MATH 223, LINEAR ALGEBRA

FALL 2007

FINAL EXAMINATION

Thursday, December 6, 2007 9:00-12:00

Examiner: Professor James Loveys
Associate Examiner: Professor Bryden Cais

Instructions:

1. No notes, books or calculators permitted.
2. Do not write anything on the separate sheet summarizing the questions.
3. This exam has eight questions. All questions carry the same weight.
4. Do all your work on the sheets provided. Do not separate sheets that have been stapled together.
5. The questions have been divided into two parts, purely to facilitate marking. Make sure you have a "white" and a "blue" set of questions. Make sure that your name, student number, and section number are on both parts. (If your instructor is Jim Loveys, you are in section 2; if your instructor is Bryden Cais, you are in section 1.)

NAME:

STUDENT NUMBER:

SECTION NUMBER:

This examination comprises two cover pages (one for each part of the exam), 10 pages of questions, and 10 extra (blank) pages. The question pages and blank pages are in two parts. There is also a sheet summarizing the problems.

PART ONE:

1. Find a basis for each of the row, column and null space of the following matrix over the complex numbers:

$$\begin{bmatrix} 1 & 1-2i & 1+i \\ i & 2+i & -1+i \\ 2-i & -5i & 4+i \\ 3 & 3-6i & 4+3i \end{bmatrix}.$$

This page is for the continuation of problem 1; it may also be used for rough work.

2. Let V be the real vector space of 3×3 matrices with real entries. Identify which of the following subsets of V are subspaces of V. Justify your answers.

   (a) $\{X \in V \mid \mathrm{tr}(X) = 0\}$ (recall that $\mathrm{tr}(X)$ is the trace of X, i.e. the sum of the diagonal entries of X).

   (b) $\{X \in V \mid X \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = X^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\}$.

   (c) $\{X \in V \mid \det(X) = 0\}$.

This page is for the continuation of problem 2; it may also be used for rough work.

3. (a) Find an invertible matrix P such that $P^{-1}AP$ is diagonal, where

$$A = \begin{bmatrix} 5 & 0 & -6 \\ 0 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}.$$

(b) Find (explicitly) A^{10} where A is from part (a) of this problem. Note that $3^{10} = 59049$.

This page is for the continuation of problem 3; it may also be used for rough work.

4. Let $P_3(t)$ be the real vector space of polynomials of degree at most 3, and let V be the subspace of $P_3(t)$ consisting of those polynomials $p(t)$ such that $p(0) = p(1)$. Define the function $L : V \to V$ by
$$L(p(t)) = t(t-1)p''(t)$$
where $p''(t)$ denotes the second derivative of $p(t)$ with respect to t.

   (a) Show that L is a linear operator on V.

   (b) Find the matrix $[L]_B$, where B be the basis of V given by
$$B := \left\{1, t^2 - t, t^3 - t^2\right\}.$$

   (c) Find bases for $\ker(L)$ and $\operatorname{im}(L)$.

   (d) Find a basis B' of V such that $[L]_{B'} = D$ is diagonal, and find D.

This page is for the continuation of problem 4; it may also be used for rough work.

5. Suppose that A is an invertible matrix and that λ is an eigenvalue of A. Show that λ^{-1} is an eigenvalue of A^{-1}.

This page is for the continuation of problem 5; it may also be used for rough work.

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PART TWO:

6. Suppose the matrices $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ d^* & e^* & f^* \\ g & h & j \end{bmatrix}$ have complex entries, $\det(A) = 1 + i$ and $\det(B) = 3 - 2i$. Find the determinant of

$$\begin{bmatrix} (1+2i)a & 2id + (1-i)d^* & g + (-6+3i)a \\ (1+2i)b & 2ie + (1-i)e^* & h + (-6+3i)b \\ (1+2i)c & 2if + (1-i)f^* & j + (-6+3i)c \end{bmatrix}.$$

Justify your answer.

This page is for the continuation of problem 6; it may also be used for rough work.

7. Let V be the real vector space of continuous real-valued functions on the interval $[-1, 1]$, and for $f, g \in V$ let

$$\langle f, g \rangle = \int_{-1}^{1} x^4 f(x) g(x) dx.$$

   (a) Verify that this defines an inner product on V.

   (b) Show that, for any $f \in V$,

$$\left(\int_{-1}^{1} x^5 f(x) dx \right)^2 \leq \frac{2}{7} \int_{-1}^{1} x^4 f(x)^2 dx.$$

   For which f does equality hold?

This page is for the continuation of problem 7; it may also be used for rough work.

8. Let W be the subspace of $\mathcal{R}^4$ spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \end{bmatrix}$.

   (a) Find an orthonormal basis for each of W and $W^\perp$.

   (b) Find the orthogonal projections $\mathrm{Proj}_W(v)$ and $\mathrm{Proj}_{W^\perp}(v)$, where

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

This page is for the continuation of problem 8; it may also be used for rough work.

9. Find a unitary matrix U such that $\overline{U}^T HU$ is diagonal, where H is the following Hermitian matrix:

$$H := \begin{bmatrix} -3 & i & 1 \\ -i & -3 & -i \\ 1 & i & -3 \end{bmatrix}.$$

[Hint: -4 is an eigenvalue of H.]

This page is for the continuation of problem 9; it may also be used for rough work.

10. Suppose that V is a real inner product space. Prove the following version of *Pythagoras' Theorem*.

    If $v, w \in V$ are orthogonal, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$

This page is for the continuation of problem 10; it may also be used for rough work.

Summary of Problems

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Justify your answer.

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