt{a^2 + b^2}, \cos(\theta) = \frac{a}{r}, \sin(\theta) = \frac{b}{r}.$$

Thus $z = r(\cos(\theta) + i\sin(\theta))$. Occasionally $cis(\theta)$ is used instead of $\cos(\theta) + i\sin(\theta)$; however, we will write $e^{i\theta}$. Thus

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

and any complex number z can be expressed in either rectangular or *polar coordinates* (these are r and θ). Going back and forth is easy. If $z = re^{i\theta}$, then $z = a + bi$, where $a = r\cos(\theta)$ and $b = r\sin(\theta)$, so, e.g.,

$$4e^{i\frac{7\pi}{6}} = 4(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})) = 4(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = -2\sqrt{3} - 2i.$$

Going backwards, if $z = -2\sqrt{3} - 2i$, we have $r = |z|$ (the *absolute value* or *modulus* of z) which is $\sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4$ and we choose θ so that $\cos(\theta) = \frac{-2\sqrt{3}}{4}$ and $\sin(\theta) = \frac{-2}{4}$. Notice that this doesn't determine θ uniquely; $\theta = \frac{7\pi}{6}$ works, but so does $-\frac{5\pi}{6}$ and so does $\frac{19\pi}{6}$, and so on. Of course, all these possibilities differ from each other by a multiple of 2π. We can write

$$-2\sqrt{3} - 2i = 4e^{i\frac{7\pi}{6}} = 4e^{-i\frac{5\pi}{6}}.$$

Note that if $a + bi = re^{i\theta}$ and $a \neq 0$, then $\tan(\theta) = \frac{b}{a}$, so if $a > 0$ we have $\theta = \arctan(\frac{b}{a})$. In a case like ours, $\frac{b}{a} = \frac{1}{\sqrt{3}}$ and $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$; we must add π to this to get the right θ; generally, for $a < 0$, $\theta = \arctan(\frac{b}{a}) + \pi$.

We can always write (for $r > 0$) $re^{i\theta} = e^{\ln r + i\theta}$, but there is usually no good reason to do so.

Polar notation is not well-suited to addition of complex numbers, but it fits in well with conjugation (if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$) and multiplication and division. Indeed, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}, \text{ and}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

We verify the first one, going through rectangular coordinates

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1(\cos(\theta_1) + i\sin(\theta_1)) r_2(\cos(\theta_2) + i\sin(\theta_2))$$

$$= r_1 r_2 \{[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)] + i[\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)]\} =$$

$$r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

using the angle-addition formulas, and this equals $r_1 r_2 e^{i(\theta_1+\theta_2)}$. (Some of us use this to *remember* the angle-addition formulas.)

Note that any number of the form $e^{i\theta}$ is of absolute value 1; geometrically, it will be on the unit circle on the complex plane.

Exponential polar notation is ideally suited to taking powers and roots of complex numbers; de Moivre's formula

$$[r(\cos(\theta) + i\sin(\theta))]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

looks like $(re^{i\theta})^n = r^n e^{in\theta}$ in this notation. Similarly, an nth root of $re^{i\theta}$ is $r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$; $r \geq 0$, so $r^{\frac{1}{n}}$ refers here to the unique nonnegative real number with nth power r. But there are n roots of z in the complex numbers if $z \neq 0$. These arise from the fact that while θ, $\theta + 2\pi$, $\theta + 4\pi$, etc., all determine the same angle, $\frac{\theta}{n}$, $\frac{\theta+2\pi}{n}$, $\frac{\theta+4\pi}{n}$, etc., don't (until we get to $\frac{\theta+(2n)\pi}{n}$, when we start to repeat ourselves).

We illustrate by finding the cube roots of i; i.e., all solutions in the complex numbers to $z^3 = i$. First, $r = 1$ here so we don't fuss with that; $i = e^{i\frac{\pi}{2}} = e^{i\frac{5\pi}{2}} = e^{\frac{9\pi}{2}}$ and dividing the exponents by 3 gives $e^{i\frac{pi}{6}}$, $e^{i\frac{5\pi}{6}}$ and $e^{\frac{9\pi}{6}}$. In rectangular coordinates, these are $\frac{\sqrt{3}}{2} + i\frac{1}{2}$, $-\frac{\sqrt{3}}{2} + i\frac{1}{2}$, and $-i$. (Of course, $e^{i\frac{\pi}{2}} = e^{i\frac{13\pi}{2}}$, also, but $e^{i\frac{\pi}{6}} = e^{i\frac{13\pi}{6}}$. We've got 'em all.)

It might be worth mentioning that we could easily solve this for rectangular coordinates algebraically; $(a + bi)^3 = i$ means that $(a^3 - 3ab^2) + (3a^2b - b^3)i = 0 + 1i$, so $a^3 - 3ab^2 = 0$ and $3a^2b - b^3 = 1$, etc. Usually, this is harder.

For us, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ is a definition and a notational convenience, but it is instructive to motivate it using Taylor series. Recall that the series for e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \cdots$$

If we trust that this should work for complex numbers, too, let's plug in $i\theta$ for x (where θ is real). We get

$$\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{4!} + \cdots$$

Now $1 = i^0 = i^4 = i^8 = \ldots$, $i = i^5 = i^9 = \ldots$, $-1 = i^2 = i^6 = i^{10} = \ldots$ and $-i = i^3 = i^7 = i^{11} = \ldots$ and so separating out the real (even) and imaginary (odd) parts, we get

$$(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$$

which (by looking again at the Taylor series) is $\cos(\theta) + i\sin(\theta)$; it seems reasonable to agree that this should then be $e^{i\theta}$.

We pause to consider the special case $\theta = \pi$, giving $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$, or

$$e^{\imath\pi} + 1 = 0.$$

Gaze in awe, mere mortals, at the spectacle — this formula brings together perhaps the five most important constants, the three most basic (positive) operations, and what is by far the most important relation in Mathematics. All in just seven symbols.