MATH 223, Linear Algebra
Fall, 2007
Assignment 3, due *in class* September 28, 2007

1. Suppose that A and B are invertible matrices of the same size.
   (a) Show that $B(A^{-1}+B^{-1}) = (A+B)A^{-1}$.
   (b) Suppose that $A+B$ is also invertible. Show that $A^{-1}+B^{-1}$ is invertible, and its inverse is $A(A+B)^{-1}B$.

2. (a) Let $i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$, a matrix over the reals. Show that $i^2 = -I$, that the set $C = \{aI + bi : a, b \in \mathcal{R}\}$ is a subspace of $M_4(\mathcal{R})$ (regarded as a vector space over $\mathcal{R}$), C is closed under matrix mutiplication, and that every nonzero matrix in C has an inverse, which is also in C. (We can identify C with the field of complex numbers in a pretty obvious manner. Reminder: $M_4(\mathcal{R})$) is the collection of all 4×4 matrices with real entries.)

   (b) Now let i be as above and $j = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $k = ij$ (the matrix product); calculate k and verify that $j^2 = k^2 = -I$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$ and $ik = -j$. Show that $H = \{aI + bi + cj + dk : a, b, c, d \in \mathcal{R}\}$ is a subspace of $M_4(\mathcal{R})$, that H is closed under matrix multiplication, and that every nonzero matrix in H has an inverse, which is also in H. [Hint for that bit: try multiplying $aI + bi + cj + dk$ by $aI - bi - cj - dk$ and see what happens.]

   The collection H then satisfies all the field axioms (with matrix addition and multiplication as operations) except that multiplication is *not* commutative. H is thus what's called a *division ring* — it's (isomorphic to) the so-called *quaternions* (or *Hamiltonians*). This last paragraph is just gratuitous information-spreading; it's not asking you to do anything.

3. Let $W = span\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where $\vec{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\vec{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 5 \\ -7 \end{pmatrix}$, $\vec{w}_3 = \begin{pmatrix} 0 \\ 5 \\ 13 \\ -2 \end{pmatrix}$ be a subspace of $\mathcal{R}^4$. For each of the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ below,

determine whether it is in W. If it is, express it as a linear combination of $\vec{w}_1$, $\vec{w}_2$ and $\vec{w}_3$; if not, explain why not. $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ -11 \\ -15 \\ 7 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ 7 \\ 20 \\ -5 \end{pmatrix}$.

(a) Show that, if $p(X)$ is any polynomial over F, A is any square matrix over F, P is any invertible matrix over F, and $B = P^{-1}AP$, then $p(B) = P^{-1}p(A)P$. [Hint: First do this in the case $p(X) = X^m$.]

(b) Suppose that $p(X)$ is a polynomial over F, that $p(0) \neq 0$, and that $p(A) = 0$ (the $n \times n$ zero matrix) for some matrix A. Show that A is invertible, and in fact there is a polynomial $q(X)$ (of degree one less than that of $p(X)$) such that $A^{-1} = q(A)$.

(c) Let $A = \begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix}$. Find a quadratic polynomial $p(X)$ over $\mathcal{R}$ such that $p(0) \neq 0$ and $p(A) = 0$; and thus a degree one polynomial $q(X)$ such that $A^{-1} = q(A)$. [To find p, first find a nontrivial dependence relation involving I, A and A^2.]

4. Find a basis for each of the null space, row space and column space of the following matrix over $\mathcal{C}$. $\begin{pmatrix} 1 & 2-3i & i & 0 \\ 1+i & 6-i & 3-2i & 4i \\ 2-2i & 1-11i & 5-5i & 4+4i \end{pmatrix}$