MATH 223, Linear Algebra
Fall, 2007
Assignment 9, due *in class* November 23, 2007

1. Let $W = Span\left\{\begin{pmatrix}1\\2\\1\\2\end{pmatrix}, \begin{pmatrix}3\\4\\3\\8\end{pmatrix}\right\}$ be a subspace of $\mathcal{R}^4$. Find an orthonormal basis for each of W and $W^\perp$. Also find the orthogonal projections $Proj_W\vec{v}$ and $Proj_{W^\perp}\vec{v}$, where $\vec{v} = \begin{pmatrix}1\\3\\5\\7\end{pmatrix}$.

2. Let $W = Span\left\{\begin{pmatrix}1\\1+i\\1\\1-i\\1\end{pmatrix}, \begin{pmatrix}4\\1+3i\\2i\\3-i\\2+i\end{pmatrix}, \begin{pmatrix}2+7i\\-1+5i\\-2-i\\5+i\\-1+3i\end{pmatrix}\right\}$ be a subspace of $\mathcal{C}^5$. Find an orthonormal basis for each of W and $W^\perp$. Also find the orthogonal projections $Proj_W\vec{v}$ and $Proj_{W^\perp}\vec{v}$, where $\vec{v} = \begin{pmatrix}2-i\\2+2i\\-2+i\\2-2i\\-1\end{pmatrix}$.

3. For each of the following Hermitian matrices H_j, find a unitary matrix U_j such that $\overline{U}_j^T H_j U$ is diagonal. Also find the diagonal matrix.
$H_1 = \begin{pmatrix}2 & 1+i\\1-i & 3\end{pmatrix}$, $H_2 = \begin{pmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{pmatrix}$, $H_3 = \begin{pmatrix}4 & 2+2i & 1-i\\2-2i & 6 & -2i\\1+i & 2i & 3\end{pmatrix}$.

4. (a) Suppose that A is a symmetric matrix over the reals with nonnegative eigenvalues. Show that there is a symmetric real matrix B such that $B^2 = A$. [A matrix B such that $B^2 = A$ is called a *square root* of A.]

   (b) Find a symmetric square root of the matrix $\begin{pmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{pmatrix}$.

5. Suppose that $(a_1, b_1), \ldots, (a_n, b_n)$ are points in the plane $\mathcal{R}^2$. If $n \geq 3$, it's unlikely that there is a line going through all of them.

   (a) Show that the line defined by $y = \alpha x + \beta$ goes through all of them if and only if $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a solution to $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix}$

and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

As mentioned, this is usually impossible if $n \geq 3$. But the system $A^T A\vec{x} = A^T\vec{b}$ frequently has a unique solution.

(b) Show that, if A is as above, and not all the values $a_1,\ldots,a_n$ are the same, then $A^T A$ has positive determinant and hence there is a unique solution to $A^T A\vec{x} = A^T\vec{b}$.

The solution $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is known as the *least squares solution* to the original system $A\vec{x} = \vec{b}$, and the line defined by $y = \alpha x + \beta$ is the *line of best fit* (to the data $(a_1, b_1),\ldots,(a_n, b_n)$).

(c) If $n = 4$ and $(a_1, b_1) = (0, 2)$, $(a_2, b_2) = (1, 3)$, $(a_3, b_3) = (2, 5)$ and $(a_4, b_4) = (3, 6)$ find the least-squares solution to $A\vec{x} = \vec{b}$; also find the line of best fit to these data.