MATH 223, Linear Algebra
Fall 2007
Midterm Solutions

1. (a) Solve the system of linear equations over the complex numbers

$$\begin{array}{rcrcr} x_1 & + & (2+i)x_2 & = & 7-3i \\ (3+i)x_1 & + & (6+6i)x_2 & = & -2+8i \end{array}$$

(b) Express the matrix

$$A = \begin{pmatrix} 1 & 2+i \\ 3+i & 6+6i \end{pmatrix}$$

as a product of elementary matrices.

(c) Find the inverse of the matrix A from part (b).

Solution: We will solve all three parts simultaneously by row-reducing the doubly-augmented matrix

$$M := \left(\begin{array}{cc|c|cc} 1 & 2+i & 7-3i & 1 & 0 \\ 3+i & 6+6i & -2+8i & 0 & 1 \end{array}\right).$$

We obtain:

$$\begin{aligned} M &\stackrel{R_2-(3+i)R_1\to R_2}{\sim} \left(\begin{array}{cc|c|cc} 1 & 2+i & 7-3i & 1 & 0 \\ 0 & 1+i & -26+10i & -3-i & 1 \end{array}\right) \\ &\stackrel{(\frac{1}{1+i})R_2\to R_2}{\sim} \left(\begin{array}{cc|c|cc} 1 & 2+i & 7-3i & 1 & 0 \\ 0 & 1 & -8+18i & -2+i & \frac{1-i}{2} \end{array}\right) \\ &\stackrel{R_1-(2+i)R_2\to R_1}{\sim} \left(\begin{array}{cc|c|cc} 1 & 0 & 41-31i & 6 & \frac{-3+i}{2} \\ 0 & 1 & -8+18i & -2+i & \frac{1-i}{2} \end{array}\right) \end{aligned}$$

(a) From our work above, we read off the unique solution $x_1 = 41-31i$ and $x_2 = -8+18i$.

(b) Note that many solutions are possible; we present one. Denoting by e_1, e_2, e_3 the three elementary row operations above, we have $e_3(e_2(e_1(A))) = I_2$, or what is the same thing

$$A = e_1^{-1}(e_2^{-1}(e_3^{-1}(I))) = E_1' E_2' E_3',$$

where $E_i' := e_i^{-1}(I)$. From this definition of E_i', we readily compute

$$E_1' = \begin{pmatrix} 1 & 0 \\ 3+i & 1 \end{pmatrix} \quad E_2' = \begin{pmatrix} 1 & 0 \\ 0 & 1+i \end{pmatrix} \quad E_3' = \begin{pmatrix} 1 & 2+i \\ 0 & 1 \end{pmatrix}$$

and so

$$A = \begin{pmatrix} 1 & 0 \\ 3+i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1+i \end{pmatrix} \begin{pmatrix} 1 & 2+i \\ 0 & 1 \end{pmatrix}.$$

(c) We read off the inverse from our initial work:

$$A^{-1} = \begin{pmatrix} 6 & \frac{-3+i}{2} \\ -2+i & \frac{1-i}{2} \end{pmatrix}.$$

2. Let $V = P(t)$ be the real vector space of polynomials with real coefficients. For each of the following subsets of $P(t)$, decide whether it is or is not a subspace of V. Justify your answers.

   (a) $S_1 = \{p \in V \mid p(7) = 0\}$.

   (b) $S_2 = \{p \in V \mid p(0) = 7 \text{ or } p \text{ is the zero polynomial}\}$.

   (c) $S_3 = \{p \in V \mid p \text{ is odd}\}$. [N.B. an odd polynomial $p(t)$ is one that satisfies $p(-t) = -p(t)$ for every real number t.]

Solution:

(a) We claim that S_1 is a subspace. Indeed, $0 \in S_1$ since the zero polynomial has value 0 everywhere (so in particular at 7). If $p, q \in S_1$ and $k \in \mathcal{R}$ then we have

$$(p+kq)(7) = p(7)+(kq)(7) = p(7)+k(q(7)) = 0+k\cdot 0 = 0+0 = 0,$$

so $p + kq \in S_1$. Thus, S_1 contains the zero vector and is closed under addition and scalar multiplication, so is a subspace.

(b) Observe that the constant polynomial $p(t) = 7$ is in S_2, but $2p(t) = 14$ is not. Thus, S_2 is not a subspace.

(c) The set S_3 is a subspace. Indeed, since $0(-t) = 0 = -0 = -0(t)$ we see that $0 \in S_3$. Moreover, we have

$$\begin{aligned}(p + kq)(-t) &= p(-t) + (kq)(-t) = p(-t) + k(q(-t)) = -p(t) - kq(t) \\ &= -(p + kq)(t)\end{aligned}$$

for any $p, q \in S_3$ and any $k \in \mathcal{R}$. It follows at once that S_3 is a subspace.

