MATH 223, Linear Algebra
Fall, 2007
Assignment 2 Solutions

1. Find the rank of each of the matrices below:

   (a)
   $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$
   (over the rationals)

   (b)
   $$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
   (over $\mathcal{Z}_2$)

   (c)
   $$\begin{bmatrix} 1 & 2-i & 3+2i & 4-3i \\ 2+i & 1 & 4+3i & 3-2i \end{bmatrix}$$
   (over $\mathcal{C}$)

   Solution: Recall that the rank of a matrix A is defined to be the number of leading ones in the reduced row echelon form of A (this is the same thing as the row canonical form of A). Thus, to find the ranks of these matrices, we need to put them into reduced row echelon form.

2. Consider the following 3×3 real matrices:
   $$A = \begin{bmatrix} -2 & 1 & 8 \\ -1 & -1 & 7 \\ 3 & 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 0 & -7 \\ 6 & 3 & -9 \\ -2 & -2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 3 & -1 \\ 2 & 4 & 5 \\ -1 & -1 & 8 \end{bmatrix}.$$
   Find (by hand!) the following expressions:

   (a) AB
   (b) BA
   (c) $AB - BA$
   (d) ABC

3. Show that
   $$x = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
   is a solution to the equation
   $$x^3 + 1 = 0.$$

Can you think of another 3×3 real matrix that is a solution to this equation? Bonus: find all *diagonal* complex matrices that are solutions to $x^3 + 1 = 0$.

4. Find elementary matrices E_1, E_2, E_3 such that

$$E_1 E_2 E_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Show that the following sets with the indicated linear structure are vector spaces over the given field:

   (a) (Over the reals) The set S of real valued functions of one real variable. Addition of two functions f and g is defined by

   $$(f+g)(x) = f(x) + g(x),$$

   and scalar multiplication is defined by

   $$(k \cdot f)(x) = k(f(x)).$$

   (b) (Over the complex numbers) The set S of polynomials in one variable with complex coefficients, equipped with the usual addition and scalar multiplication of polynomials.

6. Show that the set

$$\mathcal{Z}_6 := \{a, b, c, d, e, f\}$$

equipped with addition given by the table

$+$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	c	d	e	f	a
c	c	d	e	f	a	b
d	d	e	f	a	b	c
e	e	f	a	b	c	d
f	f	a	b	c	d	e

can not be made in to a vector space over $\mathcal{Z}_2$.

7. Explain why each of the following subsets is or is not a subspace of the given vector space:

   (a) The subset of the complex vector space $\mathcal{C}$ consisting of those complex numbers whose absolute value is at most 1.

   (b) The subset of the real vector space of real valued functions of one variable consisting of continuous functions.

(c) The subset of the complex vector space of polynomials with complex coefficients consisting of those polynomials all of whose roots in $\mathcal{C}$ are distinct.

(d) The subset of the real vector space of polynomials with real coefficients consisting of those polynomials of degree at most 10.