MATH 223, Linear Algebra
Fall, 2007
Assignment 8, due *in class* Friday November 16, 2007

1. (a) Let $V = \mathcal{R}^4$ with its usual inner product $\langle v, w\rangle = v \cdot w$. Let $u = (1, 1, 1, 1)$, $v = (2, 3, -1, 2)$, and $w = (3, 4, 0, 3)$. Determine each of

$$\|u\|,\ \|v\|,\ \|w\|, \langle u, v\rangle,\ \langle u, w\rangle,\ \langle v, w\rangle.$$

   (b) Let $V = \mathcal{C}^n$. Show that $\langle v, w\rangle = v \cdot w$ is not an inner product on V.

   (c) Now let $V = \mathcal{C}^3$, equipped with its usual inner product $\langle v, w\rangle = v \cdot \overline{w}$. Set $u = (1 + i, 2, -3 - i)$ and $v = (i, 3i, 5 - 2i)$. Find

$$\|u\|,\ \|v\|,\ \langle u, v\rangle.$$

2. Let V be any real inner product space, with inner product $\langle \cdot, \cdot\rangle$. Prove that for all $u, v \in V$

$$\langle u + v, u - v\rangle = \|u\|^2 - \|v\|^2.$$

3. Let $V = M_n(\mathcal{R})$ be the real vector space of $n\times n$ real matrices, let T be the subspace of V consisting of upper triangluar matrices, and let W be the subspace of T consisting of diagonal matrices. For any two matrices $A, B \in V$, let

$$\langle A, B\rangle = \mathrm{tr}(AB).$$

Show that $\langle \cdot, \cdot\rangle$ is *not* an inner product on V, but that its restriction to W is an inner product on W. Is the restriction of $\langle \cdot, \cdot\rangle$ to T an inner product on T? Justify your answer.

4. Let $V = M_2(\mathcal{C})$ be the complex vector space of 2×2 complex matrices, equipped with the inner product

$$\langle A, B\rangle = \mathrm{tr}(\overline{B}^T A).$$

Let $W \subseteq V$ be the subspace of diagonal matrices. Find a basis for $W^\perp$.

5. Let $V = P_3(t)$ be the real vector space of polynomials of degree at most 3 with real coefficients. For all $f, g \in V$ let

$$\langle f, g\rangle := \int_{-1}^{1} f(t)g(t)dt.$$

   (a) Show that this defines an inner product on V.

   (b) Let $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = 3t^2 - 1$, $p_4(t) = 5t^3 - 3t$. Show that $B = \{p_1, p_2, p_3, p_4\}$ is an orthogonal basis of V. Is B an orthonormal basis of V?

6. Let V be the real vector space of continuous real-valued functions on the interval $[1, 2]$, and for any $f, g \in V$ let

$$\langle f, g\rangle = \int_{1}^{2} tf(t)g(t)dt.$$

Show that this defines an inner product on V, and that for any $f \in V$ we have

$$\left(\int_1^2 t^2 f(t)dt\right)^2 \leq \frac{15}{4}\left(\int_1^2 tf(t)^2 dt\right).$$