3. The following matrix is over the reals. Find a basis for its row space, its column space, and its null space.

$$\begin{pmatrix} 1 & -3 & 4 & 0 & 3 \\ 3 & -9 & 11 & 6 & 8 \\ -2 & 6 & -7 & -6 & -5 \end{pmatrix}$$

Solution: Row reducing, we find

$$\begin{pmatrix} 1 & -3 & 4 & 0 & 3 \\ 3 & -9 & 11 & 6 & 8 \\ -2 & 6 & -7 & -6 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 0 & 24 & -1 \\ 0 & 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows immediately that a basis for the row and column spaces are

$$\{\begin{bmatrix} 1 & -3 & 0 & 24 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -6 & 1 \end{bmatrix}\}$$

and

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ -7 \end{bmatrix} \right\}$$

respectively. Using the reduced echelon form found above, we see that the null space is determined by the system of equations

$$x_1 - 3x_2 + 24x_4 - x_5 = 0 \qquad x_3 - 6x_4 + x_5 = 0.$$

Clearly, x_2, x_4, x_5 are free variables, so the parametric form of a general solution to this system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -24 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so a basis of the nullspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -24 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

4. Let W_1 and W_2 be the subspaces of $\mathcal{R}^4$ defined by

$$W_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad W_2 = \text{span}\left\{ \begin{pmatrix} 4 \\ 2 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \\ 2 \end{pmatrix} \right\}.$$

   (a) Find a basis for $W_1 + W_2$.
   (b) Find a basis for $W_1 \cap W_2$.
   (c) Compute $\dim(W_1 + W_2)$ and $\dim(W_1 \cap W_2)$.

Solution: We row reduce the block matrix whose columns are the spanning vectors in W_1, W_2:

$$\left(\begin{array}{cc|cc} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 8 & 3 \\ 0 & 2 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 2 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By looking at the columns in which the leading ones occur, we deduce that a basis for $W_1 + W_2$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \\ 4 \end{bmatrix} \right\}.$$

It follows that $\dim(W_1 + W_2) = 3$. Since $\dim(W_1) = \dim(W_2) = 2$ by inspection, we conclude from the relation

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

that the intersection $W_1 \cap W_2$ has dimension 1. Therefore, any nonzero vector in $W_1 \cap W_2$ will form a basis. To find a nonzero vector in the intersection we seek a nonzero w of the form

$$w = aC_1 + bC_2 = dC_3 + eC_4,$$

where C_i is the ith column of the matrix above. Since row reducing does not change the relationship between the columns, we easily see (by looking at the row-reduced matrix) that

$$2C_1 + 3C_2 - C_3 = C_4,$$

and hence that the vector

$$C_3 + C_4 = \begin{bmatrix} 8 \\ 3 \\ 11 \\ 6 \end{bmatrix}$$

is in $W_1 \cap W_2$; since it is visibly nonzero, a basis of $W_1 \cap W_2$ is the set

$$\left\{ \begin{bmatrix} 8 \\ 3 \\ 11 \\ 6 \end{bmatrix} \right\}.$$

5. Suppose that A is a fixed $n \times n$ matrix, $V = M_n(F)$ is the vector space of $n \times n$ matrices over the field F, and $T : V \longrightarrow V$ is the following function.

$$T(X) = AX - XA \quad \text{for each} \quad X \in V.$$

   (a) Show that T is a linear operator on V (i.e., that $T : V \to V$ is a linear mapping).

   (b) Now suppose that $n = 2$ and that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

   Let B be the standard ordered basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   of V. Find $[T]_B$.

(c) Find a basis for each of ker(T) and im(T) (using the matrix A from part (b)).

Solution:

(a) Let X, Y be arbitrary $n \times n$ matrices and k any scalar. Then

$$\begin{aligned} T(X+kY) &= A(X+kY) - (X+kY)A \\ &= AX + AkY - XA - kYA \\ &= (AX - XA) + k(AY - YA) \\ &= T(X) + kT(Y), \end{aligned}$$

where we have used repeatedly the usual rules of matrix and scalar multiplication. It follows at once that T is linear.

(b) To determine the matrix of T with respect to B, we must evaluate T on each basis vector and write the result as a linear combination of the basis vectors. We find:

$$\begin{aligned} T(e_1) &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = -2e_2 + 2e_3. \end{aligned}$$

Similarly, we compute:

$$T(e_2) = -2e_1 + 2e_4 \quad T(e_3) = 2e_1 - 2e_4 \quad T(e_4) = 2e_2 - 2e_3.$$

Therefore, we have

$$[T]_B = \begin{bmatrix} 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \end{bmatrix}.$$

(c) To find a basis of the kernel and image of T, we first find a basis of the nullspace and column space of $[T]_B$. To do this, we row reduce $[T]_B$:

$$\begin{bmatrix} 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Accordingly, a basis of the column span of $[T]_B$ is

$$\left\{ \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Moreover, from the row reduction of $[T]_B$ above, we find that

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

is a basis of the nullspace of $[T]_B$. We conclude that

$$\left\{ \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

is a basis of $\operatorname{im}(T)$ and

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

is a basis of $\ker(T)$.

6. Suppose that A is an $n \times n$ matrix over a field F. Show that the following conditions on the matrix A are equivalent.

   (a) A is invertible.

   (b) For every $n \times n$ matrix B over F, there is a solution to the matrix equation $AX = B$.

   (c) For every $n \times n$ matrix B over F, there is a *unique* solution to the matrix equation $AX = B$.

Solution: First observe that if $AX = B$ has a unique solution, then in particular it *has* a solution, so c) $\Longrightarrow$ b). Assuming b) and taking $B = I_n$, we conclude that $AX = I$ has a solution; this implies that A is invertible so b) $\Longrightarrow$ a). Now assume that a) holds. Then if B is any $n \times n$ matrix, $X = A^{-1}B$ is a solution to $AX = B$. Given another solution X', we have $AX = B = AX'$ so $A^{-1}AX = A^{-1}AX'$ whence $X = X'$, and it follows that $X = A^{-1}B$ is the unique solution to $AX = B$. Thus a) $\Longrightarrow$ c). We conclude that all three statements are equivalent.