MATH 223, Linear Algebra
Fall 2007
Midterm exam, Wednesday, October 17, 2007

Instructions:

- No notes, books or calculators permitted.
- This exam has six questions. All questions are worth the same number of marks.
- Do all of your work on the sheets provided. Do not separate sheets that have been stapled together. If you need more space for a question, use the pages at the end of each part; please indicate on those pages which question you are continuing.
- The questions have been divided into two parts, purely to facilitate marking. Make sure you have a "white" and a "blue" set of questions. Make sure that your name, student number, and section number are on both parts. (If your instructor is Bryden Cais, you are in section 1; if your instructor is Jim Loveys, you are in section 2.)
- Have fun!

PART 1.

. Name (Please PRINT clearly):

. Student number:

. Section number:

1. (a) Solve the system of linear equations over the complex numbers

$$\begin{array}{rcrcr} x_1 & + & (2+i)x_2 & = & 7-3i \\ (3+i)x_1 & + & (6+6i)x_2 & = & -2+8i \end{array}$$

(b) Express the matrix

$$A = \begin{pmatrix} 1 & 2+i \\ 3+i & 6+6i \end{pmatrix}$$

as a product of elementary matrices.

(c) Find the inverse of the matrix A from part (b).

2. Let $V = P(t)$ be the real vector space of polynomials with real coefficients. For each of the following subsets of $P(t)$, decide whether it is or is not a subspace of V. Justify your answers.

   (a) $S_1 = \{p \in V \mid p(7) = 0\}$.

   (b) $S_2 = \{p \in V \mid p(0) = 7 \text{ or } p \text{ is the zero polynomial}\}$.

   (c) $S_3 = \{p \in V \mid p \text{ is odd}\}$. [N.B. an odd polynomial $p(t)$ is one that satisfies $p(-t) = -p(t)$ for every real number t.]

3. The following matrix is over the reals. Find a basis for its row space, its column space, and its null space.

$$\begin{pmatrix} 1 & -3 & 4 & 0 & 3 \\ 3 & -9 & 11 & 6 & 8 \\ -2 & 6 & -7 & -6 & -5 \end{pmatrix}$$

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PART 2.

. Name (Please PRINT clearly):

. Student number:

. Section number:

4. Let W_1 and W_2 be the subspaces of $\mathcal{R}^4$ defined by

$$W_1 = \operatorname{span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\3\\2\end{pmatrix}\right\} \quad \text{and} \quad W_2 = \operatorname{span}\left\{\begin{pmatrix}4\\2\\8\\4\end{pmatrix}, \begin{pmatrix}4\\1\\3\\2\end{pmatrix}\right\}.$$

(a) Find a basis for $W_1 + W_2$.

(b) Find a basis for $W_1 \cap W_2$.

(c) Compute $\dim(W_1 + W_2)$ and $\dim(W_1 \cap W_2)$.

5. Suppose that A is a fixed $n \times n$ matrix, $V = M_n(F)$ is the vector space of $n \times n$ matrices over the field F, and $T : V \longrightarrow V$ is the following function.

$$T(X) = AX - XA \quad \text{for each} \quad X \in V.$$

   (a) Show that T is a linear operator on V (i.e., that $T : V \to V$ is a linear mapping).

   (b) Now suppose that $n = 2$ and that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

   Let B be the standard ordered basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   of V. Find $[T]_B$.

   (c) Find a basis for each of $\ker(T)$ and $\operatorname{im}(T)$ (using the matrix A from part (b)).

6. Suppose that A is an $n \times n$ matrix over a field F. Show that the following conditions on the matrix A are equivalent.

   (a) A is invertible.

   (b) For every $n \times n$ matrix B over F, there is a solution to the matrix equation $AX = B$.

   (c) For every $n \times n$ matrix B over F, there is a *unique* solution to the matrix equation $AX = B$.

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